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DISPERSION PROPERTIES OF TRANSVERSELY ISOTROPIC LAYERED SHELLS

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Harmonic wave propagation in thick, cylindrical, three-layered shells of infinite length was studied. Both the outer layers and the core are composites made of short strand fiberglass resin, but the planes of isotropy in the outer layers are orthogonal to the plane of isotropy at the core. A closed form solution of the exact linear equations of elasticity was sought in terms of Frobenius power series. The influence of the core thickness on the dynamics of the wave motion is estimated from numerically computed dispersion curves. Prime consideration was given the asymmetric wave motion and the different types of waves which can occur are identified over a wide range of wave numbers.

1. Stress-strain and strain — displacement relationships

Let u_i, v_i and w_i be the orthogonal components of displacement in axial (x-wise), circumferential (φ-wise) and radial (z-wise) directions of the i-th layer, respectively. The stress-strain relationship for the outer layers made of transversely isotropic material with the plane of isotropy parallel to the $x-r$ is of the form:

$$\begin{bmatrix}\sigma_{xi}\\ \sigma_{\varphi i}\\ \sigma_{ri}\\ \tau_{\varphi ri}\\ \tau_{xri}\\ \tau_{x\varphi i}\end{bmatrix} = \begin{bmatrix} C_{11i} & C_{12i} & C_{13i} & 0 & 0 & 0\\ C_{12i} & C_{22i} & C_{12i} & 0 & 0 & 0\\ C_{13i} & C_{12i} & C_{11i} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44i} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55i} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{44i}\end{bmatrix} \begin{bmatrix}\varepsilon_{xi}\\ \varepsilon_{\varphi i}\\ \varepsilon_{ri}\\ \gamma_{\varphi ri}\\ \gamma_{xri}\\ \gamma_{x\varphi i}\end{bmatrix} \tag{1.1}$$

where $C_{55i}=1/2(C_{11i}-C_{13i})$, for $i=2$ (the core). The stress-strain relationship for the middle layer made of transversely isotropic material with the plane of isotropy parallel to the $x-\varphi$ plane is given in the form:

$$\begin{bmatrix}\sigma_{xi}\\ \sigma_{\varphi i}\\ \sigma_{ri}\\ \tau_{\varphi ri}\\ \tau_{xri}\\ \tau_{x\varphi i}\end{bmatrix} \begin{matrix}=\\=\\=\\=\\=\\=\end{matrix} \begin{bmatrix}C_{11i} & C_{12i} & C_{13i} & 0 & 0 & 0\\ C_{12i} & C_{22i} & C_{12i} & 0 & 0 & 0\\ C_{13i} & C_{12i} & C_{11i} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44i} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{44i} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66i}\end{bmatrix} \begin{bmatrix}\varepsilon_{xi}\\ \varepsilon_{\varphi i}\\ \varepsilon_{ri}\\ \gamma_{\varphi ri}\\ \gamma_{xri}\\ \gamma_{x\varphi i}\end{bmatrix} \tag{1.1)''}$$

where $C_{66i}=1/2\ (C_{11i}-C_{12i})$ for $i=1,\ 3$ (inner and outer layers).

The strain-displacement relations in polar coordinates are:

$$\varepsilon_{xi}=\frac{\partial u_i}{\partial x},\qquad \gamma_{r\varphi i}=\frac{\partial v_i}{\partial x_i}+\frac{1}{r}\frac{\partial u_i}{\partial\varphi}$$

$$\varepsilon_{ri}=\frac{\partial w_i}{\partial r},\qquad \gamma_{xri}=\frac{\partial w_i}{\partial x}+\frac{\partial u_i}{\partial r} \tag{1.2}$$

$$\varepsilon_{\varphi i}=\frac{1}{r}\frac{\partial v_i}{\partial\varphi}+\frac{w_i}{r}\qquad \gamma_{\varphi ri}=\frac{1}{r}\frac{\partial w_i}{\partial\varphi}+\frac{\partial v_i}{\partial r}-\frac{v_i}{r}.$$

