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THE APPLICATIONS OF HANKEL TRANSFORM TO THE COMPUTATION OF NEARFIELDS OF CIRCULAR BAFFLED SOURCES

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In the present paper we give a new method of computation of near sound fields emitted by baffled circular sources. We have applied the Hankel transform and King's integral. Two examples were calculated — namely the piston with uniform velocity distribution and membrane for different values of the rate of the speed of elastic waves in the membrane to the speed in the medium.

1. Introduction

The computation of the near field of a baffled circular piston was given by H. Stenzel [9] and is quoted in every textbook on acoustics [e.g. 11].

Nevertheless the Stenzel's solution has two disadvantages; first — the acoustic pressure is expressed *de facto* in the form of a triple series and is therefore difficult to calculate. Second — the method itself is limited to the special case of uniform velocity distribution.

In this paper we present a different method of solving the problem of a nearfield of circular baffled sources by means of the integral Hankel transform. Application of this transform to the Helmholtz wave equation in cylindrical coordinates is due to L.U. King [5]. The idea has been continued by M.C. Junger in his monograph [4] and by M. Greenspan [3]. The quoted authors limited yet their interest to the far field, computing the proper integral by means of the saddle point method [4], or to the field on the axis of the source.

The method is quite general and can be applied to the case of an arbitrary velocity distribution given on a baffled circular piston. In this paper we have modified the method and obtained the integral representation of the nearfield of a piston with uniform velocity distribution and a membrane, of course with its "natural" Bessel distribution. Both expressions are given in the form of single integral of Bessel functions and are easily calculated numerically.

2. The general problem of an arbitrary velocity distribution on a circular baffled piston

We will consider the problem of computing the acoustical potential Φ, which in the case of harmonic vibrations is proportional to the acoustic pressure p:

$$p = i\rho_0 \omega \Phi = i\rho_0 c_0 k \Phi, \tag{1}$$

where ρ_0 is the density of the medium at rest, ω is the angular frequency of vibrations, k is the wave number and c_0 adiabatic velocity of sound.

The velocity of vibrations u, called also the acoustic velocity, is by definitiion the gradient of the acoustic potential:

$$\bar{u} = -\operatorname{grad} \Phi, \tag{2}$$

Introducing the cylindrical coordinates system, that is natural in our problem, we can write down the Helmholtz equation for the acoustic potential in the form:

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2} + k^2 \Phi = 0, \tag{3}$$

Here the field is considered to be independent of the angular coordinate φ. For a given function $u(r)$ representing the velocity distribution on the source, equation (3) has to obey the following boundary conditions:

$$\left.\frac{\partial \Phi}{\partial z}\right|_{z=0} = \begin{cases} u(r) & 0 < r \leqslant a, \\ 0 & r > a, \end{cases} \tag{4}$$

where a denotes the radius of the piston. We apply the zero order Hankel transform to both sides of the equation (3). There are several definitions of Hankel transform used in literature. In the present paper we use the form accepted in the monograph [6]. The transform of the acoustic potential $\Phi(r,z)$, denoted as $U(\rho,z)$ we write therefore in the form:

$$U(\rho,z) = \int_0^\infty r\Phi(r,z)\, J_0(\rho\, r)\, dr, \tag{5}$$

where $J_0(\,)$ is the Bessel function of zero order. The sum of the two first therms in equation (3) is transformed, according to the basic property of Hankel transform [6] into:

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} = -\rho^2 U \tag{6}$$

The transformed equation (3) takes now the form:

$$\frac{d^2 U}{dz^2} - (\rho^2 - k^2)\, U = 0, \tag{7}$$

It is, of course, the differential equation for the transformed potential i.e. $U(\rho,z)$. The Hankel transform can also be applied to the boundary condition (4). for that purpose we must transform the velocity distribution $u(r)$. That transform, denoted as $H_0(\rho)$ is equal (5):

$$H_0(\rho) = \int_0^a r\, u(r)\, J_0(\rho,r)\, dr, \tag{8}$$

We change the upper limit of the integral defining the transform from ∞ to a because $a(r) \neq 0$ only for $r \leqslant a$. Since differentiation with respect to z is independent of the integration variable in the Hankel transform, the transformed boundary conditions takes the form:

$$\left.\frac{dU}{dz}\right|_{z=0} = H_0(\rho) \tag{9}$$

The general solution of equation (7) has the form:

$$U(\rho,z) = A(\rho)\, e^{-\sqrt{\rho^2-k^2}z} + B(\rho)\, e^{\sqrt{\rho^2-k^2}z}, \tag{10}$$

The second term of the r.h.s. (10) tends to infinity when z increases and $\rho > k$. Therefore we have to put:

$$B(\rho) = 0, \tag{11}$$

and finally get:

$$U(\rho,z) = A(\rho)\, e^{-\sqrt{\rho^2-k^2}\, z}, \tag{12}$$

now we can calculate easily the l.h.s. of the boundary condition (9). We have namely from (12):

$$\left.\frac{dU}{dz}\right|_{z=0} = -A(\rho)\sqrt{\rho^2-k^2}. \tag{13}$$

Thus the unknown in (12) $A(\rho)$ is equal:

$$A(\rho) = -\frac{1}{\sqrt{\rho^2-k^2}} \left.\frac{dU}{dz}\right|_{z=0}, \tag{14}$$

