REFLECTION COEFFICIENT FOR TWO ELASTIC LAYERS JOINING TWO HOMOGENEOUS MATERIALS

Z. WESOŁOWSKI

Institute of Fundamental Technological Research
Polish Academy of Sciences
(00-049 Warszawa, Świętokrzyska 21)

Two homogeneous elastic layers are situated between two homogeneous elastic materials. The reflection coefficient for the harmonic wave depends on the elastic constants of the layers and the frequency. The formula is too complex for an analytical treatment. Two situations were analysed numerically. In the first one, thicknesses of the layers were kept constant, and the speeds leading to constant reflection coefficient were calculated. In this case the reflection coefficient either has no minimum, or its minimum equals zero. In the second situation, propagation speeds were constant, and the thickness leading to constant reflection coefficient were calculated. There exist minima equal to zero, and maxima equal to the reflection coefficient for the long-wave limit.

1. Introduction

Between two adjoining homogeneous materials usually there exists a transition zone. The incident harmonic wave arriving at the transition zone splits into the reflected and the transmitted wave. The ratio of the energy flux of the reflected wave to the energy flux of incident wave is the reflection coefficient. There exists no tool for analytic optimisation of the general continuous transition from one to the other propagation speed. In this paper the transition region is approximated by two homogeneous elastic layers. The analysis of the reflection coefficient for this situation is given. One interesting qualitative result is obtained.

2. Jump discontinuities

In general, the transition zone between two adjoining materials, due to technology (e.g. welding, glueing) is inhomogeneous. The reflection coefficient λ for such situation is a functional of the function $c(x)$, where c is the wave speed and x the distance. It is easy to write the corresponding equations, and calculate λ for a $c(x)$ given in advance. In numerous situations the analytical formula may be obtained, cf. e.g. [1]. The only difficulty is connected with finding the solutions of an ordinary differential equation

with variable coefficients. For λ given in advance many different $c(x)$ may be calculated. It is impossible, however, to find $c(x)$ leading to minimum of λ, since it is impossible to write λ as the functional of $c(x)$. This is due to the fact, that it is impossible to write explicitly the solutions of the ordinary differential equation as the function of its coefficients.

Because of this difficulty, the inhomogeneous transition zone in this paper is approximated by two homogeneous layers. Already for this very simple model interesting qualitative results are obtained. Each of the four materials considered (two fixed half-space and two layers) is identified by the subscripts 0, 1, 2, 3 (Fig. 1).
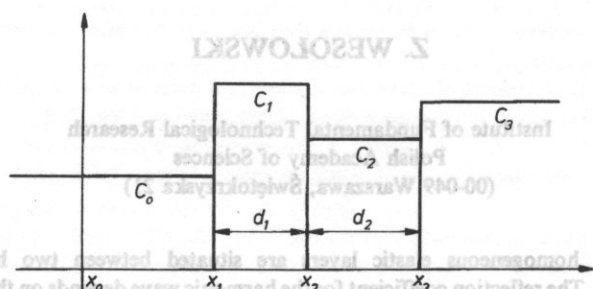

FIG. 1

The harmonic waves propagate in the direction perpendicular to the layer and the displacement u in the k-th material consists of two sinusoidal waves, the first running to the right and the second running to the left,

$$u = A_k \exp i\omega\left[t - \frac{x - x_k}{c_k}\right] + B_k \exp i\omega\left[t + \frac{x - x_k}{c_k}\right]. \tag{1.1}$$

At the boundaries between the layers both the displacement and stress are continuous. It follows that the amplitudes A_k, B_k are connected by the matrix relations (cf. e.g. [2])

$$\begin{bmatrix} A_k \\ B_k \end{bmatrix} = M_k \begin{bmatrix} A_{k-1} \\ B_{k-1} \end{bmatrix}, \tag{1.2}$$

where

$$M_k = \begin{bmatrix} (1 + \varkappa_k)\exp(-i\alpha_k) & (1 - \varkappa_k)\exp(i\alpha_k) \\ (1 - \varkappa_k)\exp(-i\alpha_k) & (1 + \varkappa_k)\exp(i\alpha_k) \end{bmatrix}. \tag{1.3}$$

