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FAR FIELD OF A CONCENTRIC RING VIBRATING WITH CONSTANT VELOCITY ON A RIGID SPHERE

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PART I – THEORY

1. Introduction

The monograph [2] presents the theory of sound radiation from such sources as: a point on sphere, a spherical cup on a sphere and a pulsating sphere. Every mentioned source has a separate theory.

In this paper the theory of sound radiation of sources with the geometry mentioned above and described with spherical coordinates is generalized. A vibrating ring on a sphere was chosen as the source. The sources under consideration in the monograph [2] can be obtained from such a source.

Part I of this paper presents the theory of sound radiation from a ring placed on a sphere. The theory is verified in several numerical examples in Part II. The directivity function was calculates in terms of the width of the ring its position on the sphere with constant width and its vibration frequency with fixed position and fixed width.

2. Geometry of the problem

A vibrating ring placed on a rigid sphere (acoustic baffle) symmetrically with respect to the z-axis (Fig. 1) was chosen as the source (vibrating surface + acoustic baffle). This ring is cut out from a sphere with radius R by two rotational cones with a common vertex and with apex angles equal to $2\theta_1$ and $2\theta_2$. Two cones also cut out a second ring for $z < 0$. Only the ring in the top part of the sphere is taken into account.

If we assume that the acoustic parameters on the ring surface are axially symmetrical, then the distribution of the field around the sphere is also axially symmetrical. Only

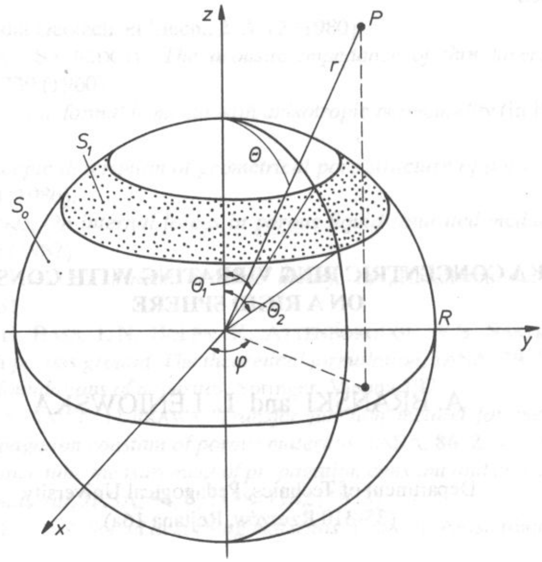

FIG. 1. Geometry of the problem.

two parameters in the spherical coordinate system r, θ, ϕ are sufficient to describe it. These are radius r and angle θ. In Fig. 1 we have S_1 – area of the ring, S_0 – area of the spherical baffle, $S_1 + S_0 = 4\pi R^2$.

3. Formulation of the boundary problem

For a steady, time-harmonic state the distribution of the field around the source is the solution to the boundary problem for a Helmholtz equation $(\Delta + k^2)\Psi = 0$, noted in the spherical coordinates [2], with the following boundary condition on a sphere with radius R

$$\frac{\partial \Psi}{\partial n} = -\upsilon_0, \quad \theta \in <\theta_1, \theta_2>,$$

$$\frac{\partial \Psi}{\partial n} = 0, \quad \theta \notin <\theta_1, \theta_2>. \tag{1}$$

where Ψ – velocity potential of acoustic field, n – unit vector of normal to the surface of the sphere, υ_0 – vibration velocity of the ring. The potential Ψ must be satisfy the Sommerfeld radiation conditions

$$\lim_{r \to \infty} = |r\,\Psi| < A, \quad A \text{ – constant},$$

$$\lim_{r - \infty} = \left(\frac{\partial \Psi}{\partial n} + ik\Psi\right) r = 0. \tag{2}$$

For the condition (1) the ring is a time-harmonic pulsating surface radial vibrations. The following sources can be obtained from a ring: in the form of a spherical cup $(\theta_1 = 0, \theta_2 \in (O, \pi/2))$, of a pulsating sphere $(\theta_1 = 0, \theta_2 = \pi$ and of a point source $(\theta_1 = 0, \theta_2 \to 0)$.