Substituting in r.h.s. (14) $\left.\frac{dU}{dz}\right|_{z=0}$ from (9) we obtain:

$$A(\rho) = -\frac{1}{\sqrt{\rho^2-k^2}}\, H_0(\rho), \tag{15}$$

While examining specific cases (examples) we know the accepted distribution $u(r)$ and therefore we can find its transform $H_0(\rho)$ (9) which allows us to compute $A(\rho)$ (15) and the transform $U(\rho)$ of the acoustic potential be means of the formula (12). Then:

$$U(\rho,z) = -\frac{1}{\sqrt{\rho^2-k^2}}\, H_0(\rho)\, e^{-\sqrt{\rho^2-k^2}\,z}. \tag{16}$$

The potential can be computed as the inverse Hankel transform of order 0:

$$\Phi(r,z) = \int_0^\infty U(\rho,z)\, J_0(\rho r)\rho\, d\rho, \tag{17}$$

and, after substituting in (17) the value $U(\rho,z)$ (16) we have finally:

$$\Phi(r,z) = -\int_0^\infty \frac{J_0(\rho r)}{\sqrt{\rho^2-k^2}}\, H_0(\rho)\, e^{-\sqrt{\rho^2-k^2}\,z}\, \rho\, d\rho \tag{18}$$

The above integral can be obtained also by means of another methods, e.g. by integrating the adequate Green function [4]. It is called the King' integral. However the method presented in this paper seems to be clearer and simpler.

In practical applications we must integrate (18) from 0 to k and from k to ∞, to obtain real and imaginary part of $\Phi(r,z)$.

3. Acoustic potential in the nearfield of a baffled rigid piston with uniform velocity distribution

The velocity distribution $u(r)$ has in our case the form:

$$u(r) = \begin{matrix} u_0 & 0<r\leqslant a \\ 0 & r>a \end{matrix} \tag{19}$$

According to the formula (8) the Hankel transform of that distribution is:

$$H_0(\rho) = u_0 \int_0^a rJ_0(\rho r)\, dr. \tag{20}$$

The r.h.s. of (20) represents an elementary integral [2] [9] and we write:

$$H_0(\rho) = u_0 a\, \frac{J_1(\rho a)}{\rho}, \tag{21}$$

where $J_1(\ )$ is a first order Bessel function. According to the formula (18) the acoustic potential takes now the form:

$$\Phi(r,z) = -u_0 a \int_0^\infty \frac{e^{-\sqrt{\rho^2-k^2}\,z}}{\sqrt{\rho^2-k^2}}\, J_1(\rho a)\, J_0(\rho r)\, d\rho. \tag{22}$$

The formula (22) can be written in a simpler form introducing a new variable:

$$x = \rho a \tag{23}$$

Then:

$$\Phi\left(\frac{r}{a},\frac{z}{a}\right) = -u_0 a \int_0^\infty \frac{e^{-\sqrt{x^2-(ka)^2}\frac{z}{a}}}{\sqrt{x^2-(ka)^2}} J_1(x)\, J_0\left(\frac{r}{a}x\right) dx \tag{24}$$

In the formula (24) we have only the relative values $\frac{r}{a}$, $\frac{z}{a}$ and the so called diffraction parameter ka. We divide the integral into two parts namely from 0 to k and from k to ∞. We separate the real and imaginary part and obtain:

$$\Phi\left(\frac{r}{a},\frac{z}{a}\right) = u_0 a \int_0^{ka} \frac{\sin\left(\sqrt{(ka)^2-x^2}\frac{z}{a}\right)}{\sqrt{(ka)^2-x^2}} J_1(x)\, J_0\left(\frac{r}{a}x\right) dx +$$

$$- u_0 a \int_{ka}^\infty \frac{e^{-\sqrt{x^2-(ka)^2}\frac{z}{a}}}{\sqrt{x^2-(ka)^2}} J_1(x)\, J_0\left(\frac{r}{a}x\right) dx + \tag{25}$$

$$+ iu_0 a \int_0^{ka} \frac{\cos\left(\sqrt{(ka)^2-x^2}\frac{z}{a}\right)}{\sqrt{(ka)^2-x^2}} J_1(x)\, J_0\left(\frac{r}{a} smx\right) dx.$$

In the formula (1) expressing the acoustical pressure p by means the potential Φ we have in our case the following coefficient before the integral namely $\omega\rho u_0 a$, which has the dimensions of a pressure. For that reason we will compute and show in figures the relative pressure equal:

$$p_{\text{rel}} = \frac{p}{\omega\rho u_0 a} \tag{26}$$

4. Circular membrane

We have a circular membrane fastened on its edge. The displacement distribution, independent of the angular variable φ has the form:

$$\xi(r,t) = AJ_0(k_n^{(r)} r)\, e^{i\omega t} = \xi_0(r)\, e^{i\omega t}. \tag{27}$$

