MUTUAL ACOUSTIC IMPEDANCE OF CIRCULAR SOURCES WITH PARABOLIC VIBRATION VELOCITY DISTRIBUTION FOR HIGH FREQUENCIES

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In the paper the mutual impedance of circular planar sources with parabolic vibration velocity distribution occurring harmoniously in time is analyzed. It is assumed that the sources set in planar rigid baffle radiate into a lossless and homogeneous gas medium. The acoustic impedance is calculated using the method based on the Fourier representation of acoustic pressure, thanks to which the mutual impedance is expressed in the Hankel representation. The real integrals in the formula are replaced by complex ones and integrates is performed round a closed and smooth integration contour. Using approximation methods, the expression for mutual impedance at high frequencies is obtained.

1. Introduction

The use of circular planar sources as a vibratory system for the reception or propagation of acoustic waves requires a knowledge of the frequency characteristics of many basic acoustic values with mutual acoustic impedance among them.

The paper [4] includes detailed mathematical considerations of the acoustic impedance problem, as well as a review of calculation methods referring to the system of planar sources at a given forced vibration velocity distribution.

The paper gives integrated formulae for the acoustic impedance of two circular sources with parabolic velocity distribution. During the calculation of the acoustic impedance, the integrand functions are developed into Lommel series and next, the Sonine and Schlafly integrals for the imaginary part are applied. In this way the final formula for the mutual impedance is presented as a double series in which the Hankel spherical function of the second kind is used. On the basis of this formula the mutual impedance for low interference parameters is calculated, with the assumption that the distance between the sources is many times longer than their radii. By applying integrated formulae, however, acoustic impedance is calculated and it is proved that in a special case, for normal

velocity distribution, the derived formulae develop into the well-known Rayleigh expression. Formulae for the acoustic impedance for low interference parameters are given as well.

This paper refers to the paper [4] and presents the expression for the mutual impedance for high interference parameters. By replacing the real variable by the complex one, the contour integral is introduced instead of the real one and further integration is done round the close contour [1]. The method of *stationary phase* and the asymptotic expansion of cylindrical functions are applied as well. The derived expressions have a simple mathematical form, what makes it possible to carry out detailed numerical calculations.

2. Mutual acoustic impedance

The system of M circular planar sources of the radius α is set in a planar rigid baffle surrounded on both sides by a gas medium with the rest density ρ. The vibration of each source is harmonic in time and its amplitude is defined by the following surface distribution

$$v_n(r) = v_{on}\left(1 - q\frac{r^2}{a^2}\right) \tag{1}$$

where n denotes a given source, v_{on} amplitude of the midpoint, r – radial variable in a chosen reference system, q – constant whose value depends on the way the source is mounted to the baffle, $0 \le q \le 1$. For such a vibratory system the mechanical impedance of the source with regard to influences from other sources can be expressed as

$$Z_s = \sum_{n=1}^{M} \frac{v_{on}}{v_{os}} Z_{sn}, \tag{2}$$

where Z_{ns} is the mutual impedance defined by the formula [4]

$$Z_{ns} = \frac{1}{v_{on}v_{os}^*} \int_{\sigma_s} \rho_{sn}(r)\, v_s^*(r)\, d\sigma_s, \tag{3}$$

in which $\rho_{sn}(r)$ denotes the amplitude of the acoustic pressure from the n–source, exerted on the s–source. Without neglecting further calculations are aimed at deriving the formula for the mutual impedance Z_s of any two sources. The above definition takes a simpler form for axially-symmetrical problem

$$Z_{sn} = 2\pi\rho c k^2 \int_0^{\pi/2 + i\infty} W_s(\vartheta)\, W_n(\vartheta)\, J_0(kl\sin\vartheta)\sin\vartheta\, d\vartheta, \tag{4}$$

where

$$W_n(\vartheta) = \int_0^a f_n(r)\, J_0(kr\sin\vartheta)\, r\, dr \tag{5}$$

is the characteristic function of the source [2], $f_n(r) = v_n(r)/v_{on}$ is the function of the vibration velocity distribution, $\vartheta = \theta + i\psi$, l is the distance between the midpoints of the sources, c – propagation velocity of sound in the medium having the density ρ, k – wave number. This formula can be obtained through the Fourier transform of acoustic pressure expressed by the Huygens–Rayleigh integral. Assuming the same vibration velocity distribution (1) on the surface of the s and n source and applying the formulae (4) and (5), the mutual impedance can be expressed as follows:

$$Z_{sn} = 2\rho c a^2 \left[(1-q)^2 I_{11} + \frac{4q(1-q)}{ka} I_{12} + \frac{4q^2}{(ka)^2} I_{22} \right], \tag{6}$$

where

$$I_{11} = \int_0^{\pi/2 + i\infty} \frac{J_1^2(ka\sin\vartheta)}{\sin\vartheta} J_0(kl\sin\vartheta)\,d\vartheta \tag{6a}$$

$$I_{12} = \int_0^{\pi/2 + i\infty} \frac{J_1(ka\sin\vartheta)J_2(ka\sin\vartheta)}{\sin^2\vartheta} J_0(kl\sin\vartheta)\,d\vartheta \tag{6b}$$

$$I_{22} = \int_0^{\pi/2 + i\infty} \frac{J_2^2(ka\sin\vartheta)}{\sin^3\vartheta} J_0(kl\sin\vartheta)\,d\vartheta \tag{6c}$$