4. Solution of the Helmholtz equation in spherical coordinates

Elementary solutions of the Helmholtz equation obtained with the Fourier method have the following form:

$$\Psi_{mn} = h_m^{(2)}(kr)P_m^n(\cos\theta)e^{in\phi}, \tag{3}$$

where $h_m^{(2)}(kr)$ spherical Hankel function of the second kind and order m, $P_m^n(\cos\theta)$ – associated Legendre function of the first kind of order m and degree n, $k = 2\pi/\lambda$.
Product

$$Y_{mn} = P_m^n(\cos\theta)e^{in\phi} \tag{4}$$

is called the surface spherical harmonics. Including Eq. (4) in Eq. (3), we achieve

$$\Psi_{mn} = h_m^{(2)}(kr)Y_{mn}, \tag{5}$$

For the axisymmetric problem $n = 0$ and the function (5) assumes a specific form:

$$\Psi_{m0} = h_m^{(2)}(kr)P_m^0(\cos\theta) = h_m P_m(\cos\theta), \tag{6}$$

where $h_m = h_m^{(2)}(kr)$, $P_m(\cos\theta) = P_m^0(\cos\theta)$.

5. Solution of the boundary problem

The solution to the problem given in paragraph 3 is picked out in the form of a series

$$\Psi = \sum_{m=0}^{\infty} A_m \Psi_m, \tag{7}$$

where $\Psi_m = \Psi_{m0}$ – formula (6).
Substituting (6) in (7) we have

$$\Psi = \sum_{m=m}^{\infty} A_m h_m(kr)P_m(\cos\theta). \tag{8}$$

Expansion coefficients A_m are calculated on the basis of the fact that the surface harmonics are orthogonal. To this end Eq. (8) is substitutes in the boundary condition (1). Thus

$$-v_0 = \sum_{m=0}^{\infty} A_m h_m'(kR)P_m(\cos\theta), \tag{9}$$

where

$$h_m'(kR) = \frac{\partial}{\partial n} h_m(kr)\bigg|_{r=R} = \frac{\partial}{\partial r} h_m(kr)\bigg|_{r=R}. \tag{10}$$

Then the formula (9) is multiplied by $P_m(\cos\theta)$ and integrated along the surface of the sphere. For an arbitrary m we obtain

$$-\upsilon_0 \int_S P_{m'}(\cos\theta) d\sigma = A_m h_m'(kR) \int_S P_m(\cos\theta) P_{m'}(\cos\theta) d\sigma, \tag{11}$$

The integral on the right hand side is

$$\int_S P_m(\cos\theta) P_{m'}(\cos\theta) d\sigma =$$

$$= R^2 \int_0^{2\pi} [\int_0^{\pi} P_m(\cos\theta) P_{m'}(\cos\theta) \sin\theta d\theta] d\phi = 2\pi R^2 \frac{2}{2m+1}. \tag{12}$$

The orthogonality relation of the Legendre polynomials [1] was applied to calculate Eq. (12)

$$\int_{-1}^{1} P_m(z) P_{m'}(z) dz = \begin{cases} \dfrac{2}{2m+1}, & m' = m, \\ 0, & m' \neq m. \end{cases} \tag{13}$$

Since υ_0 differs from zero for $\theta \in < \theta_1, \theta_2 >$ only, then the integral on the left hand side in Eq. (11) can be calculated over the surface S_1 instead of S.

$$\int_{S_1} P_m(\cos\theta) d\sigma = 2\pi R^2 \int_{\theta_1}^{\theta_2} P_m(\cos\theta) \sin\theta d\theta = \frac{2\pi R^2}{2m+1} P_m(\theta_1, \theta_2), \tag{14}$$

where

$$P_m(\theta_1, \theta_2) = P_{m+1}(\cos\theta_2) - P_{m-1}(\cos\theta_2) - P_{m+1}(\cos\theta_1) + P_{m-1}(\cos\theta_1). \tag{15}$$

The relationship [1]

$$\int_{z_1}^{z_2} P_m(z) dz = \frac{1}{2m+1} [P_{m+1}(z) - P_{m-1}(z)] \bigg|_{z_1}^{z_2} \tag{16}$$

was used to calculate the integral (14).
Substituting Eq. (12) and (14) in Eq. (11), we have

$$A_m = \frac{\upsilon_0 P_m(\theta_1, \theta_2)}{2h_m'(kR)}, \tag{17}$$

Therefore the solution of the boundary problem is given by the formula (8) with the constant A_m defined by Eq. (17).

