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WAVE PROPAGATION AND SCATTERING IN ELASTIC PLATE WITH PERIODICALLY GROOVED SURFACE

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Bragg reflection of plate waves in isotropic elastic plate with a periodically grooved surface is analyzed. Mode-coupling effect is also taken into account. The analysed problem may be applied in the construction of piezoelectric resonators and RAC filters.

1. Introduction

SAW propagation and Bragg reflection phenomena on piezoelectric substrate with periodically corrugated surface were investigated in details [1], [2], [3]. In the substrate (piezoelectric halfspace) usually propagates one surface mode [1], [4] and the Bragg condition has the form $K = 2k_v$, where K — wave number of grooves system and k_v — SAW wavenumber.

In the elastic plate case situation is more complicated because of multimodal propagation of plate waves [5], [6], [7], [8]. Different plate modes may be coupled by formula

$$K = k_1 + k_2,$$

where k_1 — wave vector of forwards propagating mode, k_2 — wave vector of backwards propagating mode.

This is Bragg reflection of the first order. In this paper Bragg reflection of slant propagating wave (with reference to groove system) is analysed. Mode conversion is also discussed. We consider isotropic elastic plate to be made of material characterised by ρ -mass density and Lamé constants λ and μ . Upper surface of plate is corrugated, the lower is flat (Fig. 1).

Surface corrugation is small, so we may apply a perturbation theory:

$$\left| \frac{h}{\Lambda} \right| \ll 1, \left| \frac{h}{d} \right| \ll 1$$

h — groove amplitude, Λ — period of groove system, d — plate thickness.

We consider waves which are propagating in any direction on a (x, z) plane. According to Floquet theorem, displacements and stresses in the plate may be written in the form

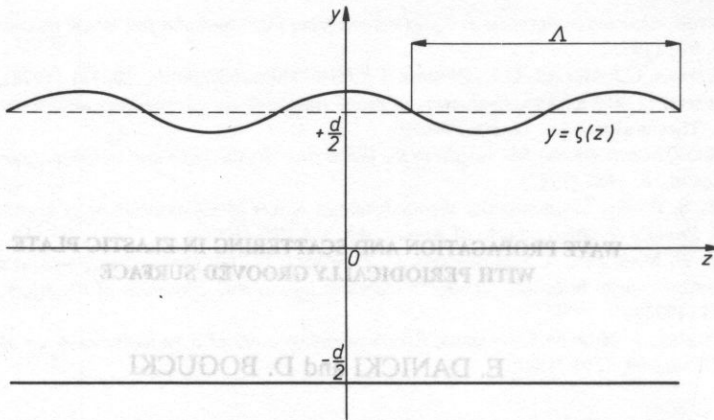


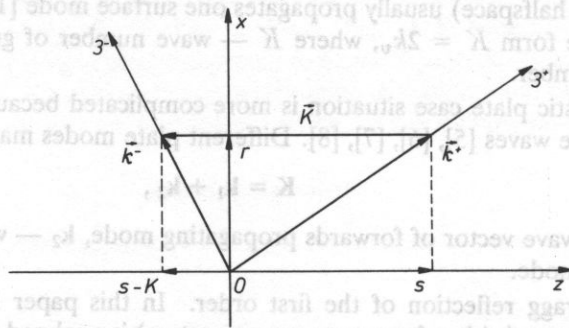
FIG. 1. Elastic plate with grooved upper surface (cross-section).

of Fourier series:

$$T_{ij} = \sum_n T_{il}^{(n)} e^{-j(s+nK)z} e^{-jrx} e^{j\omega t}, \quad (1)$$

$$u_i = \sum_n u_i^{(n)} e^{-j(s+nK)z} e^{-jrx} e^{j\omega t}, \quad (2)$$

where $i, j = x, y, z$ (1,2,3 analogously), and $r > 0$, $s > 0$ — components of wave vector of the incident wave in x and z axis direction respectively (Fig. 2).

FIG. 2. Wavevectors for $n = 0(k^+)$, $n = -1(k^-)$ Auxiliary coordinate systems-axes $k^1 || k^1$ are presented.

In this paper, for simplicity, we will take into account only the lowest harmonic components, coupled by Bragg condition i.e. components corresponding to $n = 0, -1$ (Fig. 2)

$$T_{il} = (T_{il}^+ e^{-jsz} + T_{il}^- e^{-j(s-K)z}) e^{-jrx} e^{j\omega t}, \quad (3)$$

$$u_i = (u_i^+ e^{-jsz} + u_i^- e^{-j(s-K)z}) e^{-jrx} e^{j\omega t}. \quad (4)$$

This simplification is sufficient in analysis of the first order Bragg reflection of slant propagating wave with possibility of mode conversion. In the case of propagation along the grooves (eg. for $s = 0$), it is necessary to with components account for $n = -1, 0, 1$

(which is discussed in section 3). Factor $\exp(j\omega t)$ is omitted in a further part of this article.

In section 2 basic plate equations and so-called generalized Tiersten boundary conditions are discussed. Numerical results are presented in section 3. Appendix A shows the influence of higher harmonic components in series (1) (2) on results. Appendix B presents a form of solution in the case of plate corrugated on both surfaces (upper and lower).

2. Formulation of the problem

2.1. Generalised Tiersten boundary conditions

Let us consider free material surface stretching for $y > \zeta(z)$, where

$$y = \zeta(z) = h e^{-jKz} + h^* e^{jKz} \quad (5)$$

described sinusoidally corrugated surface with period $\Lambda = 2\pi/K$ and amplitude $h \ll \Lambda$. In [1] were presented so-called generalized Tiersten boundary conditions. These are relationships between stresses on mean surface $y = 0$, and known displacements on the same surface. On other hand, generalized Tiersten boundary conditions replace homogeneous boundary conditions on corrugated surface by inhomogeneous boundary conditions on mean surface (in simplification of small corrugation).