In the above formula $k_n^{(r)}$ denotes the wave number of standing waves in the membrane, depending on the variable r. The set of admissible wave numbers is a descrete one and for that reason we have the index n. On the edge i.e. for the value $r = a$ the displacement must fulfil the condition:

$$\xi_0(a) = 0, \tag{28}$$

or:

$$J_0(k_n^{(r)} a) = 0. \tag{29}$$

That condition gives us the set of admissible wave numbers $k_n^{(r)}$. Denoting by α_{0n} the value of n-th zero of zero order Bessel function we have:

$$k_n^{(r)} = \frac{\alpha_{0n}}{a} \tag{30}$$

The velocity of vibration on the membrane equals therefore (27):

$$u(r,t) = \frac{\partial \xi}{\partial t} = i\omega A J_0(k_n^{(r)} r)\, e^{i\omega t} = i\omega A J_0\left(\alpha_{0n} \frac{r}{a}\right) e^{i\omega t} \tag{31}$$

and can be written as:

$$u(r) = \begin{cases} u_0 J_0\left(\alpha_{0n} \dfrac{r}{a}\right) & 0 \leqslant r \leqslant a \\ 0 & r > a \end{cases} \tag{32}$$

where:

$$u_0 = i\omega A \tag{33}$$

The presence of i in (33) is of course meaningless — we can, for instance, accept an imaginary value of A — it is only the problem of the phase shift.

According to the formula (8) the Hankel transform of the velocity distribution (32) has now the form:

$$H_0(\rho) = u_0 \int_0^a r J_0\left(\alpha_{0n} \frac{r}{a}\right) J_0(\rho r)\, dr \tag{34}$$

The integral in the r.h.s. (34) is given in the tables [2], [7] and we obtain:

$$H_0(\rho) = u_0 \frac{\alpha_{0n} J_1(\alpha_{0n})\, J_0(\rho a) - \rho a J_0(\alpha_{0n})\, J_1(\rho a)}{\left(\dfrac{\alpha_{0n}}{a}\right)^2 - \rho^2} \tag{35}$$

Of course $J_0(\alpha_{0n}) = 0$ and we get:

$$H_0(\rho) = u_0 \frac{\alpha_{0n} J_1(\alpha_{0n})\, J_0(\rho a)}{\left(\dfrac{\alpha_{0n}}{a}\right)^2 - \rho^2} \tag{36}$$

The wave number k in the formula (18) is the wave number of waves propagating in the medium (e.g. air). The angular frequency of these waves is:

$$\omega_n = k_n^{(r)} \cdot c_m \tag{37}$$

where c_m denotes the velocity of elastic waves on the membrane. To this frequency corresponds the wave number in the medium equal:

$$k_n = \frac{\omega_n}{c} = \frac{\alpha_{0n}}{a} \cdot \frac{c_m}{c} \tag{38}$$

where c denotes the velocity of sound in the considered medium.

Substituting $H_0(\rho)$ (34) into the formula (18) we get the acoustical potential in the nearfield of the membrane in the form:

$$\Phi(r,z) = -\alpha_{0n}\, u_0 J_1(\alpha_{0n}) \int_0^{\infty} \frac{e^{-\sqrt{\rho^2-k_n^2}\,z} J_0(\rho a)\, J_0(\rho r)}{\left[\left(\frac{\alpha_{0n}}{a}\right)^2-\rho^2\right]\sqrt{\rho^2-k_n^2}}\,\rho\, d\rho \tag{39}$$

We introduce in the formula (39) the variable x (23) and we get:

$$\Phi\left(\frac{r}{a},\frac{z}{a}\right) = -\alpha_{0n}\, u_0 J_1(\alpha_{0n}) \int_0^{\infty} \frac{e^{-\sqrt{x^2-k_n^2 a^2}\,\frac{z}{a}} J_0(x)\, J_0\!\left(\frac{r}{a}x\right)}{({\alpha_{0n}}^2-x^2)\sqrt{x^2-(k_n a)^2}}\,x\, d\, x \tag{40}$$

According to the formula (38) we have:

$$k_n a = \alpha_{0n}\frac{c_m}{c} \tag{41}$$

Therefore we can write (40) in the form:

$$\Phi\left(\frac{r}{a},\frac{z}{a}\right) = -\alpha_{0n}\, u_0\, a J_1(\alpha_{0n}) \int_0^{\infty} \frac{e^{-\sqrt{x^2-\left(\alpha_{0n}\frac{c_m}{c}\right)^2}\,\frac{z}{a}} J_0(x)\, J_0\!\left(\frac{r}{a}x\right)}{({\alpha_{0n}}^2-k^2)\sqrt{x^2-\left(\alpha_{0n}\frac{c_m}{c}\right)^2}}\,x\, dx \tag{42}$$