2. Governing differential equations

Consider a thick shell with homogeneous orthotropic layers. The three-dimensional equilibrium equations for each layer can be expressed as follows [1]:

$$\frac{\partial\sigma_{xi}}{\partial x}+\frac{1}{r}\frac{\partial\tau_{x\varphi i}}{\partial\varphi}+\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{xri})=\rho_i\frac{\partial^2 u_i}{\partial t^2}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{\varphi ri})+\frac{1}{r}\frac{\partial\sigma_{\varphi i}}{\partial\varphi}+\frac{\partial\tau_{x\varphi i}}{\partial x}=\rho_i\frac{\partial^2 v_i}{\partial t^2} \tag{2.1}$$

$$\frac{\partial\tau_{xri}}{\partial x}+\frac{1}{r}\frac{\partial\tau_{x\varphi i}}{\partial\varphi}+\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{ri})-\frac{1}{r}\sigma_{\varphi i}=\rho_i\frac{\partial^2 w_i}{\partial t^2},$$

Dimensionless variables have first been introduced in the form $r=zH$ and $x=\xi R_4$, with H being the total thickness, R_4 the outer radius of the cylinder, and $\kappa=R_4/H$ and $\lambda=R_4/L$, where L is the wavelength. The stress-strain and strain-displacement relationships (1.1)′, (1.1)″ and (1.2) are now used to obtain the governing differential equations of the three-dimensional elasticity in terms of displacement:

$$[C_{11i}D_{\zeta\zeta}+\kappa^2C_{55i}(D_{zz}+D_z/z)+\kappa^2/z^2C_{66i}D_{\varphi\varphi}]u_i+\kappa/z(C_{12i}+C_{66i})D_{\zeta\varphi}v_i+$$
$$+[\kappa/z(C_{55i}+C_{12i})D_\zeta+\kappa(C_{13i}+C_{55i})D_{z\zeta}]w_i=\rho_iR_4^2D_{tt}u_i,$$
$$\kappa/z(C_{12i}+C_{66i})D_{\zeta\varphi}u_i+[C_{66i}D_{\zeta\zeta}+\kappa^2/z^2C_{22i}D_{\varphi\varphi}+ \tag{2.2}$$

$$+\kappa^2 C_{44i}(D_{zz}+1/zD_z-1/z^2)]v_i+ \tag{2.2 [cont.]}$$

$$+[\kappa^2/z(C_{23i}+C_{44i})D_{z\varphi}+\kappa^2/z^2(C_{22i}+C_{44i})D_\varphi]w_i=\rho_i R_4^2 D_{tt}v_i,$$

$$[\kappa(C_{13i}+C_{44i})D_{z\varphi}-\kappa^2/z^2(C_{22i}+C_{44i})D_\varphi]v_i+$$

$$+[C_{55i}D_{\zeta\zeta}+\kappa^2/z^2 C_{44i}D_{\varphi\varphi}+\kappa^2 C_{33i}(D_{zz}+1/zD_z)-\kappa^2/z^2 C_{22i}]w_i=\rho_i R_4^2 D_{tt}w_i.$$

3. Boundary and continuity conditions

For the purpose of this treatment we assume the inside and outside surfaces of the layered cylinder to be stress free. Furthermore, at the internal interfaces of the adjacent layers there is equality of each displacement and of the shear and normal stress components. The stresses are given by the following equations [2]:

$$\sigma_{zi}(z,\varphi,\xi)=\left[C_{13i}\lambda u_i+\frac{C_{23i}}{z}(nv_i+w_i)+C_{33i}w_i'\right]\sin(\lambda R_4\xi)\cos(n\varphi)e^{i\Omega t} \tag{3.1}$$

$$\tau_{z\xi i}(z,\varphi,\xi)=C_{55i}(u_i'-\lambda w_i)\cos(\lambda R_4\xi)\cos(n\varphi)e^{i\Omega t}$$

$$\tau_{z\varphi i}(z,\varphi,\xi)=C_{44i}\left(v_i'-\frac{nw_i+v_i}{z}\right)\cos(\lambda R_4\xi)\sin(n\varphi)e^{i\Omega t},$$

