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ASYMPTOTIC SOLUTION OF ACOUSTIC NONLINEAR WAVE EQUATION WITH FRICTION

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A contribution to the development of asymptotic method of small parameter was made and applied to the analysis of the propagation of nonlinear acoustic waves. The result is close to strict empiric function. Asymptotic solution of acoustic nonlinear wave equation with friction was found. The result enables the propagation velocity and the pressure of an acoustic wave to be evaluated.

1. Introduction

The computation problem of longitudinal acoustic wave with friction is very important for the duct model of porous material. Absorption of porous materials is mostly tested experimentally [1, 4]. Nonlinear approximate analysis is also performed to same extent [$3 \div 5$].

Next stage of the development of wave equation analysis is the asymptotic averaging [2].

Even today the strict solution of the non-linear propagation equation of acoustic waves with friction and partial derivatives is not known. The results approach strict formula.

2. Analysis

Theoretical analysis must be connected with reality. Taking into account the duct friction of prorous material we improve the compatibility of the theory with experiments.

Sound propagation is always partly nonlinear. For high intensity levels over 100 dB the nonlinearity cannot be neglected. An approximate theoretical analysis of acoustical waves with finite amplitude is to be performed.

The first method used in the nonlinear procedure of solving nonlinear equations was the method of small parameter H. Poincaré and others, the next one was the method of Kryłowa – Bogoljubova [2] and now we present better results obained with asymptotic averaging of equations.

Nonlinear equation with friction of the propagation of longitudinal propagating in the direction of the Ox-axis can be written in the following form

$$\frac{\partial^2 u}{\partial t^2}-C_0^2\frac{\partial^2 u}{\partial a^2}=\varepsilon f\left(\frac{\partial u}{\partial a},\frac{\partial^2 u}{\partial a^2},\frac{\partial u}{\partial t}\right), \tag{2.1}$$

$$a=x-u$$

where u — displacement, C_0 — sound velocity, a — Lagrange coordinate, ε — small parameter.

The main source of nonlinearity is the adiabatic process. Friction is a nonlinear function of velocity.

Nonlinear term f can be written in the following form

$$f=\left[\left(1+\frac{\partial u}{\partial a}\right)^{-(\varkappa+1)}-1\right]\frac{\partial^2 u}{\partial a^2}-r_j\frac{\partial u}{\partial t}, \tag{2.1a}$$

where $\varkappa=1,4$ — adiabatic exponent, r_j — flow resistance.
For the value

$$\left|\frac{\partial u}{\partial a}\right|<1, \tag{2.2}$$

we can epand the first part of nonlinear formula (2.1a) into convergent power series

$$\begin{aligned}\left(1+\frac{\partial u}{\partial a}\right)^{-(\varkappa+1)}&=1-(\varkappa+1)\frac{\partial u}{\partial a}+\frac{1}{2}(\varkappa+1)(\varkappa+2)\left(\frac{\partial u}{\partial a}\right)^2\\ &-\frac{1}{3!}(\varkappa+1)(\varkappa+2)(\varkappa+3)\left(\frac{\partial u}{\partial a}\right)^3+\ldots+(\pm 1)^k\frac{1}{k!}(\varkappa+1)(\varkappa+2)\ldots\\ &\ldots(\varkappa+k)\left(\frac{\partial u}{\partial a}\right)^k\pm\ldots\end{aligned} \tag{2.3}$$

Number k determines the numerical accuracy.
After inserting Eq. (2.3) into Eq. (2.1 a), we obtain

$$\begin{aligned}f\left(\frac{\partial u}{\partial a},\frac{\partial^2 u}{\partial a^2},\frac{\partial u}{\partial t}\right)&=[-(\varkappa+1)k!\frac{\partial u}{\partial a}+(\varkappa+1)(\varkappa+2)(k-1)!\left(\frac{\partial u}{\partial a}\right)^2\\ &-\frac{1}{3!}(\varkappa+1)(\varkappa+2)(\varkappa+3)\left(\frac{\partial u}{\partial a}\right)^3+\ldots+(\pm 1)^k\frac{1}{k!}(\varkappa+1)(\varkappa+2)\ldots\\ &\ldots(\varkappa+k)\left(\frac{\partial u}{\partial a}\right)^k\pm\ldots-1]\frac{\partial^2 u}{\partial a^2}-r_j\frac{\partial u}{\partial t},\end{aligned} \tag{2.4}$$

$$\varepsilon=\frac{c_0^2}{k!}>0, \tag{2.5}$$

$$c_0=\left(\frac{P_0\varkappa}{\rho_0}\right)^{0.5}. \tag{2.6}$$

Equation (2.1) may now be rewritten as follows

$$\frac{\partial^2 u}{\partial t^2}-c_0^2\frac{\partial^2 u}{\partial a^2}=\varepsilon\Bigg(\Big[-(\varkappa+1)k!\frac{\partial u}{\partial a}+(\varkappa+1)(\varkappa+2)(k-1)!\left(\frac{\partial u}{\partial a}\right)^2$$

$$-\frac{1}{3!}(\varkappa+1)(\varkappa+2)(\varkappa+3)\left(\frac{\partial u}{\partial a}\right)^3+\ldots+(\pm1)^k\frac{1}{k!}(\varkappa+1)(\varkappa+2)\ldots \tag{2.7}$$

$$\ldots(\varkappa+k)\left(\frac{\partial u}{\partial a}\right)^k\pm\ldots-1\Big]\frac{\partial^2 u}{\partial a^2}-r_j\frac{\partial u}{\partial t}.$$

