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SIMPLE WAVES WITH FINITE AMPLITUDE IN AXIALLY-SYMMETRICAL CHANNELS WITH ANNULAR CROSS-SECTION

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Propagation of a plane progressive sound wave with finite amplitude in a waveguide with variable annular cross-section was considered. It was assumed that this waveguide was filled with loss less gaseous medium and the wave with finite amplitude was generated by a annular piston, which vibrated in harmonic motion at the inlet of the waveguide. The considerations were done in Lagrangian coordinates.

1. Introduction

According to the terminology used in literature [3], a plane progressive wave that propagates in lossless medium is called a "simple wave". The problem of propagation of simple waves with finite amplitude belongs to classical problems of hydrodynamics to which the present-day works on nonlinear acoustics are quite frequently referred. Paper [1], where the propagation of simple waves in semi-infinite cylindrical tube was considered, is an example. The waves were generated by continuous motion of a piston, which vibrated at a high amplitude.

This paper presents an attempt of consideration of propagation of the wave with a finite amplitude in the axially-symmetrical waveguides with variable annular cross-section. This problem has direct practical implications, at least in two problems:

- analysis of sound waves in the inlet and outlet channels of axial compressors [13],
- optimization of a construction of strong acoustic field axial flow generators, in which the axially-symmetrical waveguides with ring-shaped cross-section and exponential or catenoidal expansion of walls are frequently used [2, 6, 7],

In spite of the fact that in both cases the acoustic waves with high intensity can occur, this problem, up to the present has been considered only in linear approximation [6, 7, 10, 13, 14].

2. Analysis of the wave propagation equation

The waveguide subject to our consideration is presented in longitudinal section in Fig. 1. It is assumed that the waveguide is filled with a loss less gaseous medium and the width of the annular channel is so small in comparison to the wavelength that the front of the wave can be assumed to be plane. Then the layer of acoustic particles with the Lagrangian coordinate a is enclosed between plane surfaces $S_{(a)}$ and $S_{(a+da)}$. As a result of the wave disturbance, this layer is moved to the position $a+\xi$ and its thickness is changed to the value $d(a+\xi) \equiv dx \equiv [1 + (\partial\xi/\partial a]\, da$. ξ is the displacement of the acoustic particle and x is the Eulerian coordinate.

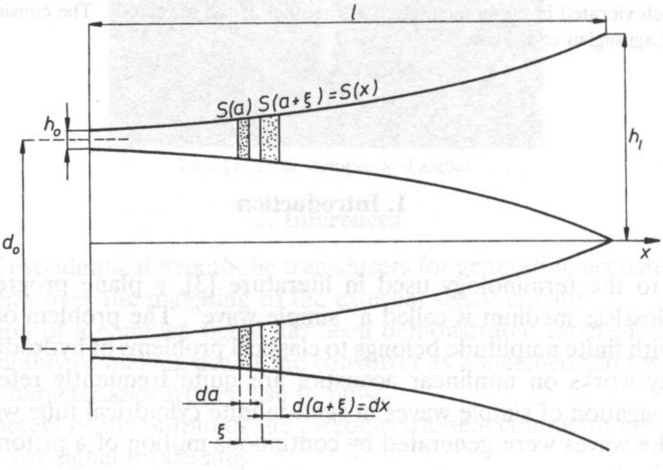

FIG. 1. The displacement of the medium layer in the waveguide.

The result of paper [8] will be a starting point for further considerations. In that paper it was shown, for the simplifying assumptions mentioned above, that the equation of propagation of the wave with finite amplitude in a waveguide with any geometry has the form

$$x'' + \frac{\zeta'}{\zeta} x' = \frac{\zeta^{\gamma-1}}{c^2} (x')^{\gamma+1} \cdot \frac{\partial^2 \xi}{\partial t^2}, \tag{2.1}$$

where

$$\zeta = \frac{S(a+\xi)}{S(a)}, \tag{2.2}$$

γ is the adiabatic exponent, c is the velocity of sound wave with infinitesimal amplitude and t is time. Commas in Eq. (2.1) denote differentiation with respect to the coordinate a.

It is assumed that the cross-section of the waveguide changes according to the formula

$$S = S_0\left[\cosh\left(\frac{x}{x_0}\right) + T\sinh\left(\frac{x}{x_0}\right)\right], \tag{2.3}$$

where $S_0 = \pi d_0 h_0$ is the surface of the inlet, x_0 is the coefficient of the wall expansion. The above formula defines the geometry of the so-called hyperbolic horns with annular section [14], which, among others, are applied to the construction of strong acoustic field flow generators. The waveguides with exponential ($T = 1$) and lately also catenoidal ($T = 0$) profiles are most frequently used. Thus these considerations are limited to the interval $T \in [0, 1]$. Then the formula (2.3) can be presented in a more compact form, which significantly simples further transformations:

$$S = \overline{S}_0 \cdot \cosh\left(\frac{x}{x_0} + \varepsilon\right), \tag{2.4}$$

where $T = \operatorname{tgh}\varepsilon$, $\overline{S}_0 = S_0/\cosh\varepsilon$. Then the value ζ in wave equation is equal to

$$\zeta = \frac{\cosh\left(\dfrac{a+\xi}{x_0} + \varepsilon\right)}{\cosh\left(\dfrac{a}{x_0} + \varepsilon\right)}. \tag{2.5}$$