$$\varkappa_k = \frac{\rho_{k-1}c_{k-1}}{\rho_k c_k}, \qquad \alpha_k = \omega\frac{x_k - x_{k-1}}{c_{k-1}}, \tag{1.4}$$

and ρ_k is the density. The transition matrix M_k is non-singular, therefore always its inverse M_k^{-1} exists. Chaining the formulae (1.2) for subsequent $K = 1, 2, 3$, the amplitudes A_3, B_3 may be expressed by A_0, B_0 and vice versa,

$$\begin{bmatrix} A_3 \\ B_3 \end{bmatrix} = M_3 M_2 M_1 \begin{bmatrix} A_0 \\ B_0 \end{bmatrix}, \qquad \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = M_1^{-1} M_2^{-1} M_3^{-1} \begin{bmatrix} A_3 \\ B_3 \end{bmatrix}. \qquad (1.5)$$

It is seen that two amplitudes may be taken at will. If we take $B_3 = 0$ and prescribe the value of A_0, then A_0, A_3, B_3 represent the amplitudes of the incident wave, the transmitted wave (both running to the right), and the reflected wave (running to the left). If we take $A_0 = 0$, then B_3, B_0, A_0 represent the amplitudes of the incident an the reflected waves both running to the left and the reflected wave running to the right.

In our problem the speeds c_0, c_3 are given in advance and the speeds c_1, c_2 are free parameters. The densities ρ_k are assumed to be equal to each other. No difficulty is connected with taking into account different densities. The positions x_k will be defined when performing the numerical calculations.

Now we take $A_0 = 0$ and consider the term proportional to B_n as the incident wave, and the terms proportional to A_n, B_0 as the reflected and transmitted waves, respectively. The other possible choice $B_3 = 0$ leads to the same reflection coefficient [1], since the system of layers has no directional properties.

In accord with the above relations, the following expressions for A_3, B_3 are obtained

$$8A_3 = B_0 \exp(-\alpha_1) \times \qquad (1.6)$$

$$\times [(1 - \varkappa_1)(1 + \varkappa_2)(1 + \varkappa_3) \exp(+\alpha_2 + \alpha_3) + (1 - \varkappa_1)(1 - \varkappa_2)(1 - \varkappa_3) \exp(+\alpha_2 - \alpha_3) +$$
$$+ (1 + \varkappa_1)(1 - \varkappa_2)(1 + \varkappa_3) \exp(-\alpha_2 + \alpha_3) + (1 + \varkappa_1)(1 + \varkappa_2)(1 - \varkappa_3) \exp(-\alpha_2 - \alpha_3)].$$

$$8B_3 = B_0 \exp(-\alpha_1) \times \qquad (1.7)$$

$$\times [(1 - \varkappa_1)(1 + \varkappa_2)(1 - \varkappa_3) \exp(+\alpha_2 + \alpha_3) + (1 - \varkappa_1)(1 - \varkappa_2)(1 + \varkappa_3) \exp(+\alpha_2 - \alpha_3) +$$
$$+ (1 + \varkappa_1)(1 - \varkappa_2)(1 - \varkappa_3) \exp(-\alpha_2 + \alpha_3) + (1 + \varkappa_1)(1 + \varkappa_2)(1 + \varkappa_3) \exp(-\alpha_2 - \alpha_3)].$$

Without restricting the generality in further calculations we assume $\alpha_1 = 0$.

The right-hand sides of (1.6), (1.7) are complex numbers. Their squared moduli are given by the formulae

$$64 A_3 \overline{A_3} = B_0 \overline{B_0} [D_1^2 + D_2^2 + D_3^2 + D_4^2 + 2(D_1 D_3 + D_2 D_4) \cos 2\alpha_2 + \qquad (1.8)$$
$$+ 2(D_1 D_2 + D_3 D_4) \cos 2\alpha_3 + 2 D_1 D_4 \cos(2\alpha_2 + 2\alpha_3) +$$
$$+ 2 D_2 D_3 \cos(2\alpha_2 - 2\alpha_3)],$$

$$64 B_3 \overline{B_3} = B_0 \overline{B_0} [D_5^2 + D_6^2 + D_7^2 + D_8^2 + 2(D_5 D_7 + D_6 D_8) \cos 2\alpha_2 + \qquad (1.9)$$
$$+ 2(D_5 D_6 + D_7 D_8) \cos 2\alpha_3 + 2 D_5 D_8 \cos(2\alpha_2 + 2\alpha_3) +$$
$$+ 2 D_6 D_7 \cos(2\alpha_2 - 2\alpha_3)],$$

where the coefficients D_1 depend only on the speed ratios $\varkappa_k$.