We separate in (42) the real and imaginary parts and get finally:

$$\Phi\left(\frac{r}{a},\frac{z}{a}\right) = \alpha_{0n}\, u_0\, a\, J_1(\alpha_{0n}) \int_0^{\alpha_{0n}\frac{c_m}{c}} \frac{\sin\left(\sqrt{\left(\alpha_{0n}\frac{c_m}{c}\right)^2-x^2}\,\frac{z}{a}\right) J_0(x)\, J_0\!\left(\frac{r}{a}\,x\right)}{({\alpha_{0n}}^2-x^2)\sqrt{\left(\alpha_{0n}\frac{c_m}{c}\right)^2-x^2}}\,x\, dx \;+$$

$$-\alpha_{0n}\, u_0\, a J_1(\alpha_{0n}) \int_{\alpha_{0n}\frac{c_m}{c}}^{\infty} \frac{e^{-\sqrt{x^2-\left(\alpha_{0n}\frac{c_m}{c}\right)^2}\,\frac{z}{a}} J_0(x)\, J_0\!\left(\frac{r}{a}x\right)}{({\alpha_{0n}}^2-x^2)\sqrt{x^2-\left(\alpha_{0n}\frac{c_m}{c}\right)^2}}\,x\, dx \;+ \tag{43}$$

$$+\; i\,\alpha_{0n}\, u_0\, a\, J_1(\alpha_{0n}) \int_0^{\alpha_{0n}\frac{c_m}{c}} \frac{\cos\left(\sqrt{\left(\alpha_{0n}\frac{c_m}{c}\right)^2-x^2}\,\frac{z}{a}\right) J_0(x)\, J_0\!\left(\frac{r}{a}\,x\right)}{({\alpha_{0n}}^2-x^2)\sqrt{\left(\alpha_{0n}\frac{c_m}{c}\right)^2-x^2}}\,x\, dx.$$

To compare the results obtained for the piston and the membrane we must normalize the formulae for the membrane in a way assuming the same output for both sources. We denote by u_{0m} the velocity amplitude for the membrane — now we must distinguish between that value for the piston and of the membrane. According to (32) the output of the membrane equals:

$$Q_m = \int_0^a \int_0^{2\pi} u_{0m} J_0\left(\frac{\alpha_{0n}}{a} r\right) r\, dr\, d\varphi = 2\Pi u_{0m} \int_0^a J_0\left(\frac{\alpha_{0n}}{a} r\right) r\, dr. \tag{44}$$

The integral in the r.h.s. (44) is an elementary one [2] and:

$$Q_m = 2\Pi u_{0m} \frac{a^2}{\alpha_{0m}} J_1(\alpha_{0n}). \tag{45}$$

The output of the piston with the constant velocity distribution u_0 equals:

$$Q_p = \Pi a^2 u_0. \tag{46}$$

Equating (46) to (45) we get:

$$u_{0m} = \frac{\alpha_{0m}}{2 J_1(\alpha_{0n})} u_0 \tag{47}$$

In the figures we represent of course the amplitude of the relative pressure (26). The numbers ka for the piston were chosen the same as for the membrane e.i. corresponding to the successive zeros of $J_0(\ )$.

5. Conclusions

We have computed numerically the normalized pressure modulus versus the relative distance from the axis (r/a) and the relative value z/a for a piston and a membrane.

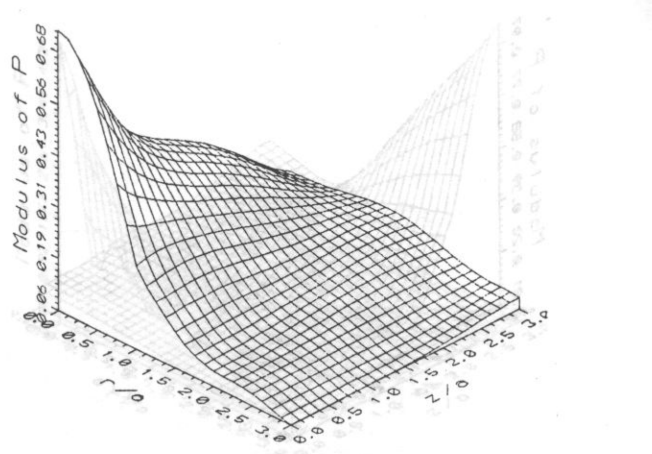
The computations were performed for the values of ka corresponding to successive zeros of the Bessel function $J_0(\ )$. Besides, for the membrane, the comutations were performed for the C_m/C equal 0.5; 1.0 and 1.5.

Of course the modulus of the pressure depends on the value of ka for both the piston and the membrane and with the increase of ka the sound field becomes more complicated — if one may use that word in the nearfield — more "directional". For the membrane the increase of the rate C_m/C deteriorates the "directivity". (Fig. 1...3)

The figures 3 and 4 represent the normalized pressure modulus of the z axis, namely for $r/a = 0$.

The numbers of figures were represented in a very clear way, namely denoted for P by the piston, $M(a)$, $M(b)$, $M(c)$ for the membrane ($C_m/C = 0.5$; 1.0; 1.5).

Fig. 1P. The normalized pressure modulus (P) as a function of normalized distance along z-axis z/a and normalized radial coordinate r/a for rigid pistone. The normalized wavenumber (diffraction parameter) $ka = 2.404...$ is equal to the value of the first zero of the Bessel function $J_0()$.