Expanding the ζ and ζ' in to series, one can obtain

$$\zeta = 1 + \frac{\xi}{x_0}\operatorname{tgh}\left(\frac{a}{x_0} + \varepsilon\right) + \frac{\xi^2}{2x_0^2} + \ldots, \tag{2.6}$$

$$\zeta' = \frac{\xi'}{x_0}\operatorname{tgh}\left(\frac{a}{x_0} + \varepsilon\right) + \frac{\varepsilon}{x_0^2}\left[1 - \operatorname{tgh}^2\left(\frac{a}{x_0} + \varepsilon\right)\right] + \frac{\xi\xi'}{x_0^2} + \ldots \tag{2.7}$$

It is assumed that the hypothetical annular piston that vibrates in harmonic motion is a source of the wave at the inlet ($a = 0$)

$$\xi_{(0,t)} = k^{-1}A \cdot \cos\omega t, \tag{2.8}$$

where ω is the angular vibration frequency (pulsation), k is the wave number. $A = 2\pi B/\lambda$, where B is the vibration amplitude and λ is the wavelength. It is known from experiment that, even for relatively high intensity of the sound, the value of $M = B/\lambda$ is rarely higher than 10^{-2} for gases[1] [17].

Thus the dimensionless amplitude A is significantly less than unity. Therefore the solution of the wave equation is assumed in the form of power series of the amplitude a:

$$\xi_{a,t} = k^{-1}[A \cdot \varphi_{1(a,t)} + A^2 \cdot \varphi_{2(a,t)} + \ldots], \tag{2.9}$$

where the functions $\varphi_{1(a,t)}$, $\varphi_{2(a,t)}$ must fulfil the boundary condition (2.8) at the inlet of the horn:

$$\varphi_{1(0,t)} = \cos\omega t, \quad \varphi_{2(0,t)} = \varphi_{3(0,t)} = \ldots = 0. \tag{2.10}$$

[1] M is the so-called Mach acoustic number [17]

Since $A \ll 1$, further considerations will be done in the second-order approximation, i.e. the terms with orders higher than A^2 will be omited. The method of calculation applied here is analogical to that used in [16] for the Bessel horns: regarding the formula (2.9) and expansions (2.6) and (2.7) in Eq. (2.1) one can obtain the wave equation of the horn in the second-order approximation. This equation can be divided into two equations: the first one contains only terms with A, the second one — those with A^2. Each equation must be fulfilled for $A \neq 0$ separately. It is found that the first equation contains only the function φ_1:

$$\varphi_1'' + \frac{\operatorname{tgh}\left(\dfrac{a}{x_0}+\varepsilon\right)}{x_0}\varphi_1' + \frac{1-\operatorname{tgh}^2\left(\dfrac{a}{x_0}+\varepsilon\right)}{x_0}\varphi_1 - \frac{1}{c^2}\ddot{\varphi}_1 = 0 \tag{2.11}$$

where the dots denote differentiation with respect to time. Second equation has the form:

$$\varphi_2'' + \frac{\operatorname{tgh}\left(\dfrac{a}{x_0}+\varepsilon\right)}{x_0}\varphi_2' + \frac{1-\operatorname{tgh}^2\left(\dfrac{a}{x_0}+\varepsilon\right)}{x_0^2}\varphi_2 - \frac{1}{c^2}\ddot{\varphi}_2 = \psi_{(a,t)} \tag{2.12}$$

where

$$\begin{aligned}\Psi_{(a,t)} = {} & \varphi_1\ddot{\varphi}_1\frac{\gamma-1}{kx_0c^2}\operatorname{tgh}\left(\frac{a}{x_0}+\varepsilon\right) + \varphi_1'\ddot{\varphi}_1\frac{\gamma}{kc^2} + \frac{1}{k}\varphi_1'\varphi_1'' + \\ & + \varphi_1^2\frac{\operatorname{tgh}\left(\dfrac{a}{x_0}+\varepsilon\right)\cdot\left[1-\operatorname{tgh}^2\left(\dfrac{a}{x_0}+\varepsilon\right)\right]}{kx_0^3} - \varphi_1\varphi_1'\frac{1-\operatorname{tgh}^2\left(\dfrac{a}{x_0}+\varepsilon\right)}{kx_0^2}.\end{aligned} \tag{2.13}$$