Relations presented in [1] are (detonation of complex amplitudes of stresses and displacements are analogous to (3) (4)).

$$\begin{bmatrix} T_{xy}^+ \\ T_{yy}^+ \\ T_{zy}^+ \end{bmatrix} = h \begin{bmatrix} G_{11} & 0 & G_{13} \\ 0 & G_{22} & 0 \\ G_{31} & 0 & G_{33} \end{bmatrix} \begin{bmatrix} u_x^- \\ u_y^- \\ u_z^- \end{bmatrix}, \quad (6)$$

where

$$\begin{aligned} G_{11} &= \rho\omega^2 - \mu \left(4 \frac{\lambda + \mu}{\lambda + 2\mu} r^2 + s(s - K) \right), \\ G_{13} &= -\mu \left(rs + \frac{2\lambda}{\lambda + 2\mu} r(s - K) \right), \\ G_{22} &= \rho\omega^2, \\ G_{31} &= -\mu \left(r(s - K) + \frac{2\lambda}{\lambda + 2\mu} rs \right), \\ G_{33} &= \rho\omega^2 - \mu \left(4 \frac{\lambda + \mu}{\lambda + 2\mu} s(s - K) + r^2 \right). \end{aligned} \quad (7)$$

Relationship for T_{yi}^- as a function of u_j^+ can be obtained from (6) (7) by replacements:

$$\begin{aligned} h &\rightarrow h^* \\ s &\leftrightarrow s - K \\ - &\leftrightarrow + \end{aligned} \quad (8)$$

For our purposes it is more useful to write relations (6)–(8) in another form [1]. We now introduce two additional coordinate systems such that axis $3^l \parallel \mathbf{k}^l$, $l^l \perp 3^l$ and $2^l \equiv y$, $l = +, -$ (Fig. 2).

In the same way as in [1] and [9], we can transform tensor components from (6)–(8) to new coordinate systems. After transformations we obtain

$$\mathbf{T}^+ = h \mathbf{g} \mathbf{u}^-, \quad (9)$$

$$\mathbf{T}^- = h^* \mathbf{g}^T \mathbf{u}^+, \quad (10)$$

where

$$\mathbf{g} = \begin{bmatrix} g_{11} & 0 & g_{13} \\ 0 & g_{22} & 0 \\ -g_{13} & 0 & g_{33} \end{bmatrix}, \quad (11)$$

$$g_{11} = \frac{1}{k^+ k^-} \{ (r^2 + s(s - K)) [\rho \omega^2 - \mu(r^2 + s(s - K))] + \mu r^2 K^2 \}$$

$$g_{13} = \frac{rK}{k^+ k^-} [\rho \omega^2 - 2\mu(r^2 + s(s - K))]$$

$$g_{22} = \rho \omega^2$$

$$g_{33} = \frac{1}{k^+ k^-} \left\{ (r^2 + s(s - K)) [\rho \omega^2 - 4\mu \frac{\lambda + \mu}{\lambda + 2\mu} (r^2 + s(s - K))] - \mu \frac{2\lambda}{\lambda + 2\mu} r^2 K^2 \right\} \quad (12)$$

where vectors \mathbf{T}^l i \mathbf{u}^l are (in short matrix notation) $\mathbf{T}^l = [T_6^l, T_2^l, T_4^l]^T$, $\mathbf{u}^l = [u_1^l, u_2^l, u_3^l]^T$ $l = +, -$, and wave numbers k^+ i k^- are such that

$$\begin{aligned} k^+ &= |\mathbf{k}^+| = \sqrt{s^2 + r^2} \\ k^- &= |\mathbf{k}^-| = \sqrt{(s - K)^2 + r^2}. \end{aligned} \quad (13)$$

2.2. Impedance relations for elastic plate

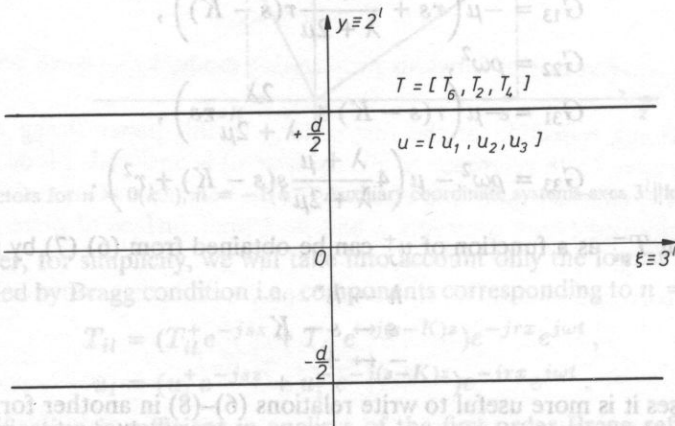


FIG. 3. Elastic plate with heterogenous boundary conditions on upper surface (cross-section).

Let us consider now the elastic plate bounded by planes $y = \pm d/2$ (Fig. 3), with its bottom surface stress-free and upper surface by harmonic stresses subjected to $\exp(-jk\xi)$.

Making use of equations of motion for elastic plate, after some transformations [10] we obtain so-called impedance relations for plate, i.e. relations between stresses and displacements on upper plate surface

$$\mathbf{u} = \mathbf{xT}, \quad (14)$$

$$\mathbf{x} = \begin{bmatrix} x_{11} & 0 & 0 \\ 0 & x_{22} & x_{23} \\ 0 & x_{32} & x_{33} \end{bmatrix}, \quad (15)$$

where: $\mathbf{T} = [T_6, T_2, T_4]^T$, $\mathbf{u} = [u_1, u_2, u_3]^T$ in our (Fig. 3) coordinate system (η, y, ξ) .