$$D_1 = (1-\varkappa_1)(1+\varkappa_2)(1+\varkappa_3), \quad D_2 = (1-\varkappa_1)(1-\varkappa_2)(1-\varkappa_3),$$
$$D_3 = (1+\varkappa_1)(1-\varkappa_2)(1+\varkappa_3), \quad D_4 = (1+\varkappa_1)(1+\varkappa_2)(1-\varkappa_3), \quad (1.10)$$
$$D_5 = (1-\varkappa_1)(1+\varkappa_2)(1-\varkappa_3), \quad D_6 = (1-\varkappa_1)(1-\varkappa_2)(1+\varkappa_3),$$
$$D_7 = (1+\varkappa_1)(1-\varkappa_2)(1-\varkappa_3), \quad D_8 = (1+\varkappa_1)(1+\varkappa_2)(1+\varkappa_3),$$

Energy flux corresponding to the wave of amplitude A_3 and speed c_3 equals $A_3\overline{A_3}/c_3$. Analogous relations hold for the remaining waves. The reflection coefficient λ equals the ratio of the reflected energy flux to the incident energy flux. Therefore

$$\lambda = \frac{A_3\overline{A_3}}{B_3\overline{B_3}}. \quad (1.11)$$

Obviously $0 < \lambda < 1$. The first inequality follows from (1.11), since both the nominator and denominator are positive, and the second follows from the energy conservation law.

We write explicity the complete formula for resulting from substitution of (1.7)−(1.9) into (1.10). We obtain

$$\lambda = [D_1^2 + D_2^2 + D_3^2 + D_4^2 + 2(D_1D_3 + D_2D_4)\cos 2\alpha_2 + \quad (1.12)$$
$$+ 2(D_1D_2 + D_3D_4)\cos 2\alpha_3 + 2 D_1D_4 \cos(2\alpha_2 + 2\alpha_3) +$$
$$+ 2 D_2D_3 \cos(2\alpha_2 - 2\alpha_3)] \times$$
$$\times [D_5^2 + D_6^2 + D_7^2 + D_8^2 + 2(D_5D_7 + D_6D_8)\cos 2\alpha_2 +$$
$$+ 2(D_5D_6 + D_7D_8)\cos 2\alpha_3 + 2 D_5D_8 \cos(2\alpha_2 + 2\alpha_3) +$$
$$+ 2 D_6D_7 \cos(2\alpha_2 - 2\alpha_3)]^{-1},$$

where D_K are defined by (1.10).

In order to find the extremum value of λ, the derivatives of the function (1.11) with respect to c_1 and c_2 must be calculated and put equal to zero. Note that D_K are functions of c_1, c_2, and therefore the corresponding system of trigonometric equations is very complex and no satisfactory analytic treatment of the equations may be expected. Therefore, we are forced to base on the numerical approach.

2. Fixed thickness, variable speeds

We intend to analyse in this chapter the value of the reflection coefficient, as a function of the two propagation speeds in the layers. Thickness of the layers are kept constant.