The boundary and continuity conditions are defined as follows:

$$\sigma_{z1}(R_1)=0,\qquad \tau_{z\xi1}(R_1)=0,\qquad \tau_{z\varphi1}(R_1)=0,$$

$$\sigma_{z1}(R_2)=\sigma_{z2}(R_2),\qquad \tau_{z\xi1}(R_2)=\tau_{z\xi2}(R_2),\qquad \tau_{z\varphi1}(R_2)=\tau_{z\varphi2}(R_2),$$

$$u_1(R_2)=u_2(R_2),\qquad v_1(R_2)=v_2(R_2),\qquad w_1(R_2)=w_2(R_2),$$

$$u_2(R_3)=u_3(R_3),\qquad v_2(R_3)=v_3(R_3),\qquad w_2(R_3)=w_3(R_3),$$

$$\sigma_{z2}(R_3)=\sigma_{z3}(R_3),\qquad \tau_{z\xi2}(R_3)=\tau_{z\xi3}(R_3),\qquad \tau_{z\varphi2}(R_3)=\tau_{z\varphi3}(R_3),$$

$$\sigma_{z3}(R_4)=0,\qquad \tau_{z\xi3}(R_4)=0,\qquad \tau_{z\varphi3}(R_4)=0.$$

4. Solutions of differential equations

As the system of differential equations (2.2) is singular at $z=0$, the general solution is sought in terms of generalized power series (Frobenius series):

$$u_i=\sum_{j=1}^{j=6}A_{ij}\sum_{k=0}^{k=\infty}a_{kj}z^{k+\alpha_j}[\sin(\lambda R_{4\xi})\cos(n\varphi)\exp(i\Omega t)], \tag{4.1}$$

$$v_i = \sum_{j=1}^{j=6} A_{ij} \sum_{k=0}^{k=\infty} b_{kj} z^{k+\alpha_j} [\cos(\lambda R_{4\xi}) \sin(n\varphi) \exp(i\Omega t)], \tag{4.1 [cont.]}$$

$$w_i = \sum_{j=1}^{j=6} A_{ij} \sum_{k=0}^{k=\infty} c_{kj} z^{k+\alpha_j} [\cos(\lambda R_{4\xi}) \cos(n\varphi) \exp(i\Omega t)].$$

The coefficients a_{kj}, b_{kj}, c_{kj} and the indices α_j in the Frobenius power series are to be determined such that the differential equations (2.2) are satisfied. The procedure has been treated in paper [1] and will not be repeated here.

The purpose of this paper is to find the dispersion curves when the cylinder consists of two types of transversely isotropic materials to form a sandwich configuration. By inserting the solutions (4.1) into the set of differential equations (2.2) and the boundary conditions (3.2) and by using relations (3.1) we can accomplish the task.

The wavespeed ratio $c_w = (\Omega/\lambda)/\sqrt{(C_{55i}/\rho)}$, where λ is the actual wavenumber in the longitudinal direction and is implicitly involved in the coefficients of the Frobenius power series. For a non-trial solution, the determined of the final 18 homogenous linear algebraic equations must vanish. This results in the dispersion (or frequency) equation of the form:

$$|\ \beta_{ij}(n, c_w, \lambda \ldots)\ | = 0, \qquad \text{for } i, j = 1, 2, \ldots 18. \tag{4.2}$$

5. Numerical evaluation and discussion of results

The dispersion equation (4.2) is a function of all the geometrical and material parameters of the layered shell. We have considered a rather thick shell in which $\kappa = R_4/H = 4$ for all computations. The only geometrical parametr varied is the ratio of the core thickness to the total thickness H of the shell, $ph2 = h_2/(2h_1 + h_2)$. Extreme values of the parameter $ph2$ are $ph2 = 0$ (corresponding to a single layered shell made of the material defined by Eq. (1a)) and $ph2 = 1$ (corresponding to a single layered shell made of material defined by Eq. (1b)). The matrices of elastic constants are given as follows:

Material according to (1.1)′:

$$\begin{bmatrix} 21 & 9 & 7 & 0 & 0 & 0 \\ 9 & 21 & 7 & 0 & 0 & 0 \\ 7 & 7 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

Material according to (1.1)″:

$$\begin{bmatrix} 21 & 7 & 9 & 0 & 0 & 0 \\ 7 & 10 & 7 & 0 & 0 & 0 \\ 9 & 7 & 21 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

These values have been kept constant in the numerical experiment. With this arrangement the dispersion equation is a function of only four nondimensional