Nonzero elements of matrix \mathbf{x} are

$$x_{11} = \frac{-\cos \beta d}{\mu \cdot \Delta_T} \quad (16)$$

$$x_{22} = \frac{1}{2\mu \cdot \Delta_L} \alpha(k^2 + \beta^2) \left(\sin \alpha \frac{d}{2} \sin \beta \frac{d}{2} \Delta_1 - \cos \alpha \frac{d}{2} \cos \beta \frac{d}{2} \Delta_2 \right) \quad (17)$$

$$x_{33} = \frac{1}{2\mu \cdot \Delta_L} \beta(k^2 + \beta^2) \left(\sin \alpha \frac{d}{2} \sin \beta \frac{d}{2} \Delta_2 - \cos \alpha \frac{d}{2} \cos \beta \frac{d}{2} \Delta_1 \right) \quad (18)$$

$$x_{23} = j \frac{k}{2\mu \cdot \Delta_L} \left((2\alpha\beta\Delta_2 + (k^2 - \beta^2)\Delta_1) \sin \beta \frac{d}{2} \cos \alpha \frac{d}{2} + \right. \\ \left. + (2\alpha\beta\Delta_1 + (k^2 - \beta^2)\Delta_2) \sin \alpha \frac{d}{2} \cos \beta \frac{d}{2} \right) \quad (19)$$

$$x_{32} = x_{23}^* \quad (20)$$

where

$$\alpha^2 = \left(\frac{\omega}{V_L} \right)^2 - k^2, \quad \beta^2 = \left(\frac{\omega}{V_T} \right)^2 - k^2, \quad V_T = \sqrt{\frac{\mu}{\rho}}, \quad V_L = \sqrt{\frac{2\mu + \lambda}{\rho}}. \quad (21)$$

Signs of α and β may be assumed both plus or minus because relations (16)–(20) are even with respect to this variables.

$$\Delta_1 = (k^2 - \beta^2) \sin \alpha \frac{d}{2} \cos \beta \frac{d}{2} + 4k^2\alpha\beta \cos \alpha \frac{d}{2} \sin \beta \frac{d}{2} \quad (22)$$

$$\Delta_2 = (k^2 - \beta^2) \sin \beta \frac{d}{2} \cos \alpha \frac{d}{2} + 4k^2\alpha\beta \cos \beta \frac{d}{2} \sin \alpha \frac{d}{2} \quad (23)$$

$$\Delta_L = \Delta_1 \cdot \Delta_2 \quad (24)$$

$$\Delta_T = \beta \cdot \sin \beta. \quad (25)$$

Formula (14) is the Fourier transform of Green's function for elastic plate (k is Fourier transform variable). Dispersion relations for plate waves may be obtained, if we put zero to determinant of the inverse matrix from (15) ($\mathbf{T} = 0$ is boundary condition in this case). Element x_{11} describes propagation of SH plate waves, other nonzero elements — propagation of Lamb modes.

2.3. Dispersion relations

For isotropic plate, relations (15) are independent of the wave propagation direction. So if the system of coordinates on Fig. 3 is such that axis $\xi \equiv 3^+(y \equiv 2^+)$ then by substitution $k = k^+$ we have relations for incident wave.

$$\mathbf{u}^+ = \mathbf{x}^{(0)} \mathbf{T}^+. \quad (26)$$

If we put $k = k^- (\xi \equiv 3^-, y \equiv 2^-)$ we obtain relations for reflected wave.

$$\mathbf{u}^- = \mathbf{x}^{(-1)} \mathbf{T}^-. \quad (27)$$

Relations (9), (10), (26), (27) are sufficient for the determination of dispersion relations of waves propagating in corrugated plate. Substituting (10) to (27) and (11) to (26) we obtain

$$\begin{cases} \mathbf{u}^+ = h \mathbf{x}^{(0)} g \mathbf{u}^- \\ \mathbf{u}^- = h^* \mathbf{x}^{(-1)} g^T \mathbf{u}^+ \end{cases} \quad (28)$$

So we have a system of 6 equations with 6 unknowns $[u_i^+, u_i^-] \ i = 1, 2, 3$. Dispersion relations may be determined by putting to zero determinant of equations set (28). It is easy to reduce set (28) to equivalent set 3×3 equations:

$$\{\mathbf{I} - |h|^2 \mathbf{A}\} \mathbf{u}^+ = \mathbf{0}, \quad (29)$$

where:

$$\mathbf{A} = \mathbf{x}^{(0)} g \mathbf{x}^{(-1)} g^T. \quad (30)$$

Finally, the dispersion relation is

$$\det\{\mathbf{I} - |h|^2 \mathbf{A}\} = D = 0. \quad (31)$$

To solve equation (31) generally is quite difficult and only numerically possible, but equation (31) is much simpler in two special cases:

1. for $r = 0$, i.e. normal incidence of wave onto grooves
2. for $s = 0$, i.e. wave propagation along the grooves Those special cases are investigated in the next part of this paper.

3. Numerical results

3.1. Normal incidence onto grooves

In numerical calculations presented in this section it is assumed that:

- $s \in (0, K)$, i.e. s is in the first Brillouin zone;
- angular frequency ω is normalised to $\Omega = \omega/V_T$, $\Omega \in (0, K)$;
- it is assumed that $h = 0.01\Lambda$;
- it was assumed $V_L = 2V_T$;
- plate thickness d is normalised $\Lambda d = m\Lambda$, in our case: $m = 1$.

For $r = 0$, i.e. in the case of normal incidence of wave onto grooves the non-zero

elements of A matrix are

$$\begin{aligned} A_{11} &= x_{11}^{(0)} x_{11}^{(-1)} g_{11}^2 \\ A_{22} &= x_{22}^{(0)} x_{22}^{(-1)} g_{22}^2 + x_{23}^{(0)} x_{32}^{(-1)} g_{22} g_{23} \\ A_{23} &= x_{22}^{(0)} x_{23}^{(-1)} g_{22} g_{33} + x_{23}^{(0)} x_{33}^{(-1)} g_{33}^2 \\ A_{32} &= x_{32}^{(0)} x_{22}^{(-1)} g_{22}^2 + x_{33}^{(0)} x_{32}^{(-1)} g_{22} g_{33} \\ A_{33} &= x_{32}^{(0)} x_{23}^{(-1)} g_{22} g_{33} + x_{33}^{(0)} x_{33}^{(-1)} g_{33}^2. \end{aligned} \quad (32)$$