Introducing to Eq. (2.11)

$$\varphi_{1(a,t)} = \Phi_{1(a)}\cdot e^{i\omega t}, \tag{2.14}$$

one obtains the propagation of the first harmonic wave:

$$\phi_1'' + \frac{\operatorname{tgh}\left(\dfrac{a}{x_0}+\varepsilon\right)}{x_0}\phi_1' + \left[1 - \frac{\operatorname{tgh}^2\left(\dfrac{a}{x_0}+\varepsilon\right)}{x_0} + k^2\right]\phi_1 = 0. \tag{2.15}$$

Substituting

$$\phi_{1(z)} = (\cosh z)^{-\frac{1}{2}}\cdot\eta_{(z)}, \tag{2.16}$$

where

$$z = \frac{a}{x_0} + \varepsilon, \tag{2.17}$$

the wave equation (2.15) can be transformed to the form

$$\eta'' + [\mu^2 - V_{(z)}]\eta = 0, \tag{2.18}$$

where

$$\mu = kx_0, \tag{2.19}$$

$$V_{(z)} = \frac{3}{4}\operatorname{tgh}^2 z - \frac{1}{2}, \tag{2.20}$$

and this time the commas denote differentiation with respect to the variable z.

The propagation equation of the first harmonic wave written in the form (2.18) is known in the linear theory of acoustic horns [11] and, therefore, it will not be considered here in details. It is known that, for the frequency higher than cut-off frequency of the horn, the solution of this equation can be presented in the form [15]

$$\eta = B_1 \cdot e^{i\overline{K}z} + B_2 \cdot e^{-i\overline{K}z}, \tag{2.21}$$

where

$$\overline{K} = \left\{ \frac{1}{z_1 - \varepsilon} \int_{\varepsilon}^{z_1} [\mu^2 - V_{(z)}] dz \right\}^{\frac{1}{2}} \tag{2.22}$$

$z_l = (l/x_0) + \varepsilon$, where l is the length of the horn. From that, on the basis Eqs. (2.19) and (2.20) one can obtain

$$\overline{K} = \left\{ k^2 x_0^2 - \frac{1}{4} + \frac{3x_0}{4l} \left[\operatorname{tgh} \varepsilon - \operatorname{tgh} \left(\frac{1}{x_0} + \varepsilon \right) \right] \right\}^{\frac{1}{2}}. \tag{2.23}$$

Subsequently, taking into account Eqs. (2.14), (2.16), (2.17) and considering only the real part of the solution, the following equation can be obtained for the wave running from the inlet to the outlet:

$$\varphi_{1(a,t)} = \frac{C_1}{\sqrt{\cosh(\frac{a}{x_0} + \varepsilon)}} \cos \left[\omega t - \overline{K} \left(\frac{a}{x_0} + \varepsilon \right) \right] - \frac{C_2}{\sqrt{\cosh(\frac{a}{x_0} + \varepsilon)}} \sin \left[\omega t - \overline{K} \left(\frac{a}{x_0} + \varepsilon \right) \right]. \tag{2.24}$$

The constants C_1 and C_2 can be calculated from the boundary condition (2.10)

$$C_1 = \sqrt{\cosh \varepsilon} \cdot \cos(\overline{K}\varepsilon), \tag{2.25}$$

$$C_2 = \sqrt{\cosh \varepsilon} \cdot \sin(\overline{K}\varepsilon) \tag{2.26}$$

and that allows us to write Eq. (2.24) in a more compact form:

$$\varphi_{1(a,t)} = \sqrt{\frac{\cosh \varepsilon}{\cosh(\frac{a}{x_0} + \varepsilon)}} \cdot \cos \left(\omega t - \overline{K} \frac{a}{x_0} \right). \tag{2.27}$$

The function $\varphi_{2(a,t)}$ that occurs in the second term of the solution (2.9) can be found from Eq. (2.12). This equation has a structure similar to Eq. (2.11) except that on the right-hand side the term $\Psi_{(a,t)}$ determined by the solution found in the first approximation see Eq. (2.13) appears. Applying Eqs. (2.24)–(2.26), this term can be presented in the form

$$\psi = \sigma_{(a)} \cdot \cos 2 \left(\omega t - \overline{K} \frac{a}{x_0} \right) + \delta_{(a)} \cdot \sin 2 (\omega t - \overline{K} \frac{a}{x_0}) + \Omega_{(a)}, \tag{2.28}$$

where

$$\sigma_{(a)} = \frac{-15 \operatorname{tgh}^3(\frac{a}{x_0} + \varepsilon) + [(2-\gamma)4k^2x_0^2 + 12\overline{K}^2 + 14] \operatorname{tgh}(\frac{a}{x_0} + \varepsilon)}{16kx_0^3 \cosh(\frac{a}{x_0} + \varepsilon)} \cosh \varepsilon, \tag{2.29}$$

$$\delta_{(a)} = \frac{18 \operatorname{tgh}^2(\frac{a}{x_0} + \varepsilon) - 8\overline{K}^2 - 8k^2x_0^2\gamma - 12}{16kx_0^3 \cosh(\frac{a}{x_0} + \varepsilon)} \overline{K} \cosh \varepsilon, \tag{2.30}$$

$$\Omega_{(a)} = \frac{-15\,\mathrm{tgh}^3(\frac{a}{x_0}+\varepsilon) + [(2-\gamma)4k^2x_0^2 - 4\overline{K}^2 + 14]\,\mathrm{tgh}(\frac{a}{x_0}+\varepsilon)}{16kx_0^3\cosh(\frac{a}{x_0}+\varepsilon)}\cosh\varepsilon. \tag{2.31}$$