The knowledge of the boundary conditions is necessary for solving Eq. (2.7). Displacement and velocity depend only on the Lagrange coordinate a

$$u(a,0)=\psi(a),\quad \frac{\partial u(a,0)}{\partial t}=\phi(a), \tag{2.8}$$

$$0<a<1. \tag{2.9}$$

The displacements at the ends of the duct vanishes

$$u(0,t)=0,\quad u(1,t)=0, \tag{2.10}$$

where: 1 — duct length of porous layer.
The overall solution has the form

$$u=\sum_{n=1}^{\infty}A_n(t,\varepsilon)\cos\left[\omega_n^{(0)}t+\delta_n(t,\varepsilon)\right]\sin\frac{n\Pi a}{l}, \tag{2.11}$$

$$\frac{\partial u}{\partial a}=\frac{\Pi}{1}\sum_{n=1}^{\infty}nA_n(t,\varepsilon)\cos\left[\omega_n^{(0)}t+\delta_n(t,\varepsilon)\right]\cos\frac{n\Pi a}{l}, \tag{2.12}$$

$$\frac{\partial^2 u}{\partial a^2}=-\frac{\Pi^2}{l^2}\sum_{n=1}n^2A_n(t,\varepsilon)\cos\left[\omega_n^{(0)}t+\delta_n(t,\varepsilon)\right]\sin\frac{n\Pi a}{l}, \tag{2.13}$$

$$\frac{\partial u}{\partial t}=\Bigg(\sum_{n=1}^{\infty}\frac{\partial A_n(t,\varepsilon)}{\partial t}\cos[\omega_n^{(0)}t+\delta_n(t,\varepsilon)]$$

$$-\sum_{n=1}\left[\omega_n^{(0)}+\frac{\partial_n(t,\varepsilon)}{\partial t}\right]A_n(t,\varepsilon)\sin\left[\omega_n^{(0)}t+\delta_n(t,\varepsilon)\right]\Bigg)\sin\frac{n\Pi a}{l}, \tag{2.14}$$

$$\frac{\partial^2 u}{\partial t^2}=\left(\sum_{n=1}^{\infty}\frac{\partial^2 A_n(t,\varepsilon)}{\partial t^2}\cos[\omega_n^{(0)}t+\delta_n(t,\varepsilon)]\,(+)\right.$$

$$-\sum_{n=1}^{\infty}\frac{\partial A_n(t,\varepsilon)}{\partial t}\left[\omega_n^{(0)}+\frac{\partial\delta_n(t,\varepsilon)}{\partial t}\right]\sin[\omega_n^{(0)}t+\delta_n(t,\varepsilon)]\,(+) \tag{2.15}$$

$$-\left(\sum_{n=1}^{\infty}\frac{\partial^2\delta_n(t,\varepsilon)}{\partial t^2}A_n(t,\varepsilon)\sin[\omega_n^{(0)}t+\delta_n(t,\varepsilon)]+\right.$$

$$+\sum_{n=1}^{\infty}\left[\omega_n^{(0)}+\frac{\partial\delta_n(t,\varepsilon)}{\partial t}\right]\frac{\partial A_n(t,\varepsilon)}{\partial t}\sin[\omega_n^{(0)}t+\delta_n(t,\varepsilon)]+$$

$$\left.+\sum_{n=1}^{\infty}\left[\omega_n^{(0)}+\frac{\partial\delta_n(t,\varepsilon)}{\partial t}\right]^2 A_n(t,\varepsilon)\cos[\omega_n^{(0)}t+\delta_n(t,\varepsilon)]\right)\right)\cdot\sin\frac{n\Pi a}{l}$$

Taking into account

$$\omega_n^{(0)}t+\delta_n(t,\varepsilon)=\varphi_n(t,\varepsilon) \tag{1.15a}$$

we obtain the nonlinear part

$$\varepsilon f\equiv\varepsilon\left(\left[-(\varkappa+1)k!\frac{\Pi}{l}\sum_{n=1}^{\infty}n\,A_n(t,\varepsilon)\cos\varphi_n(t,\varepsilon)\cos\frac{n\Pi a}{l}+\right.\right. \tag{2.16}$$

$$\left.+\ldots-1\right]\left(-\frac{\Pi^2}{l^2}\right)\sum_{n=1}^{\infty}(n^2 A_n(t,\varepsilon)\cos\varphi_n(t,\varepsilon)\sin\frac{n\Pi a}{l}-r_j)\cdot$$

$$\cdot\left(\sum_{n=1}^{\infty}\frac{\partial A_n(t,\varepsilon)}{\partial t}\cos\varphi_n(t,\varepsilon)-\sum_{n=1}^{\infty}\left[\omega_n^{(0)}+\frac{\partial\delta_n(t,\varepsilon)}{\partial t}\right]A_n(t,\varepsilon)\cdot\sin\varphi_n(t,\varepsilon)\right)\sin\frac{n\Pi a}{l}\Bigg).$$