6. Specific solutions

To check the validity of the solution for a ring, its specific form can be compared with the solutions given in the monograph [2] for a spherical cup, a point on a sphere and a vibrating sphere. For example, for a spherical cup $\theta_1 = 0$, $\theta_2 \in (0, \pi/2)$. From the expression (17): we have ($P_m(1) = 1$ for every m)

$$A_m = \frac{v_0}{2h_m'(kR)}[P_{m+1}(\cos\theta_2) - P_{m-1}(\cos\theta_2)]. \tag{18}$$

Substituting Eq. (18) in Eq. (8), we obtain the solution to the problem of sound radiation of a spherical cup. The same solution is given in the monograph [2]. Chapter XX, formulae (22) and (39). The solution (8) described the acoustic field for an arbitrary distance $r > R$. The specific form of this solution describes the far field: for $r \to \infty$ in accordance with [2]

$$h_m(kr) = \exp\frac{-i[kr - (m+1)\frac{\pi}{2}]}{kr}, \tag{19}$$

Equations (8) and (17) lead to

$$\Psi = \sum_{m=0}^{\infty}\frac{v_0}{2h_m'(kR)}\exp\frac{-i[kr - (m+1)\frac{\pi}{2}]}{kr}P_m(\theta_1, \theta_2)P_m(\cos\theta) \tag{20}$$

The formula (20) was used to calculate of numerical examples.

PART II – NUMERICAL CALCULATIONS

7. Frame of numerical calculations

First, verifying calculations were carried out:
– since the series in the formula (8) is infinite, the number of terms ensuring adequate accuracy of results was numerically determined,
– the distance from the sphere which can be assumed as the approximate boundary of the far field was estimated also numerically. Furthermore the directivity function was calculated in terms of:

the width of the ring,

the position of the ring with constant width,

the dimensionless wave number *ka* for a fixed position of the ring on the sphere and for a fixed width.

Up to now there are no papers concerning the directivity function of a ring placed on a sphere. Therefore the validity of the computer programme was checked by comparing the directional function calculated for a spherical cup with the characteristic for such a source given in the monograph [2].

8. Directivity function of a source

This means the far field defined by

$$D = \frac{|p|}{|p_0|} \tag{21}$$

where: $|p|$ – pressure amplitude measured in an arbitrary direction, $|p_0|$ – amplitude of maximal pressure.

The formula (21) express the pressure drop in an arbitrary direction in dimensionless units. It is convenient to express this drop in dB. Then

$$D_{\mathrm{dB}} = 20\,\lg_{10}\frac{|p|}{|p_0|} \tag{22}$$

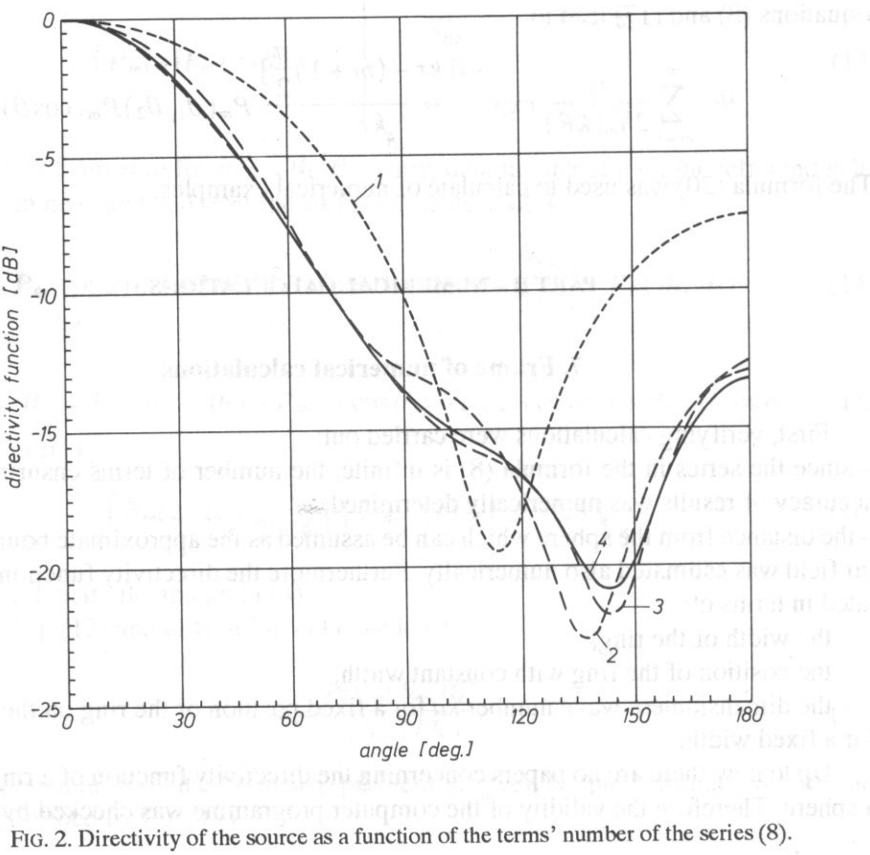

FIG. 2. Directivity of the source as a function of the terms' number of the series (8).