The fact that the term Ψ is determined by the function $\varphi_{1(a,t)}$ reflects the fact that the second harmonic wave is generated as the result of a disturbance of a medium in the waveguide caused by waves with the fundamental frequency.

The function $\varphi_{2(a,t)}$, as an integral of Eq. (2.12), is a sum of two components [4]; the first one is the general solution of a homogeneous equation coupled with Eq. (2.12). This solution has a form similar to Eq. (2.24)

$$\varphi_{21} = \frac{C_3}{\sqrt{\cosh(\frac{a}{x_0}+\varepsilon)}}\cos\left[2\omega t - \overline{K}_1\left(\frac{a}{x_0}+\varepsilon\right)\right] + \\ - \frac{C_4}{\sqrt{\cosh(\frac{a}{x_0}+\varepsilon)}}\sin\left[2\omega t - \overline{K}_1\left(\frac{a}{x_0}+\varepsilon\right)\right], \tag{2.32}$$

where

$$\overline{K}_1 = \left\{4k^2x_0^2 - \frac{1}{4} + \frac{3x_0}{4l}\left[\mathrm{tgh}\,\varepsilon - \mathrm{tgh}\left(\frac{1}{x_0}+\varepsilon\right)\right]\right\}^{\frac{1}{2}}. \tag{2.33}$$

The second component is the singular of Eq. (2.12) and has the form which results from that of the free term of i.e.

$$\varphi_{22} = g_{(a)}\cos 2\omega t + f_{(a)}\cdot\sin 2\omega t + u_{(a)}. \tag{2.34}$$

Introducing φ_{22} in to Eq. (2.12), it is possible to find $g(a)$, $f(a)$ and $u(a)$, subsequently, for instance by the method of constant variations [4]. Finally, knowing $\varphi_{1(a,t)}$ and $\varphi_{2(a,t)}$, one can determine the particle displacement ξ in the waveguide in the second approximation on the basic of Eq. (2.9).

Determination of the particle velocity and acoustic pressure for the wave with finite amplitude will be the next step. So, the particle velocity of the medium in Lagrangian coordinates is obtained after differentiating Eq. (2.9) with respect to time

$$v_{(a,t)} = k^{-1}[A\cdot\dot{\varphi}_{1(a,t)} + A^2\cdot\dot{\varphi}_{2(a,t)} + \ldots]. \tag{2.35}$$

Using the relation between the Lagrangian and Eulerian coordinates [17], the vibration velocity at the point x can be determined

$$v_{(x,t)} = v_{(a,t)} - \frac{\partial v}{\partial a}\cdot\xi_{(a,t)} + \ldots \tag{2.36}$$

The acoustic pressure is a surplus of the pressure P in a vinicity of the particle a, over the so-called static pressure P_0 which occurs in the absence of a wave disturbance:

$$p = P_{(a+\xi)} - P_0. \tag{2.37}$$

To find a relation between the acoustic pressure p and the particle displacement ξ it is possible to use the equation of continuity:

$$S_{(a)}\cdot\varrho_0 = S_{(a+\xi)}\cdot(1+\xi')\cdot\varrho_{(a+\xi)}, \tag{2.38}$$

and the thermodynamic equation:

$$P_{(a+\xi)} = P_0\left[\frac{\varrho_{(a+\xi)}}{\varrho_0}\right]^{\gamma}, \tag{2.39}$$

where ρ denotes the medium density and, in particular, $\rho_{(a)} = \rho_0$ is the static density.

From Eq. (2.38) and (2.39) one obtains

$$P_{(a+\xi)} = P_0[\zeta(1+\xi')]^{-\gamma}. \tag{2.40}$$

From that, after using Eq. (2.37),

$$p = P_0\{[\zeta(1+\xi')]^{-\gamma} - 1\}, \tag{2.41}$$

A transformation into Eulerian coordinates is analogical to that in Eq. (2.36).

Now it is possible to analyze the first and second harmonic waves in a selected waveguide with the determined profile of walls.