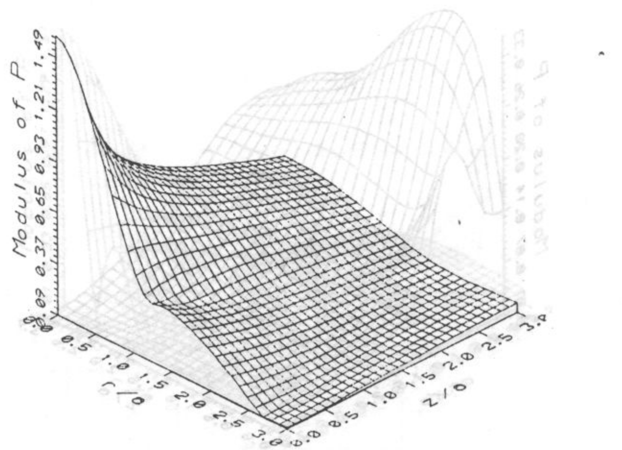


Fig. 1M(a) The normalized pressure modulus (P) as a function of normalized distance along z-axis z/a and normalized radial coordinate r/a for membrane. The normalized wavenumber (diffraction parameter) $ka = 2.404...$ is equal to the value of the first zero of the Bessel function $J_0()$. Coefficient $C_m/C = 0.5$

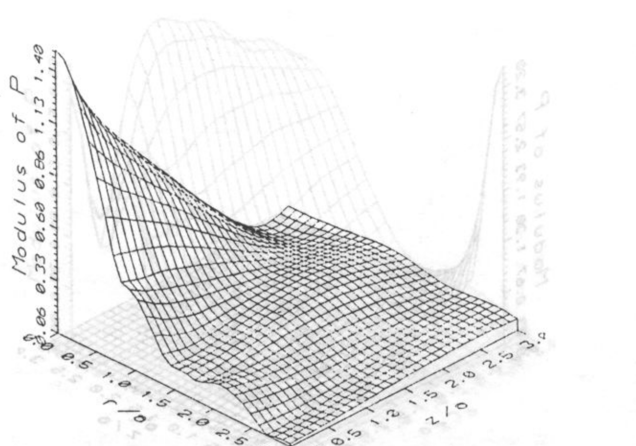


Fig. 1M(b) Same as Fig. 1M(a) but coefficient $C_m C = 1$.

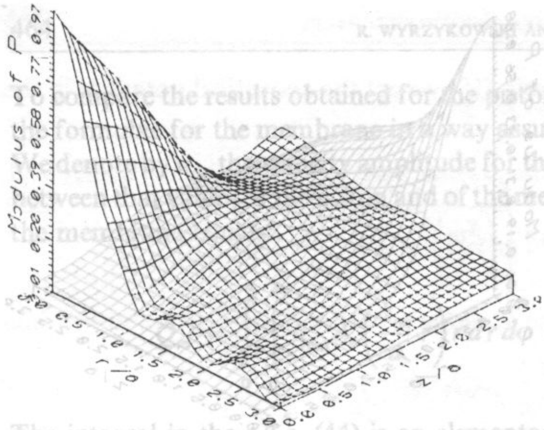


Fig. 1M(c) Same as Fig. 1M(a) but coefficient $C_m C = 1.5$.

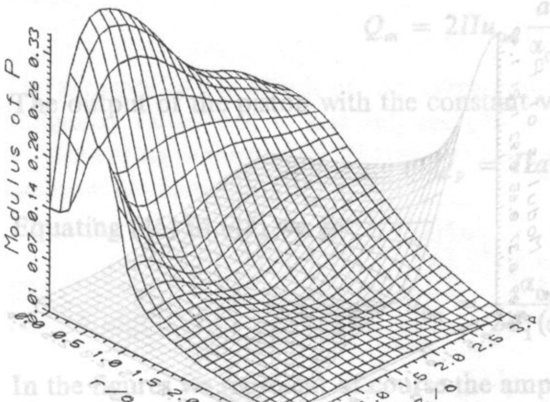


Fig. 2P The normalized pressure modulus (P) as a function of normalized distance along z-axis z/a and normalized radial coordinate r/a for rigid pistone. The normalized wavenumber (diffraction parameter) $ka = 5.520...$ is equal to the value of the second zero of the Bessel function $J_0()$.