Assume $\rho_k = $ const., $x_0 = x_1 = 0$, $x_2 = d$, $x_3 = 2d$. The speed ratio for the homogeneous materials is assumed to be equal two, $c_3 = 2c_0$. The ratios c_1/c_0 and c_2/c_0 are the two independent variables. Figure 2 gives the curves of constant λ for fixed $\omega d/c_0 = 2/3$. It is seen that no minimum exists for c_1/c_0, $c_2/c_0 < 4$. Numerical analysis for larger speeds c_1, c_2 proves that also in other intervals there exists no minimum. Note that only the minimum value is interestings. The maximum value $\lambda = 1$ may be reached in the trivial reflection from the free end of from the rigid support.
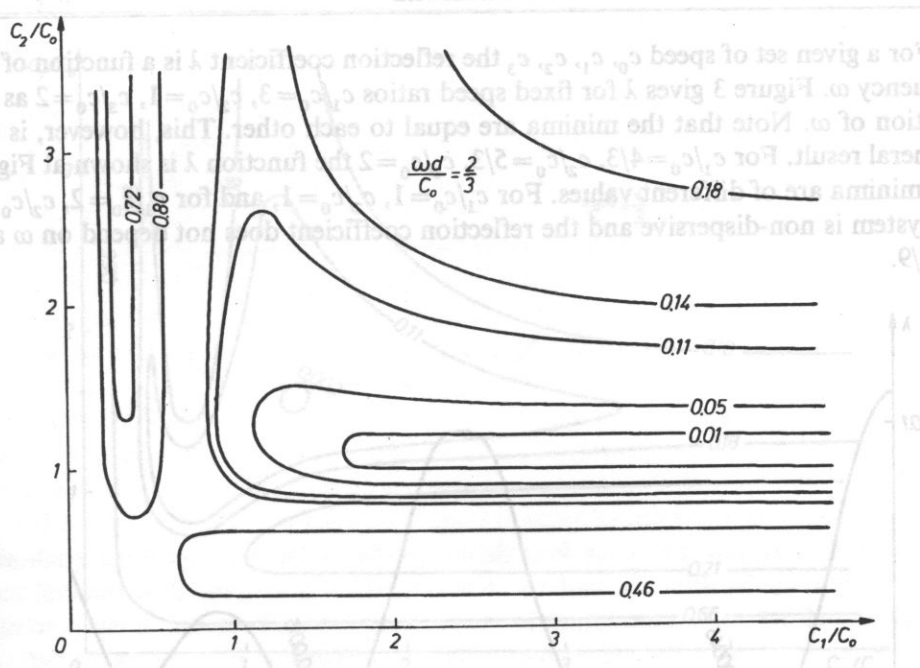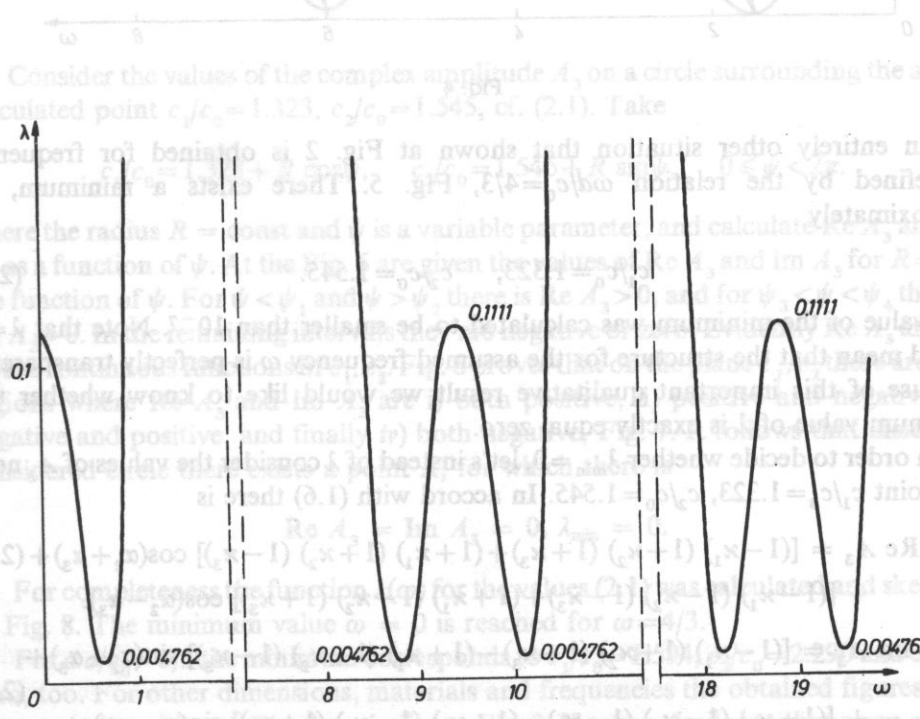
$$\frac{\omega d}{C_0} = \frac{2}{3}$$