3. The exponential waveguide with ring-shaped cross-section

Amond the considered wave guides with the ring-shaped cross-section those with exponential expansion of walls have been used most frequently. In this case $\varepsilon \to \infty$ ($T = 1$), thus Eq. (2.4) take the form

$$S = S_0 \cdot e^{\frac{x}{x_0}}, \tag{2.42}$$

where e is the base of natural logarithms.

The function $\varphi_{1(a,t)}$, after taking into account $\varepsilon \to \infty$ in Eq. (2.27) assumes the form

$$\varphi_{1(a,t)} = e^{-\frac{a}{2x_0}} \cdot \cos\left(\omega t - \overline{K}\frac{a}{x_0}\right), \tag{2.43}$$

while $\overline{K}$ given by Eq. (2.23) is simplified to the form:

$$\overline{K} = \sqrt{\mu^2 - \frac{1}{4}}. \tag{2.44}$$

Equation (2.12) for the function $\varphi_{2(a,t)}$ is also simplified,

$$\varphi_2'' + \frac{1}{x_0}\varphi_2' - \frac{1}{c^2}\ddot{\varphi}_2 = \psi_{(a,t)} \tag{2.45}$$

where Eq. (2.28) for Ψ the limits of Eq. (2.29)–(2.31) for $\varepsilon \to \infty$ being accounted for, can be expressed as

$$\psi = e^{-\frac{a}{x_0}}\left\{\frac{(5-\gamma)\mu^2 - 1}{4kx_0^3}\cos 2\left(\omega t - \overline{K}\frac{a}{x_0}\right) + \right.$$

$$\left. + \sqrt{\mu^2 - \frac{1}{4}} \cdot \frac{1-(\gamma+1)\mu^2}{2kx_0^3}\sin 2\left(\omega t - \overline{K}\frac{a}{x_0}\right) + \frac{(1-\gamma)k}{4x_0}\right\}. \tag{2.46}$$

General solution of the homogeneous equation coupled with Eq. (2.45) for the pulsation 2ω has form

$$\varphi_{21} = e^{-\frac{a}{2x_0}}\left[C_3\cos\left(2\omega t - \overline{K}\frac{a}{x_0}\right) - C_4\sin\left(2\omega t - \overline{K}\frac{a}{x_0}\right)\right], \tag{2.47}$$

where (see Eq. (2.33))

$$\overline{K}_1 = \sqrt{4\mu^2 - \frac{1}{4}}. \tag{2.48}$$

The singular solution of Eq. (2.45) results from the formula for the agree term in Eq. (2.46)

$$\varphi_{22} = e^{-\frac{a}{x_0}}\left[Qa + L\cos 2\left(\omega t - \overline{K}\frac{a}{x_0}\right) + N\sin 2\left(\omega t - \overline{K}\frac{a}{x_0}\right)\right]. \tag{2.49}$$

The coefficient Q, L, N can be found by introducing φ_{22} into Eq. (2.45)

$$Q = \frac{\gamma - 1}{4}k, \tag{2.50}$$

$$L = \frac{\gamma + 1}{4}\mu - \frac{\gamma}{8\mu}, \tag{2.51}$$

$$N = \frac{2-\gamma}{4\mu}\overline{K}. \tag{2.52}$$

Next, from the boundary condition (2.10), the constants C_3 and C_4 can be found. Thus, for $a = 0$ one gets.

$$\varphi_{2(0,t)} = \varphi_{12(0,t)} + \varphi_{22(0,t)} = 0. \tag{2.53}$$

This condition is fulfilled when

$$C_3 = -L, \quad C_4 = N. \tag{2.54}$$

Finally, Eq. (2.9) of the acoustic particle displacement takes the following form written in the second approximation:

$$\begin{aligned}\xi = k^{-1}\Bigg\{ & A\cdot e^{-\frac{a}{2x_0}}\cdot\cos\left(\omega t - \overline{K}\frac{a}{x_0}\right) + \\ & + A^2\cdot e^{-\frac{a}{2x_0}}\left[-L\cos\left(2\omega t - \overline{K}_1\frac{a}{x_0}\right) - N\sin\left(2\omega t - \overline{K}_1\frac{a}{x_0}\right)\right] + \\ & + A^2\cdot e^{-\frac{a}{x_0}}\left[Qa + L\cos 2\left(\omega t - \overline{K}\frac{a}{x_0}\right) + N\sin 2\left(\omega t - \overline{K}\frac{a}{x_0}\right)\right]\Bigg\}\end{aligned} \tag{2.55}$$

It can be noticed that in above equation the nonperiodic component equal to $k^{-1}A^2\exp -a/x_0\, Qa$ occurs besides the periodic components, which corresponds to the first and second harmonic waves.