So equation (31) takes the form:

$$D = D_T D_L = 0, \quad (33)$$

where:

$$D_T = 1 - |h|^2 A_{11}, \quad (34)$$

$$D_L = 1 - |h|^2 (A_{22} + A_{33}) + |h|^4 (A_{22} A_{33} - A_{23} A_{32}). \quad (35)$$

Relation (34) describes the propagation of SH (transverse) plate modes, and (35) the propagation of Lamb modes. It is worth to note that in $r = 0$ case SH and Lamb waves propagation is independent, so may be independently analysed. Fig. 4a shows dispersion curves $\Omega = \Omega(s)$ for free (uncorrugated $h = 0$) plate, at $m = 1 (d = \lambda)$. In this case in our structure always (independent of plate thickness) propagate three modes: — Antisymmetric A_0 and symmetric S_0 Lamb modes, — SH_0 transverse mode.

In corrugated plate ($h \neq 0$), corresponding relations are more complicated — Fig. 4b. Existence of forbidden frequency bands — marked A, B, C, D, E on figure, is characteristic for this case. In those frequency bands wavenumber has complex values — which is connected with effect of Bragg reflection: band A is connected with reflection of SH_0 mode backwards to SH_0 , band B : $A_0 \rightarrow A_0$, and band C : $S_0 \rightarrow S_0$. For bands $A - C$ solutions for s may be written in the form $s = K/2 \pm j \operatorname{Im}\{s\}$, where $\operatorname{Im}\{s\}$ is small in relation to K (and dependent on h magnitude). Bands D and E are connected with Bragg reflection and mode conversion ($S_0 \leftrightarrow A_0$). It is interesting that in those cases inside frequency band $\operatorname{Re}\{s\} \neq \text{const} \neq K/2$, because of the different velocity of coupled modes. For SH_0 mode, we approximate the form of dispersion relation. Assuming that

$$\sin \beta d \approx \beta d; \quad \cos \beta d \approx 1, \quad (45)$$

we obtain from (34)

$$(\Omega^2 - s^2)(\Omega^2 - (s - K)^2) - \kappa^2(\Omega^2 - s(s - K))^2 = 0, \quad (36)$$

where $\kappa = |h|/d$, so complex solutions for s we obtain if

$$\Omega \in \left(\frac{K}{2} \sqrt{\frac{1 - \kappa}{1 + \kappa}}, \frac{K}{2} \sqrt{\frac{1 + \kappa}{1 - \kappa}} \right), \quad (37)$$

(then $\operatorname{Re}\{s\} = K/2$). Finally width of frequency band is

$$\Delta\Omega = K \frac{\kappa}{\sqrt{1 - \kappa^2}}. \quad (38)$$

For plates thicker then $m = 3$, higher plate modes may propagate and corresponding dispersion relations are much more complicated.

3.2. Propagation along the grooves

As it was mentioned in Sect. 1, in this case we must regard 3 harmonic components in series (1) i (2) that is $n = -1, 0, 1$

$$T_{il} = (T_{il}^+ e^{-jsz} + T_{il}^- e^{-j(s-K)z} + T_{il}^{\sim} e^{-j(s+k)z}) e^{-jrx}. \quad (39)$$

$$u_i = (u_i^+ e^{-jsz} + u_i^- e^{-j(s-K)z} + u_i^- e^{-j(s+K)z}) e^{-jrx}. \quad (40)$$

In our case (Fig. 5)

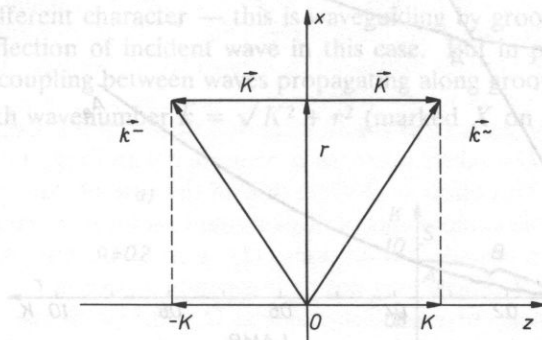


FIG. 5. Wavevectors taken under consideration in the case of propagation along the grooves

$$k^+ = r$$

$$k^\sim = k^- = \bar{k} = \sqrt{K^2 + r^2}. \quad (41)$$

Generalized Tiersten boundary conditions have now form

$$\mathbf{T}^+ = h\mathbf{g}\mathbf{u}^- + h^*\mathbf{g}^T\mathbf{u}^{\sim}, \quad (42)$$

$$\mathbf{T}^- = h^* \mathbf{g}^T \mathbf{u}^+, \quad (43)$$

$$\mathbf{T}^{\sim} = h\mathbf{g}\mathbf{u}^+, \quad (44)$$

where

$$g_{11} = \frac{r}{k}(\rho\omega^2 - \mu(r^2 - K^2)), \quad (45)$$

$$g_{13} = \frac{K}{f}(\rho\omega^2 - 2\mu r^2), \quad (46)$$

$$g_{22} = \rho\omega^2, \quad (47)$$

$$g_{33} = \frac{r}{k} \left((\rho\omega^2 - 4\mu \frac{\lambda + \mu}{\lambda + 2\mu} r^2) - \mu \frac{2\lambda}{\lambda + 2\mu} K^2 \right) \quad (48)$$

at $\mathbf{T}^l = [T_6^l, T_2^l, T_4^l]^T$, $\mathbf{u}^l = [u_6^l, u_2^l, u_4^l]^T$, where $l = +, -, \sim$.

Impedance relations for $n = +1$ may be find analogously as in section 2.3

$$\mathbf{u}^{\sim} = \mathbf{x}^{(l)} \mathbf{T}^{\sim} \quad (49)$$

it is easy to find from (41) that for $n = -1$ in our case $\mathbf{x}^{(-1)} = \mathbf{x}^{(1)}$. Finally we obtain an analogous to (28) system 9 equations with 9 unknowns.