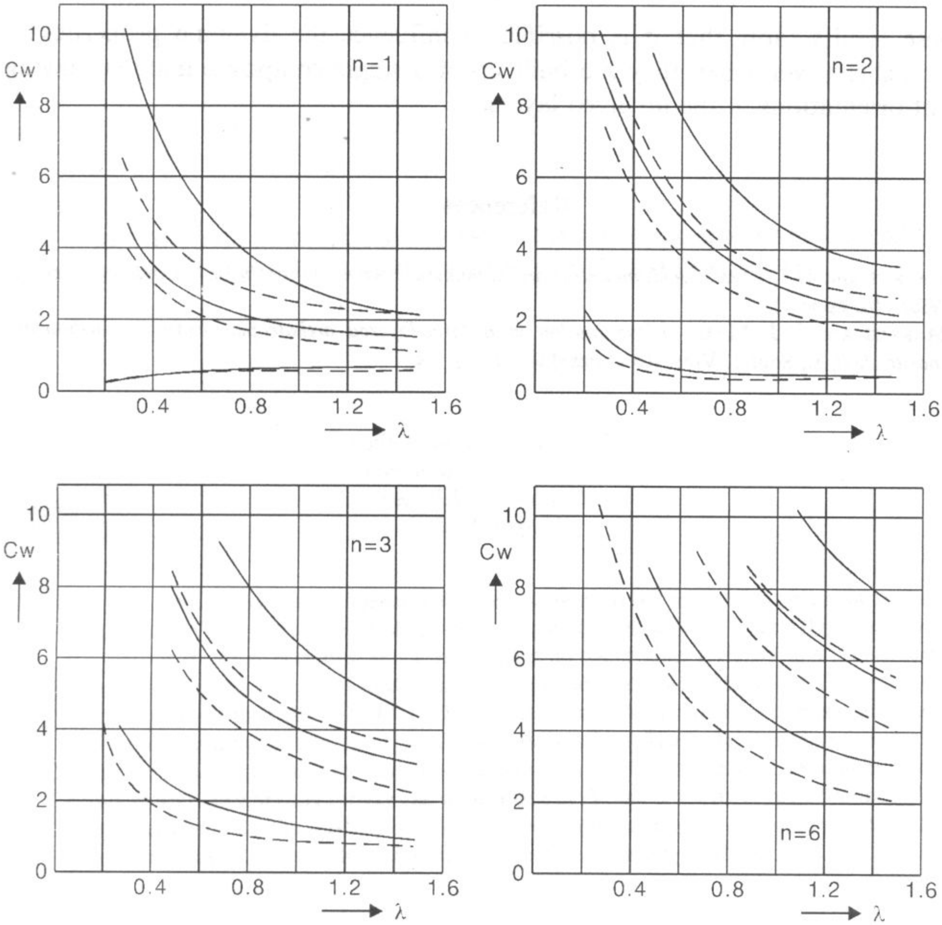

Fig. 1. First three branches of the dispersion curves c_w (λ, ph2) for n = 1, 2, 3, 6. Solid lines: ph2 = 0, dotted lines — ph2 = 1.

parameters: the circumferential wavenumber n, longitudinal wavenumber λ, wavespeed c_w and the nondimensional core thickness ratio $ph2$.

In actual calculation the left side of the dispersion equation (4.2) has been evaluated for fixed values of n, λ and $ph2$ cover a range of wavespeeds c_w. The interval have been sought in which a sign change was encountered. To identify the precise roots, the bisection method was used. In most search areas near the roots the values of the frequency determinant fluctuated violently within limits as large as $\pm 10^{99}$.

The first three branches of the dispersion curves for $n = 1, 2, 3$ and 6 are shown in Fig. 1. The solid lines correspond to ph2 = 0 (the limiting case of single layered shell made of material according to (1.1)'). Dotted lines correspond to ph2 = 1 which is the other limiting case for a single layered shell made of material according to (1.1)".

These results show that it is possible to influence the dynamic properties of sandwich shells even when they are built up of a single composite material having different orientations in the involved layers.

References

[1] Š. Markuš and D.J. Mead, *Axisymmetric and asymmetric wave motion in orthotropic cylinders,* J. Sound Vibr., (to appear).

[2] Š. Markuš and D.J. Mead, *Wave motion in a three-layered orthotropic—isotropic—orthotropic composite shell,* J., Sound. Vibr. (to appear).