By differentiating Eq. (2.55) with respect to time, the expression for the vibration velocity of the particles in the Lagrangian coordinates can be obtained

$$\begin{aligned}v = & -Ace^{-\frac{a}{2x_0}}\cdot\sin\left(\omega t - \overline{K}\frac{a}{x_0}\right) + \\ & + 2cA^2\Bigg\{e^{-\frac{a}{2x_0}}\left[L\sin\left(2\omega t - \overline{K}_1\frac{a}{x_0}\right) + N\sin\left(2\omega t - \overline{K}_1\frac{a}{x_0}\right)\right] + \\ & + e^{-\frac{a}{x_0}}\left[-L\sin 2\left(\omega t - \overline{K}\frac{a}{x_0}\right) + N\cos 2\left(\omega t - \overline{K}\frac{a}{x_0}\right)\right]\Bigg\}.\end{aligned} \tag{2.56}$$

It is noticeable that in the second approximation, the term with frequency ω (first harmonic)

$$v_1 = -Ace^{-\frac{a}{2x_0}} \sin\left(\omega t - \overline{K}\frac{a}{x_0}\right) \tag{2.57}$$

is supplemented with the component of frequency 2ω (second harmonic). After transformations this component can be presented in the form

$$v_2 = 2cA^2 \cdot e^{-\frac{a}{2x_0}} D \cdot \sin(2\omega t + \alpha), \tag{2.58}$$

where

$$D = \sqrt{(L^2 + N^2)[1 + e^{-\frac{a}{x_0}} - 2e^{-\frac{a}{2x_0}} \cos(\overline{K}_1 - 2\overline{K})\frac{a}{x_0}}, \tag{2.59}$$

$$\operatorname{tg}\alpha = \frac{e^{-\frac{a}{2x_0}}(L \sin 2\overline{K}\frac{a}{x_0} + N \cos 2\overline{K}\frac{a}{x_0}) - (L \sin \overline{K}_1\frac{a}{x_0} + N \cos \overline{K}_1\frac{a}{x_0})}{L \cos \overline{K}_1\frac{a}{x_0} - N \sin \overline{K}_1\frac{a}{x_0} - e^{-\frac{a}{2x_0}}(L \cos 2\overline{K}\frac{a}{x_0} - N \sin 2\overline{K}\frac{a}{x_0})}. \tag{2.60}$$

The ratio of the velocity amplitudes of both harmonics is equal to

$$\frac{\widehat{v}_2}{\widehat{v}_1} = 2AD. \tag{2.61}$$

This acoustic pressure in the waveguide can be found from Eq. (2.41) and, on the basis of Eq. (2.2) and (2.42), for an exponential horn

$$\zeta = e^{\frac{\xi}{x_0}}. \tag{2.62}$$

From that it follows that

$$p = P_0[(1 + \xi')^{-\gamma} \cdot e^{\frac{-\gamma\xi}{x_0}} - 1]. \tag{2.63}$$

In the second approximation, when Eq. (2.9) is taken into account, the following relationship can be obtained:

$$p = P_0\left[\frac{-\gamma A}{kx_0}\varphi_1 - \frac{-\gamma A^2}{kx_0}\varphi_2 + \frac{\gamma^2}{2k^2x_0^2}A^2\varphi_1^2 - \frac{\gamma A}{k}\varphi_1' - \frac{\gamma A^2}{k}\varphi_2' + \right.$$
$$\left. + \frac{\gamma^2 A^2}{k^2 x_0}\varphi_1\varphi_1' + \frac{\gamma(\gamma+1)}{2k^2}A^2\varphi_1'^2\right]. \tag{2.64}$$

From that, finally, on the basis of Eqs. (2.43) and (2.45), one can get:

$$\frac{p}{P_0} = -\gamma Ae^{-\frac{a}{2x_0}} \sin\left(\omega t - \overline{K}\frac{a}{x_0} + \beta\right) + \frac{\gamma A^2}{2}e^{-\frac{a}{x_0}} + \gamma A^2 e^{-\frac{a}{2x_0}} G \sin(2\omega t + \Delta), \tag{2.65}$$

where

$$\operatorname{tg}\beta = \frac{1}{2\overline{K}}, \tag{2.66}$$

$$G = \left\{R^2 + W^2 + e^{-\frac{a}{x_0}}(X^2 + Z^2) + 2e^{-\frac{a}{2x_0}}\left[(RZ + WX)\cos(\overline{K}_1 - 2\overline{K})\frac{a}{x_0} + \right.\right.$$
$$\left.\left. + (RX - WZ)\sin(\overline{K}_1 - 2\overline{K})\frac{a}{x_0}\right]\right\}^{\frac{1}{2}}, \tag{2.67}$$

$$\operatorname{tg}\Delta = \frac{R\cos\overline{K}_1\frac{a}{x_0} - W\sin\overline{K}_1\frac{a}{x_0} + e^{-\frac{a}{x_0}}\left(Z\cos 2\overline{K}\frac{a}{x_0} - X\sin 2\overline{K}\frac{a}{x_0}\right)}{W\cos\overline{K}_1\frac{a}{x_0} + R\sin\overline{K}_1\frac{a}{x_0} + e^{-\frac{a}{2x_0}}\left(X\cos 2\overline{K}\frac{a}{x_0} + Z\sin 2\overline{K}\frac{a}{x_0}\right)} \tag{2.68}$$