Since $p(t) = \rho_0 \dfrac{\partial \Psi}{\partial t}$, then time harmonic radiation the spatial distribution of acoustical pressure is $p = i\omega\rho_0 \Psi$, where Ψ is defined by the formula (8).

9. Examples

9.1. The shape of the far field was investigated as a function of the number of terms in the sum (8).

A spherical cup defined by angles $\theta_1 = 0$, $\theta_2 = 60$ and $kR = 3$, $R = 0.1$, $k = 30$, $f = 1600$ [Hz] was assumed. The results of calculations are presented in Fig. 2. Line "1" was plotted for $m = 2$, line "2" for $m = 3$ line "3" for $m = 5$ and line "4" for $m = 10$. Examination of Fig. 2 indicates that the difference between the directivity function calculated for $m = 5$ and $m = 10$ are small. Calculations were carried out on an IBM PC/AT computer, so even when much greater values of m were taken into consideration, e.g. $m = 50$, the calculating time was not very much longer, $m = 10$ was accepted for further calculations.

9.2. The distance from the surface of the sphere which can be assumed as the boundary of the Fraunhofer zone. The following values were assumed: $kR = 3$, $R = 0.1$,

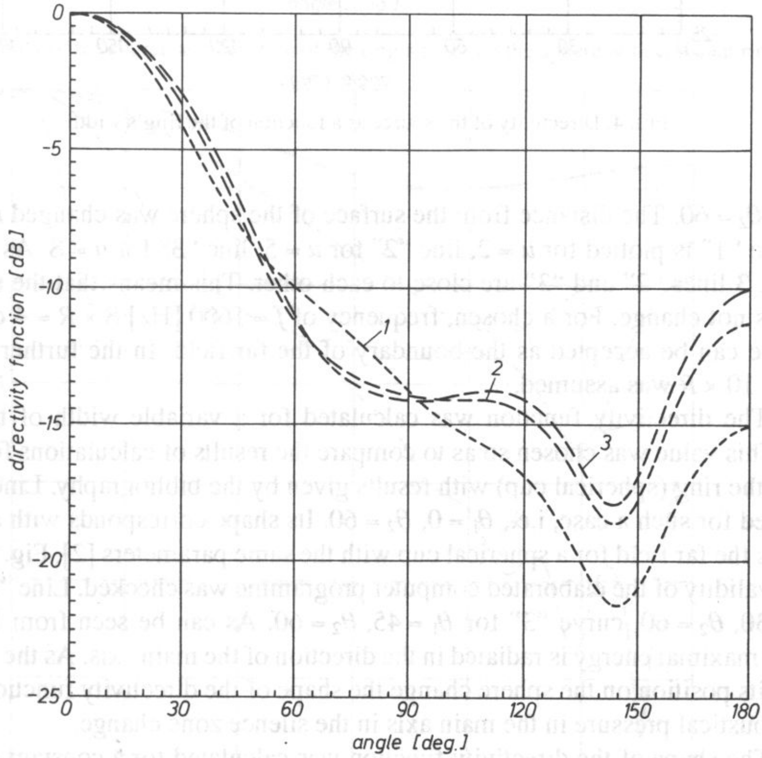

FIG. 3. Directivity of the source as a function of the distance r.