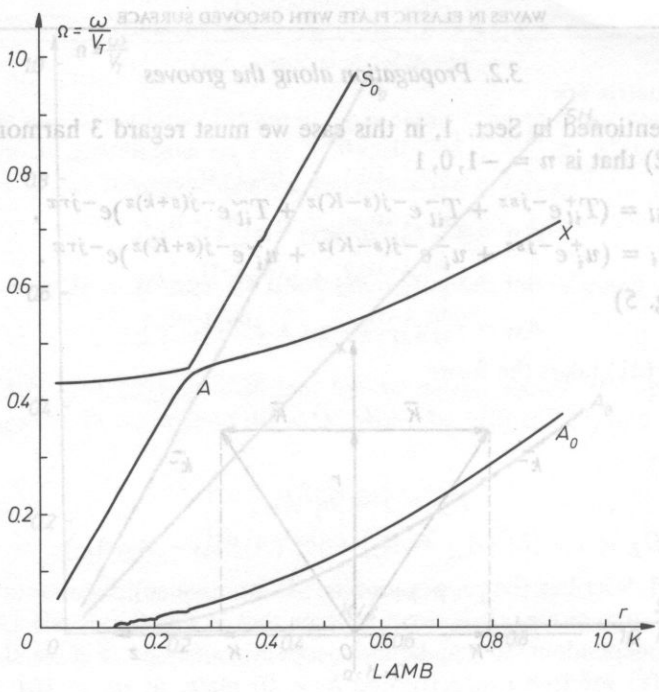


Fig. 5. Wavevector taken under consideration in the case of propagation along the grooves.

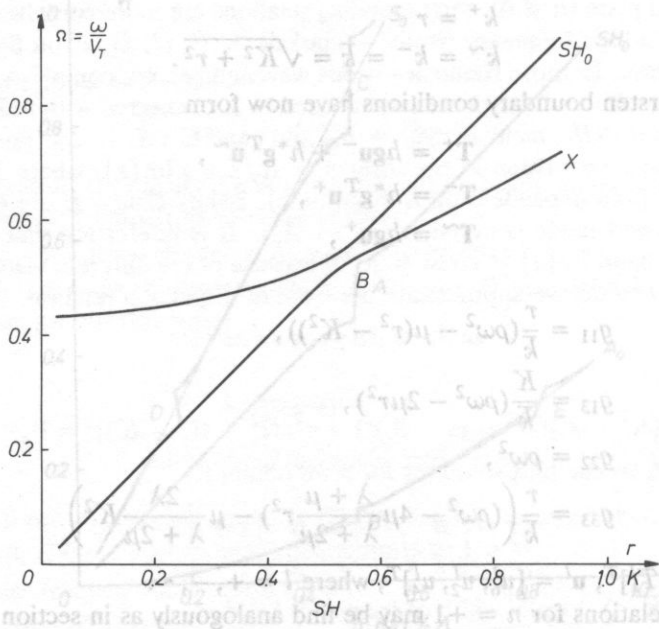


Fig. 6. Dispersion curves for propagation along the grooves ($s = 0$), $d = \Lambda$ a) SH modes b) Lambda modes.

$$\begin{cases} \mathbf{u}^+ = h\mathbf{x}^{(0)} = \mathbf{g}\mathbf{u}^- + h^*\mathbf{x}^{(0)}\mathbf{g}^T\mathbf{u}^-, \\ \mathbf{u}^- = h^*\mathbf{x}^{(-1)}\mathbf{g}^T\mathbf{u}^+, \\ \mathbf{u}^- = h\mathbf{x}^{(-1)}\mathbf{g}\mathbf{u}^+. \end{cases} \quad (50)$$

This set may be easily reduced to an equivalent system 3×3 analogous (29) with

$$\mathbf{A} = \mathbf{x}^{(0)}\mathbf{g}\mathbf{x}^{(-1)}\mathbf{g}^T + \mathbf{x}^{(0)}\mathbf{g}^T\mathbf{x}^{(-1)}\mathbf{g}. \quad (51)$$

It is important that dispersion equation has analogous form to (33), but solutions have a quite different character — this is waveguiding by groove system (Fig. 6). There is no Bragg reflection of incident wave in this case. But in parts marked A and B on Fig. 6 we have coupling between waves propagating along grooves (S_0, SH_0) with modes propagating with wavenumber $\bar{k} = \sqrt{K^2 + r^2}$ (marked X on figure). Those modes are