The utility of the asymptotic method can be seen after averaging and analysing the convergence of trigonometric terms

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T(t,A_n,\varphi_n)\sin\varphi_n\,dt=X_{on}(A_n,\delta_n),$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f_{on}(t,A_n,\varphi_n)\sin\varphi_n\,dt=Y_{on}(A_n,\delta_n), \tag{2.17}$$

where: f_{on} — Fourier coefficients.

On the basis of Eqs. (2.15)–(2.17) we can write

$$f_{on}^{(1)}=(\varkappa+1)k!\frac{\Pi^3}{l^3}n^3A_n^2\cos^2\varphi_n^{(0)}-\left[\omega_n^{(0)}+\frac{\partial\delta_n(t,\varepsilon)}{\partial t}\right]A_n\sin\varphi_n^{(0)}. \tag{2.18}$$

Convergence of trigonometric terms leads to

$$\underset{t}{M}(\cos^2\varphi_n^{(0)}\sin\varphi_n^{(0)})\equiv\lim\frac{1}{T}\int_0^T\cos^2\varphi_n^{(0)}\sin\varphi_n^{(0)}\,dt=0. \tag{2.19}$$

$$\underset{t}{M}\{\cos\varphi_n^{(0)}\sin\varphi_n^{(0)}\}=0. \tag{2.20}$$

$$\underset{t}{M}\{\sin^2\varphi_n^{(0)}\}=0.5. \tag{2.21}$$

Often

$$\frac{\partial\delta_n(t,\varepsilon)}{\partial t}=\text{const}=C. \tag{2.22}$$

The amplitude A_n may be derived as A_n' from averaging

$$\frac{dA_n'}{dt}=-\frac{\varepsilon}{\omega_n^{(0)}}X_{on}. \tag{2.23}$$

$$\frac{dA_n'}{dt}=-0.5\,\varepsilon[\omega_n^{(0)}]^{-1}[\omega_n^{(0)}+C]A_n'. \tag{2.24}$$

After integration we obtain

$$A_n'=A_{no}\exp\{-0.5[\omega_n^{(0)}]^{-1}[\omega_n^{(0)}+C]\}. \tag{2.25}$$

Let us analyse the third part of expanding Eq. (2.17), (2.16)

$$f_{on}^{(0)}=\frac{1}{3!}(\varkappa+1)(\varkappa+2)(\varkappa+3)\frac{\Pi^5}{l^5}n^5A_n^4\cos^4\varphi_n^{(0)}, \tag{2.26}$$

$$\underset{t}{M}\{\cos^4\varphi_n^{(0)}\}=\frac{3}{8}. \tag{2.27}$$

Mean value of δ_n is denoted by δ_n'. It is involved in the following equation:

$$\frac{d\delta_n'}{dt}=-\frac{\varepsilon}{\omega_n^{(0)}}Y_{on}, \tag{2.28}$$

After averaging A_{no} is replaced by $A_n^{(0)}$ and

$$\frac{d\delta_n'}{dt}=-\frac{\varepsilon}{\omega_n^{(0)}}\frac{1}{16}(\varkappa+1)(\varkappa+2)(\varkappa+3)\frac{\Pi^5}{l^5}n^5{A_n^{(0)}}^4. \tag{2.29}$$

$$\delta_n'=-685\frac{\varepsilon n^5}{\omega_n^{(0)}l^5}{A_n^{(0)}}^4\,t. \tag{2.30}$$

Differentiating the formula (2.30) we obtain the mean value of constant (2.22)

$$c = c' = \frac{\partial \delta_n'}{\partial t} = -685 \frac{\varepsilon n^2 A_n^{(0)^4}}{\omega_n^{(0)} l^5} \tag{2.31}$$

The solution of equation (2.1) has the form

$$u = \sum_{n=1} A_{no} \exp\left\{ -0.5[\omega_n^{(0)}]^{-1} \left[\omega_n^{(0)} - 685 \frac{\varepsilon n^5 A_n^{(0)^4}}{\omega_n^{(0)} l^5} \right] \right\} \cdot \cos\left[\omega_n^{(0)} - 685 \frac{\varepsilon n^5 A_n^{(0)^4}}{\omega_n^{(0)} l^5} \right] t \cdot \sin \frac{n\Pi a}{l} \tag{2.32}$$

3. Concluding remarks

After averaging and convergence analysis, the asymptotic solution is obtained to the nonlinear equation of longitudinal wave propagation. The obtained formula is a good approximation of the exact solution and is in good fitness with experimental result [1, 5]. The pressure level is very important parameter in this analysis since from formula (2.32).

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