FIG. 2

FIG. 3

For a given set of speed c_0, c_1, c_2, c_3 the reflection coefficient λ is a function of the frequency ω. Figure 3 gives λ for fixed speed ratios $c_1/c_0=3$, $c_2/c_0=1$, $c_3/c_0=2$ as the function of ω. Note that the minima are equal to each other. This, however, is not a general result. For $c_1/c_0=4/3$, $c_2/c_0=5/3$, $c_3/c_0=2$ the function λ is shown at Fig. 4. The minima are of different values. For $c_1/c_0=1$, $c_2/c_0=1$, and for $c_1/c_0=2$, $c_2/c_0=2$ the system is non-dispersive and the reflection coefficient does not depend on ω and $\lambda=1/9$.
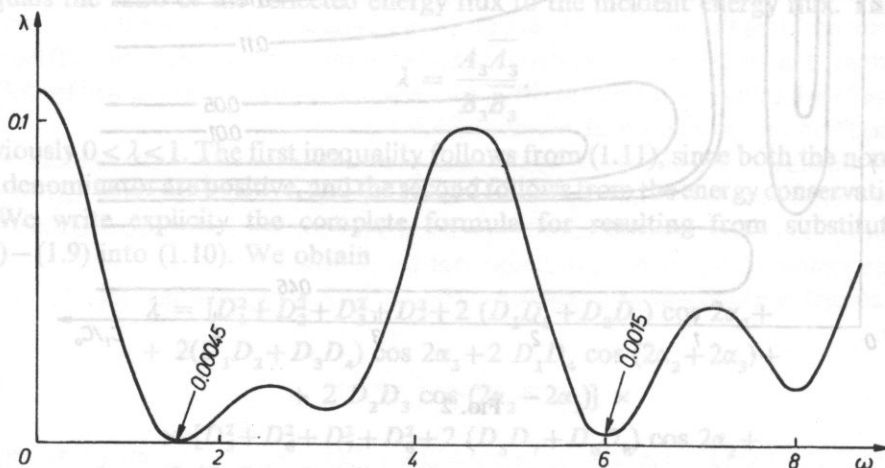

FIG. 4

An entirely other situation that shown at Fig. 2 is obtained for frequency ω defined by the relation $\omega d/c_0=4/3$, Fig. 5. There exists a minimum, at approximately

$$c_1/c_0=1.323, \qquad c_2/c_0=1.545. \tag{2.1}$$

The value of the minimum was calculated to be smaller than 10^{-7}. Note that $\lambda=0$ would mean that the structure for the assumed frequency ω is perfectly transparent. Because of this important qualitative result we would like to know whether the minimum value of λ is exactly equal zero.

In order to decide whether $\lambda_{\min}=0$, let's instead of λ consider the values of A_3 near the point $c_1/c_0=1.323$, $c_2/c_0=1.545$. In accord with (1.6) there is

$$8\,\mathrm{Re}\,A_3 = [(1-\varkappa_1)\,(1+\varkappa_2)\,(1+\varkappa_3)+(1+\varkappa_1)\,(1+\varkappa_2)\,(1-\varkappa_3)]\,\cos(\alpha_2+\alpha_3)+ \tag{2.2}$$
$$+ [(1-\varkappa_1)\,(1-\varkappa_2)\,(1-\varkappa_3)+(1+\varkappa_1)\,(1-\varkappa_2)\,(1+\varkappa_3)]\,\cos(\alpha_2-\alpha_3).$$