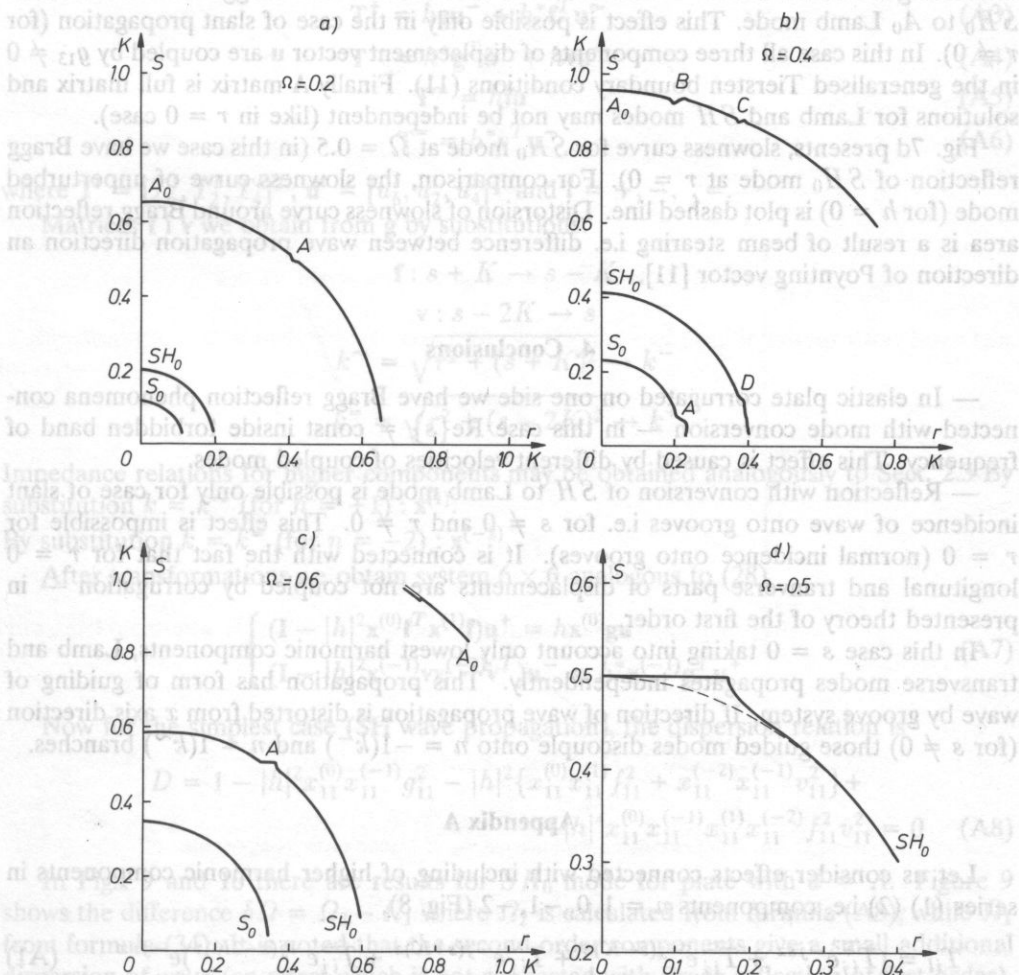


FIG. 7. Slowness curves for $d = \Lambda$.

connected with harmonic components $n = \pm 1$ ($k^- = k^{\sim}$). Numerical calculations show that in the case of small distortion (about 10%) influence of components $n = +1$ i.e. k^{\sim} may be omitted.

Results presented in this paper are essentially good agree with results published in [2].

3.3. General case (slant incidence onto grooves)

In this case $D = D(r, s, \Omega)$ depends, in a complicated way, on the frequency and wave propagation direction. The most natural is to analyse D on (r, s) plane, for $\Omega = \text{const}$ — in that way we obtain so-called slowness curves (Fig. 7). In our problem those curves have the form of concentric circles disfigured in regions where we have Bragg reflection, i.e. in area of complex values of wavenumber.

In Figs. 7a, 7b, 7c slowness curves for Ω increased with step 0.2 are presented. Very interesting are bands C i D — their existence is connected with Bragg reflection of mode SH_0 to A_0 Lamb mode. This effect is possible only in the case of slant propagation (for $r \neq 0$). In this case all three components of displacement vector \mathbf{u} are coupled by $g_{13} \neq 0$ in the generalised Tiersten boundary conditions (11). Finally \mathbf{A} matrix is full matrix and solutions for Lamb and SH modes may not be independent (like in $r = 0$ case).

Fig. 7d presents, slowness curve for SH_0 mode at $\Omega = 0.5$ (in this case we have Bragg reflection of SH_0 mode at $r = 0$). For comparison, the slowness curve of unperturbed mode (for $h = 0$) is plot dashed line. Distorsion of slowness curve around Bragg reflection area is a result of beam steering i.e. difference between wave propagation direction and direction of Poynting vector [11].

4. Conclusions

— In elastic plate corrugated on one side we have Bragg reflection phenomena connected with mode conversion — in this case $\text{Re}\{s\} \neq \text{const}$ inside forbidden band of frequency. This effect is caused by different velocities of coupled modes.

— Reflection with conversion of SH to Lamb mode is possible only for case of slant incidence of wave onto grooves i.e. for $s \neq 0$ and $r \neq 0$. This effect is impossible for $r = 0$ (normal incidence onto grooves). It is connected with the fact that for $r = 0$ longitudinal and transverse parts of displacements are not coupled by corrugation — in presented theory of the first order.

In this case $s = 0$ taking into account only lowest harmonic components, Lamb and transverse modes propagates independently. This propagation has form of guiding of wave by groove system. If direction of wave propagation is distorted from x axis direction (for $s \neq 0$) those guided modes decouple onto $n = -1$ (k^-) and $n = 1$ (k^{\sim}) branches.

Appendix A

Let us consider effects connected with including of higher harmonic components in series (1) (2) i.e. components $n = 1, 0, -1, -2$ (Fig. 8),

$$T_{il} = (T_{ij}^+ e^{-jsz} + T_{ij}^- e^{-j(s-K)z} + T_{ij}^{\sim} e^{-j(s+K)z} + T_{ij}^{\bar{\sim}} e^{-j(s-2K)z}) e^{-jrx} \quad (\text{A1})$$

$$u_i = (u_i^+ e^{-jsz} + u_i^- e^{-j(s-K)z} + u_i^{\sim} e^{-j(s+K)z} + T_{ij}^{\bar{\sim}} e^{-j(s-2K)z}) e^{-jrx} \quad (\text{A2})$$

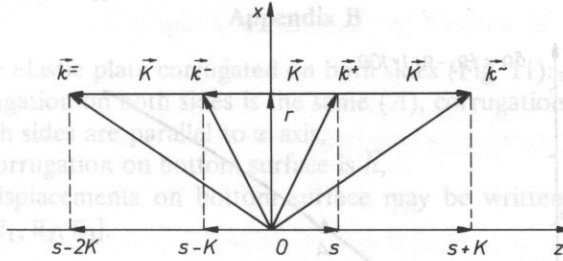


FIG. 8. Wavevectors taken under consideration in analysis of influence of second order components.