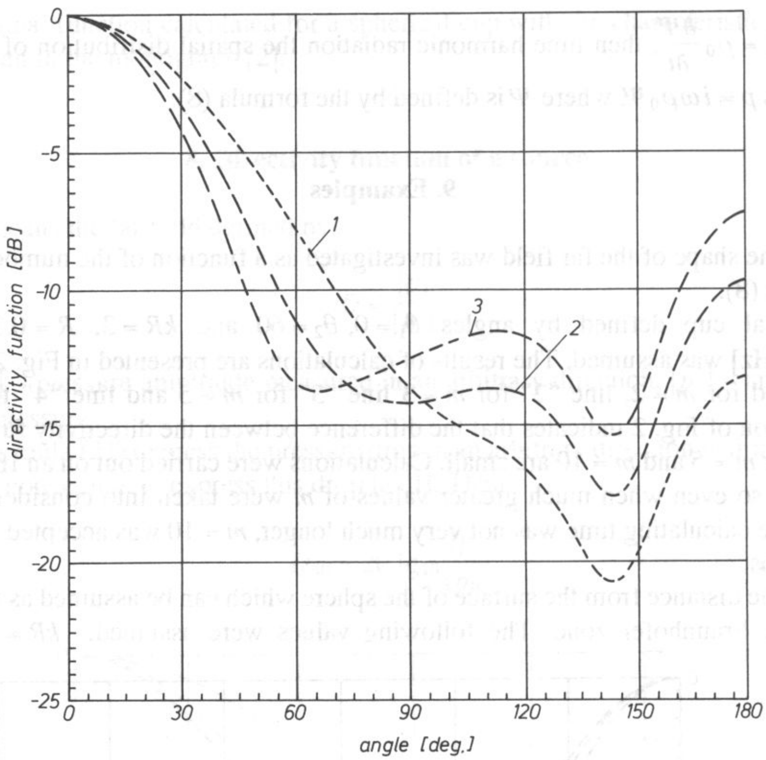

FIG. 4. Directivity of the source as a function of the ring's width.

$\theta_1 = 30$, $\theta_2 = 60$. The distance from the surface of the sphere was changed $r = u \times R$. In Fig. 3 line "1" is plotted for $u = 2$, line "2" for $u = 5$, line "3" for $u = 8$. As can be seen from Fig. 3 lines "2" and "3" are close to each other. This means that the shape of the field does not change. For a chosen, frequency of $f = 1600$ [Hz] $8 \times R = 4$ diameters of the source can be accepted as the boundary of the far field. In the further part of this paper $r = 10 \times R$ was assumed.

9.3. The directivity function was calculated for a variable width of the ring, for $kR = 3$. This value was chosen so as to compare the results of calculations for a specific shape of the ring (spherical cup) with results given by the bibliography. Line "1" in Fig. 4 is plotted for such a case, i.e., $\theta_1 = 0$, $\theta_2 = 60$. Its shape corresponds with a line which illustrates the far field for a spherical cup with the same parameters [2], Fig. 20.7. In this way the validity of the elaborated computer programme was checked. Line "2" is plotted for $\theta_1 = 30$, $\theta_2 = 60$, curve "3" for $\theta_1 = 45$, $\theta_2 = 60$. As can be seen from Fig. 4 in all cases the maximal energy is radiated in the direction of the main axis. As the width of the ring and its position on the sphere change the shape of the directivity function and value of the acoustical pressure in the main axis in the silence zone change.

9.4. The shape of the directivity function was calculated for a constant width of the ring and various positions of the ring on the sphere ($kR = 3$). In Fig. 5 line "1" is for a

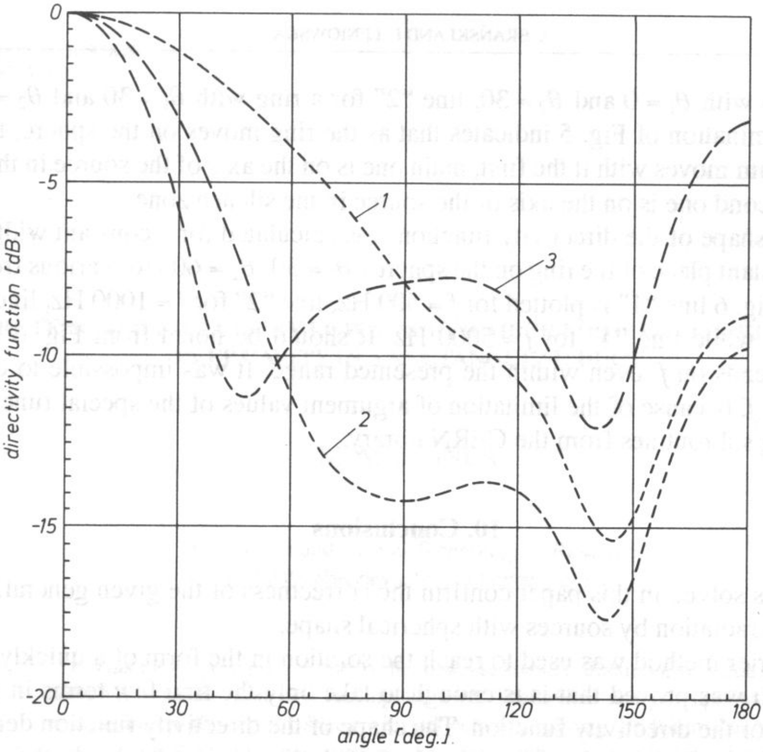

FIG. 5. Directivity of the source as a function of the ring's place on the sphere with constant ring width.