and

$$R = \frac{4(\gamma-2)\overline{K}\,\overline{K}_1 - \gamma}{16\mu^2} + \frac{\gamma+1}{8}, \tag{2.69}$$

$$W = \left(\frac{\gamma+1}{4} - \frac{\gamma}{8\mu^2}\right)\overline{K}_1 + \frac{2-\gamma}{8\mu^2}\overline{K}, \tag{2.70}$$

$$X = \left(\frac{2\gamma-1}{4\mu^2} - \frac{\gamma+1}{2}\right)\overline{K}, \tag{2.71}$$

$$Z = \frac{2\gamma-1}{8\mu^2} - \frac{3}{4}(1-\gamma). \tag{2.72}$$

From Eq. (2.65) it results that the ratio of the pressure amplitudes of the second harmonic to the first one is given by

$$\frac{\widehat{p}_2}{\widehat{p}_1} = AG. \tag{2.73}$$

Furthermore, in Eq. (2.65) for the acoustic pressure, the component independent of time occurs

$$\overline{p} = \frac{1}{2}\gamma A^2 P_0 \cdot e^{-\frac{a}{x_0}}. \tag{2.74}$$

In the nonlinear theory of a plane wave in a lossless infinite medium, the component independent of time also occurs in the equation of the pressure, but in that case it is also independent of the position in the sounds wave. That is the so-called radiation stress [5]. It is seen that in the case of the waveguide considered this component decreases exponentially, like $\exp(-a/x_0)$, while the amplitude of the first harmonic decreases slower, like $\exp(-a/2x_0)$.

At the end, the exponential waveguide with typical dimensions for a construction of axial flow generators is taken into account in the numerical example:

- width of the channel at the inlet $h_0 = 1.5 \cdot 10^{-3}$ m,
- mean diameter of the annular channel $d_0 = 10^{-1}$ m,
- width of the channel at the outlet $h_l = 10^{-1}$ m,
- length of the waveguide $l = 1.5 \cdot 10^{-1}$ m.

The waveguide with these dimensions has the cut-off frequency $f_{gr} = 760$ Hz.

On the basis of Eq. (2.57)–(2.59), the relations between the amplitude of the vibration velocities for the first ($n = 1$) and second ($n = 2$) harmonic and the particle displacement of the medium, in the waveguide, when the frequency of a piston that initiates the harmonic wave at the inlet is equal to $f = 3$ kHz, are presented in Fig. 2. The amplitudes of the vibration velocities are related to that of the piston $v_{10} = Ac$. Appearance of the second harmonic for $a > 0$ (it would be higher harmonics in subsequent approximations) is caused by nonlinear properties of the medium in the waveguide and explains the de-

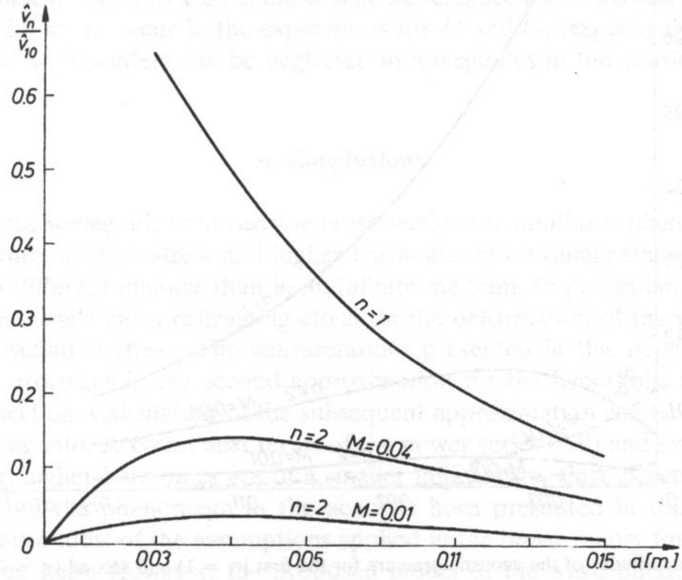

FIG. 2. The amplitudes of the particle velocities of the medium in the waveguide for the first ($n = 1$) and second ($n = 2$) harmonics. The generation frequency of the sinusoidal wave at the inlet is $f = 3$ kHz.

formation of the wave front when the wave moves along the horn. It can be seen that the second harmonic increases in the part near the inlet, and subsequently decreased as the walls of the waveguide expand. It means that deformation of the wave-front develops in the inlet part, and then it stops. So the course of the nonlinear phenomena is different than that in the case of propagation of a plane wave in a lossless infinite medium, or in a cylindrical tube when the wave deformation constantly increases and the amplitude of the second harmonic is proportional to the distance from a source [5].