Other assumptions are the same as in previous section. Generalised Tiersten boundary conditions then have the form (higher powers of h are omitted)

$$\mathbf{T}^+ = h\mathbf{g}\mathbf{u}^- + h^*\mathbf{f}^T\mathbf{u}^- \quad (\text{A3})$$

$$\mathbf{T}^- = h^*\mathbf{g}^T\mathbf{u}^+ + h\mathbf{v}\mathbf{u}^- \quad (\text{A4})$$

$$\mathbf{T}^- = h\mathbf{f}\mathbf{u}^+ \quad (\text{A5})$$

$$\mathbf{T}^- = h^*\mathbf{v}^T\mathbf{u}^- \quad (\text{A6})$$

where $\mathbf{T}^l = [T_6^l, T_2^l, T_4^l]^T$, $\mathbf{u}^l = [u_6^l, u_2^l, u_4^l]^T$ and $l = +, -, =$.

Matrices \mathbf{f} i \mathbf{v} we obtain from \mathbf{g} by substitutions:

$$\mathbf{f} : s + K \rightarrow s - K$$

$$\mathbf{v} : s - 2K \rightarrow s$$

$$k^- = \sqrt{r^2 + (s + K)^2} \rightarrow k^-$$

$$k^- = \sqrt{r^2 + (s - 2K)^2} \rightarrow k^+$$

Impedance relations for higher components may be obtained analogously to Sect. 2.3 By substitution $k = k^-$ (for $n = +1$) : $\mathbf{x}^{(1)}$.

By substitution $k = k^-$ (for $n = -2$) : $\mathbf{x}^{(-2)}$.

After transformations we obtain system 6×6 analogous to (28)

$$\begin{cases} (\mathbf{I} - |h|^2 \mathbf{x}^{(0)} \mathbf{f}^T \mathbf{x}^{(1)} \mathbf{f}) \mathbf{u}^+ = h \mathbf{x}^{(0)} \mathbf{g} \mathbf{u}^- \\ (\mathbf{I} - |h|^2 \mathbf{x}^{(-1)} \mathbf{v} \mathbf{x}^{(-2)} \mathbf{v}^T) \mathbf{u}^- = h^* \mathbf{x}^{(-1)} \mathbf{g}^T \mathbf{u}^+ \end{cases} \quad (\text{A7})$$

Now for the simplest case (SH wave propagation), the dispersion relation is

$$D = 1 - |h|^2 x_{11}^{(0)} x_{11}^{(-1)} g_{11}^2 - |h|^2 \{ x_{11}^{(0)} x_{11}^{(1)} f_{11}^2 + x_{11}^{(-2)} x_{11}^{(-1)} v_{11}^2 \} + |h|^4 x_{11}^{(0)} x_{11}^{(-1)} x_{11}^{(1)} x_{11}^{(-2)} f_{11}^2 v_{11}^2 = 0 \quad (\text{A8})$$

In Figs. 9 and 10 there are results for SH_0 mode for plate with $d = \Lambda$. Figure 9 shows the difference $\delta\Omega = \Omega_2 - \Omega_1$ where Ω_2 is calculated from formula (A8), while Ω_1 from formula (34). It is noted that the second order components give a small additional dispersion of wave (an effect which is not connected with Bragg reflection of first order), and little change in the width of frequency band (A — Fig. 9).

Appendix B

Let us consider elastic plate corrugated on both sides (Fig. 11): period of corrugation on both sides is the same (Λ), corrugation is small, grooves on both sides are parallel to x axis, amplitude of corrugation on bottom surface is \bar{h} , stresses and displacements on bottom surface may be written in the form: $\bar{\mathbf{T}} = [\bar{\mathbf{T}}_6, \bar{\mathbf{T}}_2, \bar{\mathbf{T}}_4]$, $\bar{\mathbf{u}} = [\bar{u}_1, \bar{u}_2, \bar{u}_3]$.

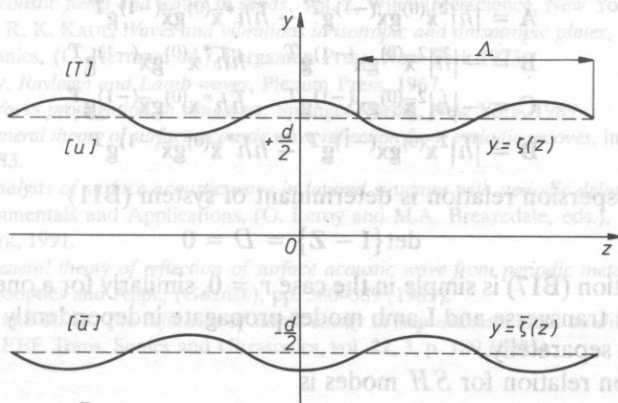


FIG. 11. Elastic plate corrugated on both sides (cross-section).

Generalized Tiersten boundary condition in the case of double corrugation have the form

$$\mathbf{T}^+ = h\mathbf{g}\mathbf{u}^- \quad \mathbf{T}^- = h^*\mathbf{g}^T\mathbf{u}^+ \quad y = +\frac{d}{2} \quad (\text{B1})$$

$$\bar{\mathbf{T}}^+ = \bar{h}\bar{\mathbf{g}}\bar{\mathbf{u}}^- \quad \bar{\mathbf{T}}^- = \bar{h}^*\bar{\mathbf{g}}^T\bar{\mathbf{u}}^+ \quad y = -\frac{d}{2} \quad (\text{B2})$$

and impedance relations (B3)

$$\mathbf{u} = \mathbf{x}\mathbf{T} + \bar{\mathbf{x}}\bar{\mathbf{T}} \quad (\text{B4})$$

$$\bar{\mathbf{u}} = -\bar{\mathbf{x}}\mathbf{T} - \mathbf{x}\bar{\mathbf{T}} \quad (\text{B5})$$