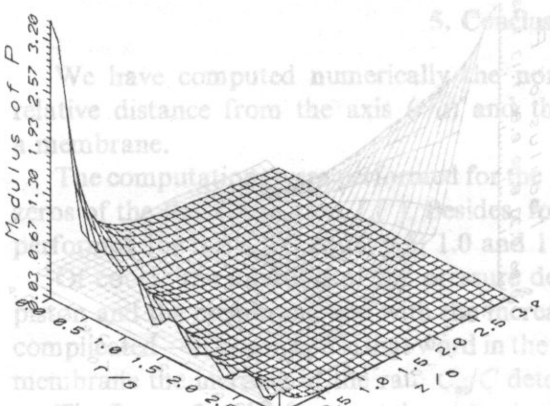


Fig. 2M(a) The normalized pressure modulus (P) as a function of normalized distance along z-axis z/a and normalized radial coordinate r/a for membrane. The normalized wavenumber (diffraction parameter) $ka = 5.520...$ is equal to the value of the first zero of the Bessel function $J_0()$. Coefficient $C_m/C = 0.5$

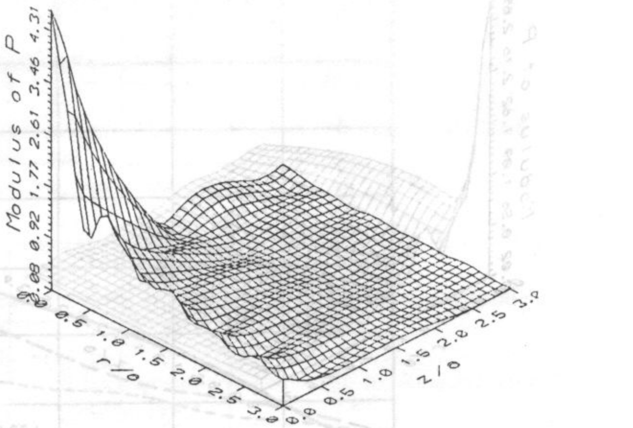


Fig. 2M(b) Same as Fig. 2M(a) but coefficient $C_m C = 1$.

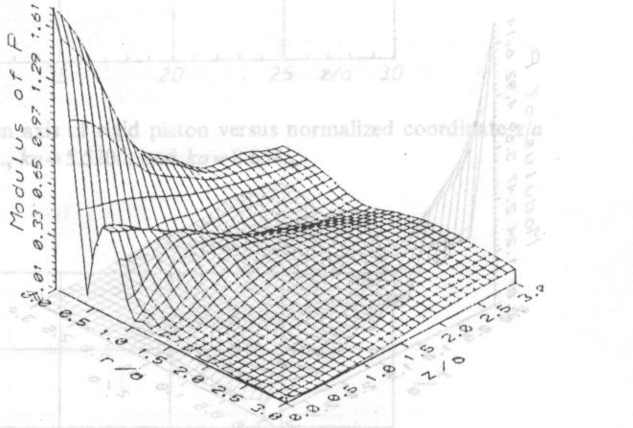


Fig. 2M(c) Same as Fig. 2M(a) but coefficient $C_m C = 1.5$.

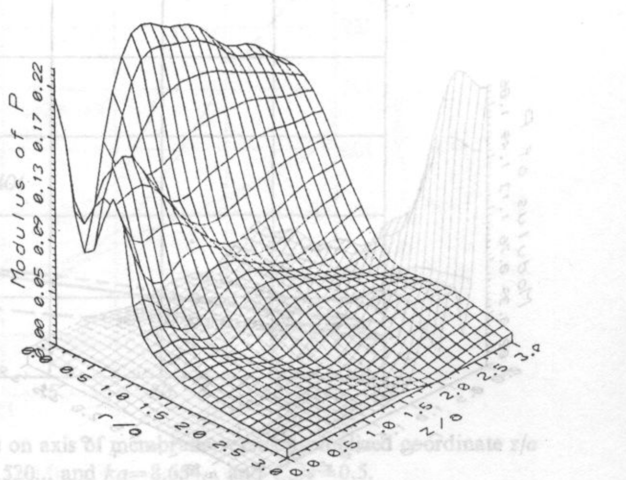


Fig. 3P The normalized pressure modulus (P) as a function of normalized distance along z-axis z/a and normalized radial coordinate r/a for rigid pistone. The normalized wavenumber (diffraction parameter) $ka = 8.654...$ is equal to the value of the third zero of the Bessel function $J_0()$.

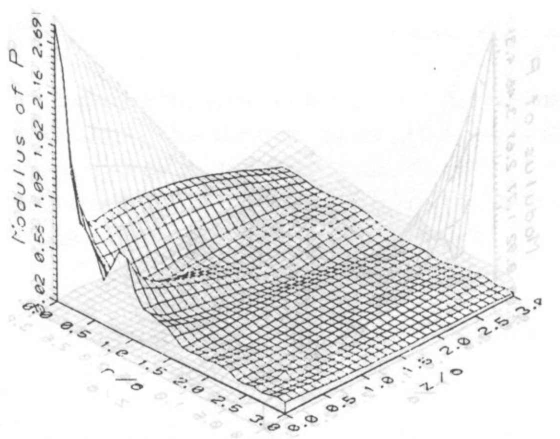


Fig. 3M(a) The normalized pressure modulus (P) as a function of normalized distance along z-axis z/a and normalized radial coordinate r/a for membrane. The normalized wavenumber (diffraction parameter) $ka = 8.654...$ is equal to the value of the third zero of the Bessel function $J_0()$. Coefficient $C_m/C = 0.5$