$$8\,\mathrm{Im}\,A_3 = [(1-\varkappa_1)\,(1+\varkappa_2)\,(1+\varkappa_3)-(1+\varkappa_1)\,(1+\varkappa_2)\,(1-\varkappa_3)]\,\sin(\alpha_2+\alpha_3)+$$
$$\tag{2.3}$$
$$+ [(1-\varkappa_1)\,(1-\varkappa_2)\,(1-\varkappa_3)-(1+\varkappa_1)\,(1-\varkappa_2)\,(1+\varkappa_3)]\,\sin(\alpha_2-\alpha_3).$$
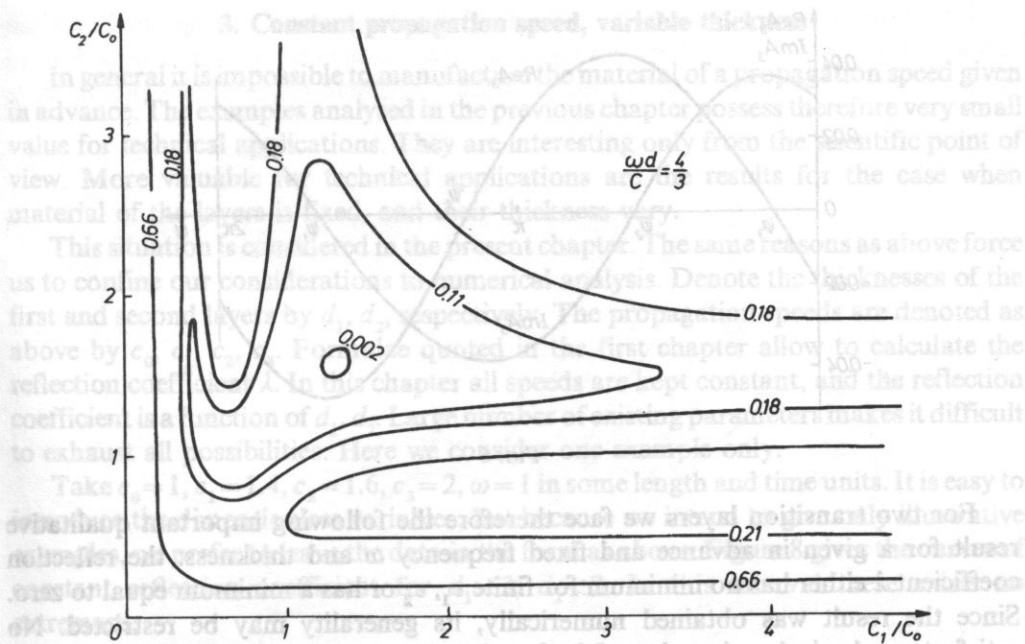
FIG. 5

Consider the values of the complex amplitude A_3 on a circle surrounding the above calculated point $c_1/c_0 = 1.323$, $c_2/c_0 = 1.545$, cf. (2.1). Take

$$c_1/c_0 = 1.323 + R\cos\psi, \qquad c_2/c_0 = 1.545 + R\sin\psi, \qquad 0 \leqslant \psi < 2\pi. \qquad (2.4)$$

where the radius R = const and ψ is a variable parameter, and calculate Re A_3 and Im A_3 as a function of ψ. At the Fig. 6 are given the values of Re A_3 and Im A_3 for $R=1$ as the function of ψ. For $\psi < \psi_1$ and $\psi > \psi_2$ there is Re $A_3 > 0$, and for $\psi_3 < \psi < \psi_4$ there is Im $A_3 > 0$. In the remaining intervals they are negative or zero. Evidently Re A_3 and Im A_3 are continuous functions of c_1, c_2. Fig. 6 proves that on the plane c_1, c_2 there are four regions where Re A_3 and Im A_3 are i) both positive, ii) positive and negative, iii) negative and positive, and finally iv) both negative, Fig. 7. It follows that inside the considered circle there exists a point K, for which there is