Non-zero elements of matrix $\bar{\mathbf{x}}$ may be determined analogously to [10]

$$\bar{x}_{11} = \frac{1}{\mu \cdot \Delta_T} \quad (\text{B6})$$

$$\bar{x}_{22} = \frac{1}{2\mu \cdot \Delta_L} \alpha(k^2 + \beta^2) \left(\sin \alpha \frac{d}{2} \sin \beta \frac{d}{2} \Delta_l + \cos \alpha \frac{d}{2} \cos \beta \frac{d}{2} \Delta_2 \right) \quad (\text{B7})$$

$$\bar{x}_{33} = \frac{1}{2\mu \cdot \Delta_L} \beta(k^2 + \beta^2) \left(\sin \alpha \frac{d}{2} \sin \beta \frac{d}{2} \Delta_2 + \cos \alpha \frac{d}{2} \cos \beta \frac{d}{2} \Delta_l \right) \quad (\text{B8})$$

$$\begin{aligned} \bar{x}_{23} = j \frac{k}{2\mu \cdot \Delta_L} & \left((-2\alpha\beta\Delta_2 + (k^2 - \beta^2)\Delta_l) \sin \beta \frac{d}{2} \cos \alpha \frac{d}{2} + \right. \\ & \left. + (2\alpha\beta\Delta_L - (k^2 - \beta^2)\Delta_2) \sin \alpha \frac{d}{2} \cos \alpha \frac{d}{2} \right) \quad (\text{B9}) \end{aligned}$$

$$\bar{x}_{32} = \bar{x}_{23} \quad (\text{B10})$$

Our problem may be reduced to system of 6×6 equations

$$\{I - Z\}u = 0 \quad (B11)$$

where $u = [u_i^+; \bar{u}_i^+]^T, i = 1, 2, 3$.

While Z is matrix 6×6 build of 4 blocks.

$$[Z] = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (B12)$$

$$A = |h|^2 x^{(0)} g x^{(-1)} g^T - \bar{h} h^* \bar{x}^{(0)} g \bar{x}^{(-1)} g^T \quad (B13)$$

$$B = -|\bar{h}|^2 \bar{x}^{(0)} g x^{(-1)} g^T - h \bar{h}^* x^{(0)} g \bar{x}^{(-1)} g^T \quad (B14)$$

$$C = -|h|^2 x^{(0)} g \bar{x}^{(-1)} g^T - \bar{h} h^* x^{(0)} g \bar{x}^{(-1)} g^T \quad (B15)$$

$$D = |\bar{h}|^2 \bar{x}^{(0)} g x^{(-1)} g^T - h \bar{h}^* \bar{x}^{(0)} g x^{(-1)} g^T \quad (B16)$$

Finally the dispersion relation is determinant of system (B11)

$$\det\{I - Z\} = D = 0 \quad (B17)$$

Solution of equation (B17) is simple in the case $r = 0$, similarly for a one-sided corrugated plate in that case transverse and Lamb modes propagate independently of each other and may be analysed separately.

The dispersion relation for SH modes is

$$D = (1 - A_{11})(1 - D_{11}) - B_{11}C_{11} = 0 \quad (B18)$$

Assuming that

$$\bar{h} = h_2 \exp\left\{-j\frac{\varphi}{2}\right\}; h = h_1 \exp\left\{j\frac{\varphi}{2}\right\} \quad (B19)$$

where $h_1 > 0$ i $h_2 > 0$ are the amplitudes of corrugation on upper and lower surface, respectively. Equation (B18) may be written in the form

$$D = 1 - \{(h_1^2 + h_2^2)x_{11}^{(0)}x_{11}^{(-1)} - 2h_1h_2 \cos \varphi \cdot \bar{x}_{11}^{(0)}\bar{x}_{11}^{(-1)}\}g_{11}^2 + h_1^2h_2^2(\bar{x}_{11}^{(0)2} - x_{11}^{(0)2})(\bar{x}_{11}^{(-1)2})g_{11}^4 = 0 \quad (B20)$$

We may approximate the frequency band width — analogously it was made in Sect. 3.1. So we obtain the equation

$$D = (\Omega^2 - s^2)(\Omega^2 - (s - K)^2) + \chi - \kappa^2(\Omega^2 - s(s - K))^2 = 0 \quad (B21)$$

where

$$\kappa = \frac{1}{d} \sqrt{h_1^2 + h_2^2 - 2h_1h_2 \cos \varphi}, \quad (B22)$$

$$\chi = h_1^2h_2^2(\Omega^2 - s(s - K))^4, \quad (B23)$$

when $h_1 \neq 0$ and $h_2 \neq 0$ solution depends on angle φ .

There are two special cases:

1. $\varphi = 0$ then $\kappa = |h_1 - h_2|/d$, when $h_2 = h_1$ equation (B21) describes propagation of uncoupled modes — there is no Bragg reflection.

2. $\varphi = \pi$ then $\kappa = |h_1 + h_2|/d$, when $h_1 = h_2$ small component χ may be omitted, so we have a case of equivalent plate corrugated on one side but with amplitude $2h$.

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1. Introduction

Frequently the airflow over a cavity is accompanied by generation of discrete sound, i.e. the acoustic signal consisting of discrete frequency components. This feature, exploited in musical instruments [1–3], is undesirable in wind tunnels with slotted walls [4, 5], in the industrial flow installations [6] and during the motion of various objects (aircraft, for example) in a static medium [7, 8] (detailed discussion of the sound generation phenomenon during flow over cavities can be found in review articles [9, 10]). So far, main research effort was placed on understanding the mechanism of the discrete sound generation and the role of the cavity in this process (shallow cavity [11–14], deep cavity [15–22], the cavity in the form of a Helmholtz resonator [23–27]) as well as the flow structure (laminar or turbulent flow [21, 28]). In case of the airflow over a deep cavity the experimental studies [19, 20] revealed the occurrence of the three different kinds of discrete components: pipetones, sheartones and turbulence-generated tones; the latter constitute a "passive response of cavity" to the turbulences in the airflow. The mechanism of pipetones generation, analogous to the process in organ pipes, is based on the mutual interactions between shear layer disturbances and cavity-resonance modes. In the process of the sheartone generation, the basis is the "leading edge — trailing edge interaction", i.e. the effect of the acoustic perturbations occurring at the leading edge, on the shear