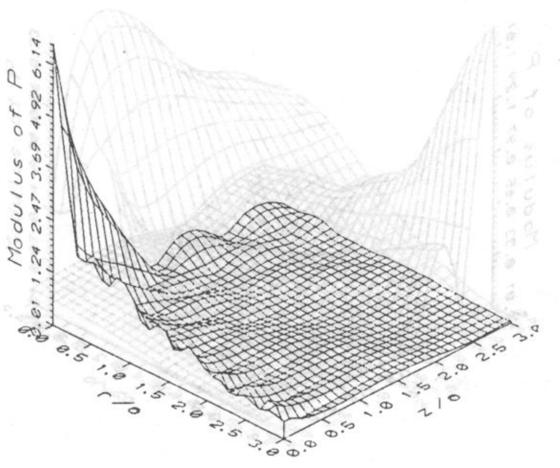


Fig. 3M(b) Same as Fig. 3M(a) but coefficient $C_mC = 1$.

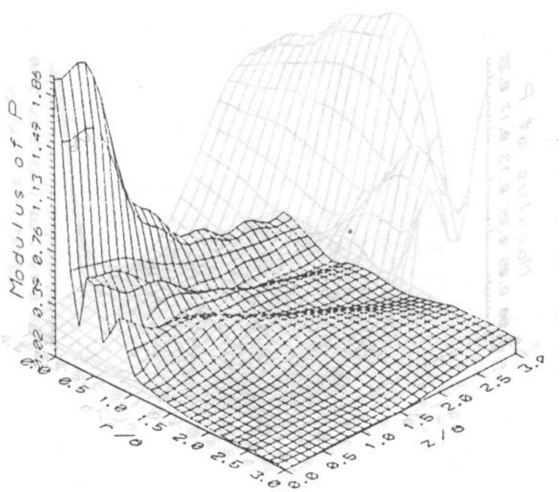


Fig. 3M(c) Same as Fig. 3M(a) but coefficient $C_mC = 1.5$.

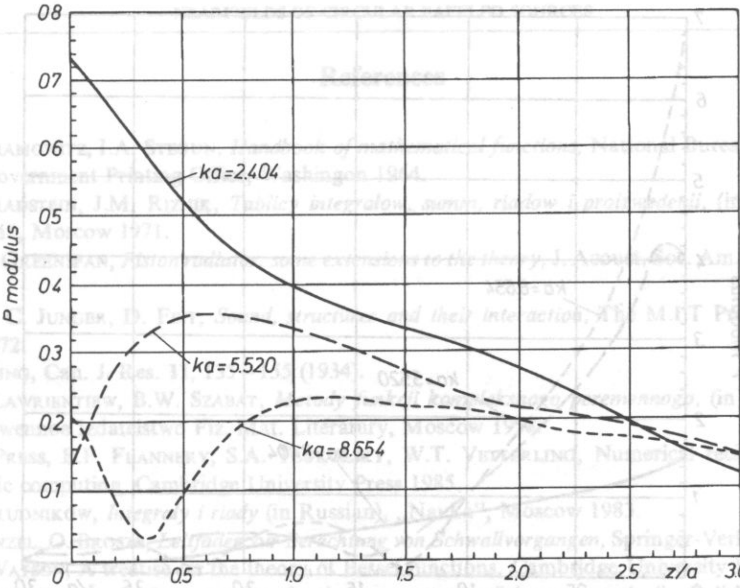


Fig. 4P The normalized pressure modulus (*P*) on axis of rigid piston versus normalized coordinate *z*/*a* (*r*/*a*=0) for *ka*=2.404..., *ka*=5.520... and *ka*=8.654...

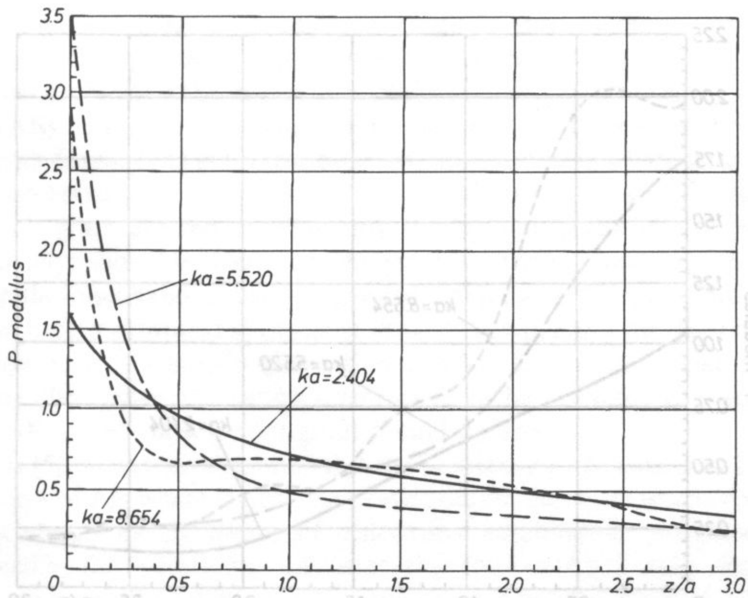


Fig. 4M(a) The normalized pressure modulus (*P*) on axis of membrane versus normalized coordinate *z*/*a* (*r*/*a*=0) for *ka*=2.404..., *ka*=5.520... and *ka*=8.654... and $C_m/c=0.5$.