$$\text{Re } A_3 = \text{Im } A_3 = 0, \quad \lambda_{\min} = 0. \qquad (2.5)$$

For completeness the function $\lambda(\omega)$ for the values (2.1) was calculated and sketched at Fig. 8. The minimum value $\omega = 0$ is reached for $\omega = 4/3$. For $\omega d/c_0 = 8/3$ the minimum corresponds to $c_1/c_0 = 1.594$, $c_2/c_0 = 2.290$ and equals zero, too. For other dimensions, materials and frequencies the obtained figures were always similar either to Fig. 2 or similar to Fig. 3. If a map similar to that given at Fig. 3 was obtained, then the minimum would be equal to zero.
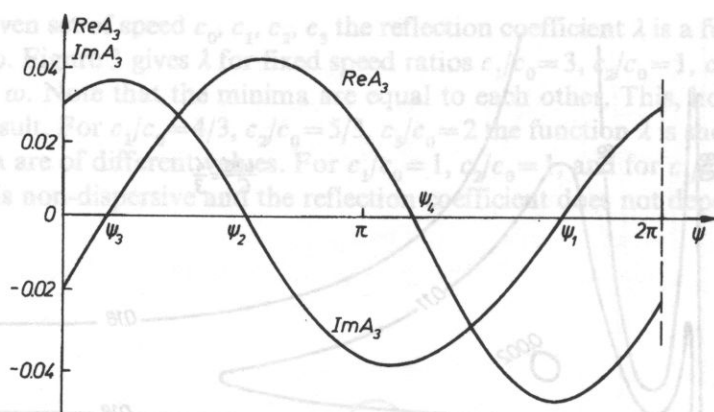
FIG. 6

For two transition layers we face therefore the following important qualitative result for a given in advance and fixed frequency ω and thickness: the reflection coefficient λ either has no minimum for finite c_1, c_2 or has a minimum equal to zero. Since the result was obtained numerically, its generality may be restricted. No satisfactory physical explanation of the fact that the minimum equals zero is known to the author.
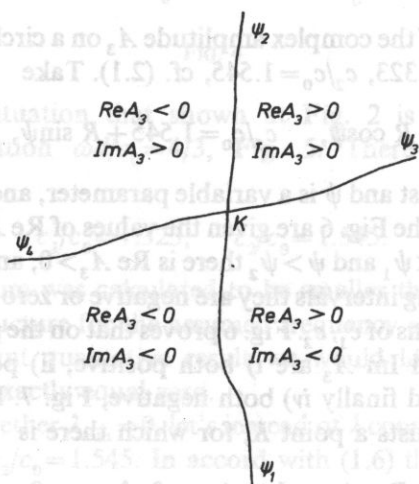

FIG. 7

For larger number of layers the above qualitative result was numerically checked in few numerical examples. It seems that the following qualitative result holds: either there exist no minimum for finite speeds, ot there exists the minimum equal to zero. This result demands further analysis. It is not known, if it is general.

3. Constant propagation speed, variable thickness

In general it is impossible to manufacture the material of a propagation speed given in advance. The examples analysed in the previous chapter possess therefore very small value for technical applications. They are interesting only from the scientific point of view. More valuable for technical applications are the results for the case when material of the layers is fixed, and their thickness vary.

This situation is considered in the present chapter. The same reasons as above force us to confine our considerations to numerical analysis. Denote the thicknesses of the first and second layers by d_1, d_2, respectively. The propagation speeds are denoted as above by c_0, c_1, c_2, c_3. Formulae quoted in the first chapter allow to calculate the reflection coefficient λ. In this chapter all speeds are kept constant, and the reflection coefficient is a function of d_1, d_2. Large number of existing parameters makes it difficult to exhaust all possibilities. Here we consider one example only.