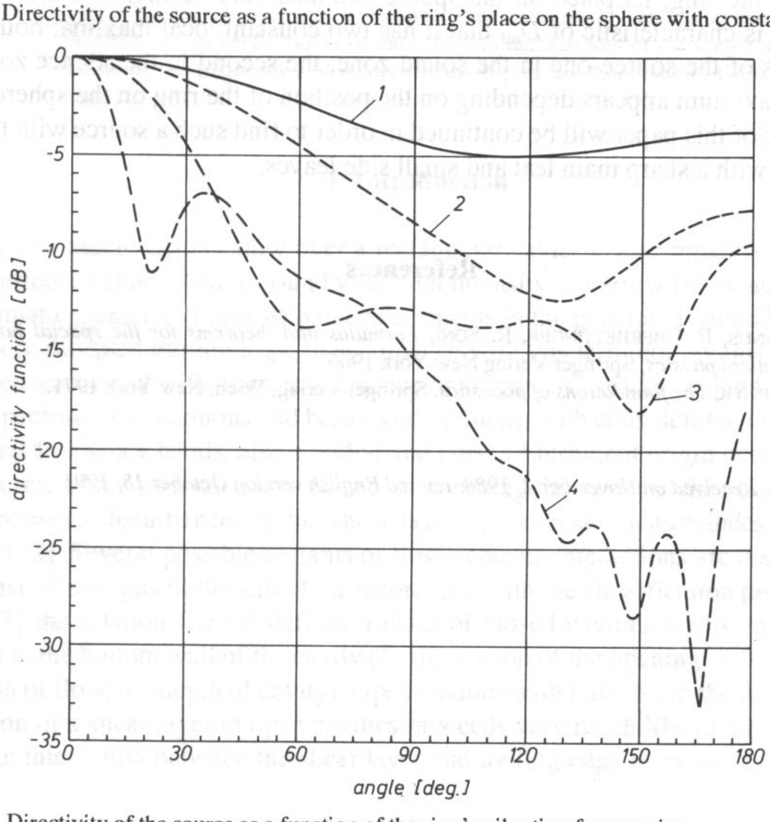

FIG. 6. Directivity of the source as a function of the ring's vibration frequencies.

spherical cup with $\theta_1 = 0$ and $\theta_2 = 30$, line “2” for a ring with $\theta_1 = 30$ and $\theta_2 = 60$ and $\theta_2 = 90$. Examination of Fig. 5 indicates that as the ring moves on the sphere, the third local maximum moves with it the first, main one is on the axis of the source in the sound zone. The second one is on the axis of the source in the silence zone.

9.5. The shape of the directivity function was calculated for a constant width of the ring and constant place of the ring on the sphere ($\theta_1 = 30$, $\theta_2 = 60$) for various frequency f values. In Fig. 6 line “1” is plotted for $f = 500$ Hz, line “2” for $f = 1000$ Hz, line “3” for $f = 2000$ Hz, while line “4” for $f = 5000$ Hz. It should be noted from Fig. 6 that D_{dB} strongly depends on f, even within the presented range. It was impossible to calculate D_{dB} for other f, because of the limitation of argument values of the special function calculated using subroutines from the CERN library.

10. Conclusions

Examples solved in this paper confirm the correctness of the given generalized theory of sound radiation by sources with spherical shape.

The Fourier method was used to reach the solution in the form of a quickly convergent series. It was proved that is is enough to take only the first few terms in practical calculations of the directivity function. The shape of the directivity function depends on the width of the ring, its place on the sphere and also very strongly on the vibration frequency. It is characteristic of D_{dB} that it has two constant local maxima; both are on the main axis of the source-one in the sound zone, the second in the silence zone. The third local maximum appears depending on the position of the ring on the sphere.

The topic of this paper will be continued in order to find such a source with the most directive i.e. with a sharp main leat and small side leaves.

References

[1] W. Magnus, F. Oberhettinger, R. Soni, *Formulas and theorems for the spacial functions of mathematical physics*, Springer Verlag New York 1966.

[2] R. Skudrzyk, *The foundations of acoustics*, Springer Verlag, Wien, New York 1971.

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