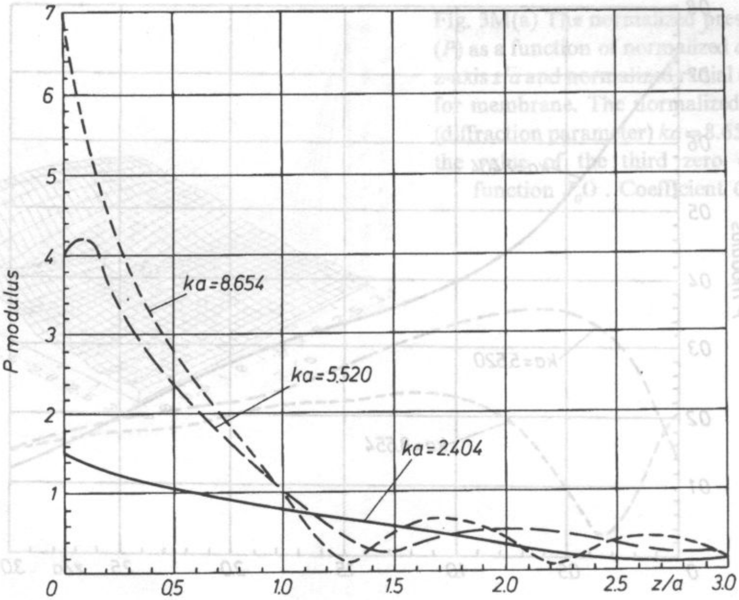


Fig. 4M(b) Same as Fig. 4M(a) but coefficient $C_m/c = 1$.

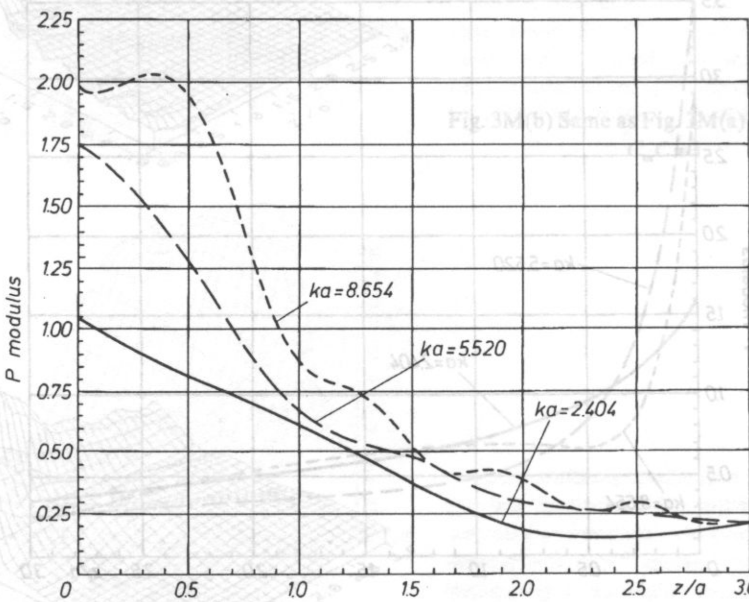


Fig. 4M(c) Same as Fig. 4M(a) but coefficient $C_m/c = 1.5$.

References

[1] M. Abramowitz, I.A. Stegun, *Handbook of mathematical functions*, National Bureau of Standards, U.S. Government Printing Office, Washingon 1964.

[2] J.S. Gradstein, J.M. Rizhik, *Tablicy integralow, summ, riadow i proizwedenii*, (in Russian), Izd. „Nauka", Moscow 1971.

[3] Martin Greenspan, *Piston radiator: some extensions to the theory*, J. Acoust. Soc. Am. **65**, 3, 608–621 (1979).

[4] Miguel C. Junger, D. Feit, *Sound, structures and their interaction*, The M.I.T Press, Cambridge, MA 1972.

[5] L.V. King, Can. J. Res. **11**, 135–155 (1934).

[6] M.A. Lawrientiew, B.W. Szabat, *Metody funkcji kompleksnogo peremennogo*, (in Russian), Gossudarstwennoe Izdatelstwo Fiz.-Mat. Literatury, Moscow 1958.

[7] W.H. Press, B.P. Flannery, S.A. Teukolsky, W.T. Vetterling, Numerical recipes: The art of scientific computing, Cambridge University Press 1985.

[8] A.P. Prudnikow, *Integraly i riady* (in Russian), „Nauka", Moscow 1983.

[9] H. Stenzel, O. Brosze, *Leitfaden zur Berechtung von Schwallvorgangen*, Springer-Verlag, Berlin 1958.

[10] G.N. Watson, A treatise on the theory of Bessel functions, Cambridge University Press, 1944.

[11] R. Wyrzykowski, *Liniowa teoria pola akustycznego ośrodków gazowych*, (in Polish), RTPN-WSP Rzeszów 1972.

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