It can be seen from the Fig. 3 that the plot of the acoustic pressure amplitudes for both harmonics (solid lines) is similar to that of the velocity amplitudes, as shown in Fig. 2. Furthermore, in Fig. 3, dashed line marks the course of the time-independent component $\overline{p}$ of the acoustic pressure. It appears that this component decreases significantly faster that the amplitude of the second harmonic pressure. it should be mentioned that the amplitude of both harmonics and the value of the component $\overline{p}$ are related here to pressure amplitude of the first harmonic at the inlet: $\widehat{p}_{10} = \gamma A P_0$.

It is shown in Fig. 4 that, for a given layer of medium particles in the waveguide, the vibration velocity amplitude of the second harmonic increases compared with that of the first one when the frequency increases. This increase is faster when the amplitude of the piston which initiates the wave is greater. A similar conclusion can be formulated with respect to the harmonic components of the pressure; plot of $\widehat{p}_2/\widehat{p}_1$ as a function of the frequency has a similar course, as that of the vibration velocity, and therefore it is not presented here.

The medium in the acoustic horn is dispersive [11] and that can cause a characteristic pulsation of amplitudes of the higher harmonics [9]. Equation (2.58), (2.59) and

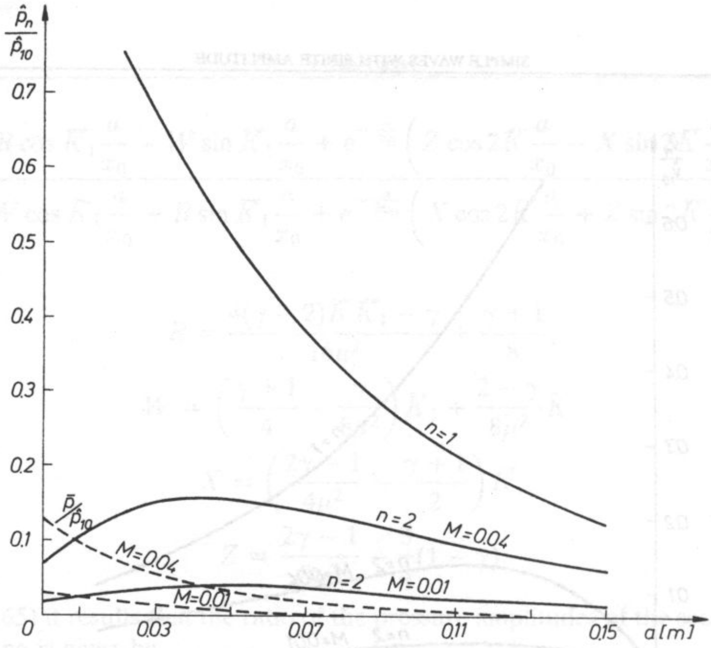

FIG. 3. The amplitudes of the acoustic pressure for the first ($n = 1$) and second ($n = 2$) harmonics (solid lines); the nonperiodic component of the acoustic pressure (dashed lines).

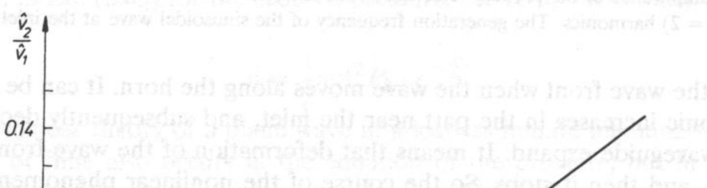

FIG. 4. The ratio of the vibration velocity amplitudes of the second harmonic to the first one for the layer of medium particles $a = 1.5 \cdot 10^{-1}$ m.

(2.65)–(2.67) indicate that this kind of effect is possible. Since the terms $\cos(\overline{K}_1 - 2\overline{K})a/x_0$ and $\sin(\overline{K}_1 - 2\overline{K})a/x_0$ occur in the expressions for D and G, respectively. However, as the Figs. 2–4 show, this effect can be neglected in waveguides in the considered range of frequencies.

4. Conclusions

In the acoustic waveguide with variable cross-section the nonlinear phenomena such as generation of the radiation stress and higher harmonics in a primary sinusoidal wave can take place in a different manner than in an infinite medium. In particular, the expansion of the waveguide walls has a restraining effect on the deformation of the wave-front and decreases the radiation stress. The considerations presented in this paper enable us to describe these processes in the second approximation for the hyperbolic horns with the annular cross-section. Calculation of the subsequent approximation can be done in a similar way by taking into. Account next terms of the power series (2.9) and expansions (2.6), (2.7). However, higher harmonics are of a smaller importance, since practically $|A| \ll 1$.

The course of the phenomena in the acoustic horn presented in this paper is still simplified, because most of the assumptions applied in the linear theory for infinitely long horns have been kept. However, the proposed model of the wave phenomena is more advanced than the linear one.

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