Take $c_0 = 1$, $c_1 = 1.4$, $c_2 = 1.6$, $c_3 = 2$, $\omega = 1$ in some length and time units. It is easy to introduce the dimensionless variables, but because we intend to give only illustrative examples, we prefer to leave the data in the form as above. Figure 8 gives the curves of constant reflection coefficient for $d_1 < 3$, $d_2 < 2$. In this interval there exists no extremum.
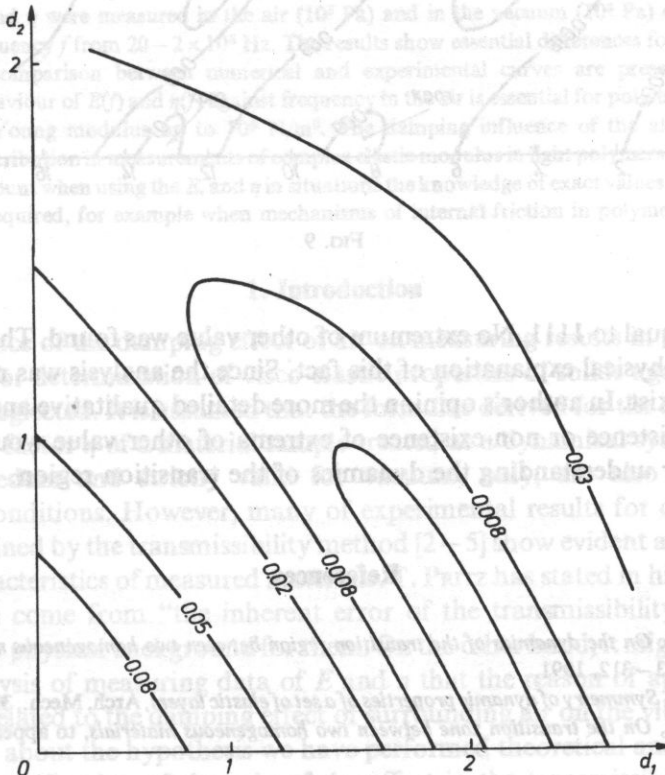

FIG. 8

For large thicknesses the picture is very irregular, since for large thickness the motion of the transition region dominates the behaviour of the system. Figure 9 gives some curves of the constant reflection coefficient λ. There are minima equal to zero,
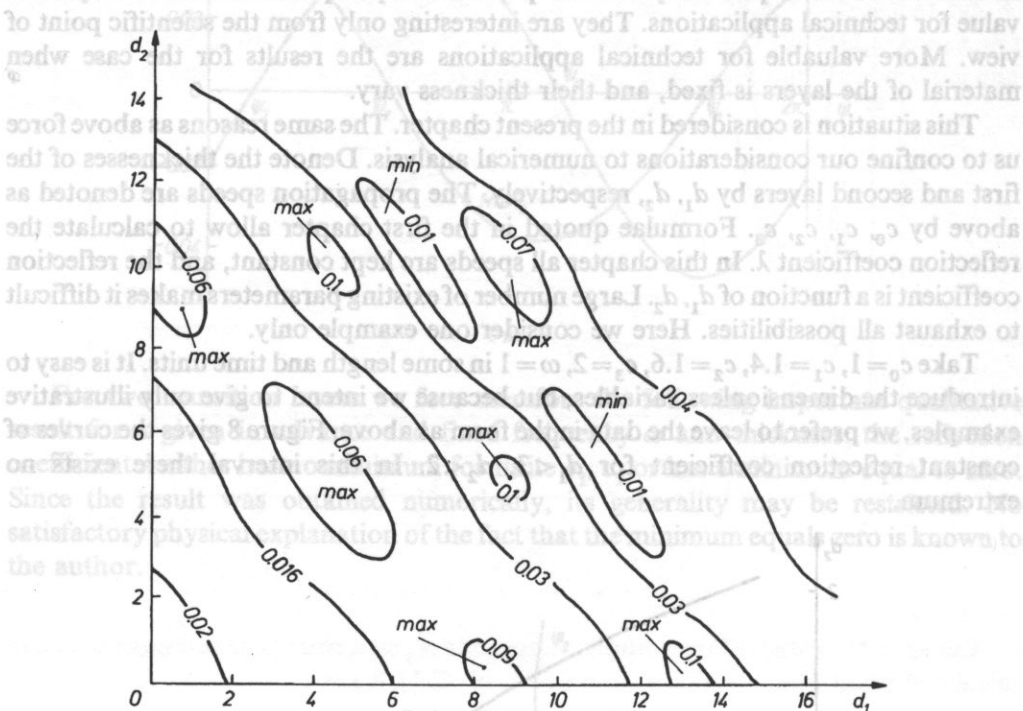

FIG. 9

and maxima equal to 1111. No extremum of other value was found. The author is not aware of any physical explanation of this fact. Since the analysis was numerical such extrema may exist. In author's opinion the more detailed qualitative analysis aimed at proving the existence or non-existence of extrema of other value would be of large importance for understanding the dynamics of the transition region.

References

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