The integrals included in Eq. (6a, b, c) have no exact analytical solutions. Thus there is a need to find a method to solve them, which would be proper for a given range of the interference parameter ka. The method presented below is based on the use of the asymptotic expansion of cylindrical functions as well as on integration using the method of constant phase. The obtained results are valid for sufficiently high frequencies. Each integral (6a, b, c) is a complex function of a complex variable. Separation of variables is done by replacing $\gamma = \theta + i\psi$ what results in

$$\sin(\vartheta) = \sin(\theta + i\psi) = \sin\theta\cos i\psi + \cos\theta\sin i\psi \tag{7}$$

Taking into consideration the chosen integral contour $0 \leq \theta \leq \pi/2$, $\psi = 0$, $0 \leq \psi < \infty$, $\theta = \pi/2$, the first integral (6a) has the following real and imaginary part:

$$\mathrm{Re}(I_{11}) = \int_0^{\pi/2} \frac{J_1^2(ka\sin\theta)}{\sin\theta} J_0(kl\sin\theta)\,d\theta \tag{8}$$

$$\mathrm{Im}(I_{11}) = \int_0^{\infty} \frac{J_1^2(ka\cosh\psi)}{\cosh\psi} J_0(kl\cosh\psi)\,d\psi \tag{9}$$

In order to obtain the solution of the above integrals, let us apply the replacement $x = \sin\theta$ for Eq. (8) and $x = \cosh\psi$ for Eq. (9)

$$\text{Re}(I_{11}) = \int_0^1 \frac{J_1^2(kax)}{x\sqrt{1-x^2}} J_0(klx)\,dx \tag{10}$$

$$\text{Im}(I_{11}) = \int_0^\infty \frac{J_1^2(kax)}{x\sqrt{x^2-1}} J_0(klx)\,dx \tag{11}$$

Let us now consider the real part of the first integral (6a). The remaining ones are transformed analogically in regard to the similarity of their integrands. The integral (6a) is replaced for a complex one by introducing the auxiliary complex function

$$F(z) = J_1^2(kaz) H_0^{(1)}(klz) \tag{12}$$

We choose the smooth, closed integral contour along the positive part of the x axis, omitting the branch cut. Then we follow along the quadrant with an infinite radius, which joins the positive semi-axes x and y. The value of this integral, according to the Cauchy theorem, equals zero. The integration itself is done by adding the integrals around the consecutive parts of the contour.

Let us now consider the real part of the applied contour integral. Since both the real part of the integral along the axis y and the part around the circle with an infinite radius equal zero, we obtain the following equation

$$\text{Re}\left\{ \int_c \frac{F(z)}{z\sqrt{1-z^2}} \right\} = \int_0^1 \frac{J_1^2(kax)}{x\sqrt{1-x^2}} J_0(klx)\,dx - \int_1^\infty \frac{J_1^2(kax)}{x\sqrt{x^2-1}} N_0(klx)\,dx = 0 \tag{13}$$

Let us now compare the integrals included in the above equation. The first is Eq. (8) calculated before

$$\int_0^1 \frac{J_1^2(kax)}{x\sqrt{1-x^2}} J_0(klx)\,dx = \int_1^\infty \frac{J_1^2(kax)}{x\sqrt{x^2-1}} N_0(klx)\,dx \tag{14}$$

Applying the asymptotic expansion of the cylindrical functions

$$J_1(kax) = \left[\frac{2}{\pi kax}\right]^{\frac{1}{2}} \cos(kax - 3/4\,\pi) \tag{15}$$

$$N_0(klx) = \left[\frac{2}{\pi klx}\right]^{\frac{1}{2}} \sin(klx - \pi/4) \tag{16}$$

and integrating using the method of stationary phase, we obtain

$$\text{Re}(I_{11}) = \frac{\sqrt{2/\pi kl}}{\pi ka} \int_1^\infty \frac{[1 - \sin(2kax)]\sin(klx - \pi/4)}{x^{5/2}\sqrt{x^2-1}}\,dx =$$

$$= \frac{1}{2\pi ka}\left(\frac{2\sin(kl)}{kl} - \frac{\cos k(l - 2a)}{k\sqrt{l(l - 2a)}} + \frac{\cos k(l + 2a)}{k\sqrt{l(l + 2a)}} \right) \tag{17}$$

As opposed to the real parts, the imaginary parts of the integrals (6a, b, c) do not require transformations connected with the change of integration limits. This enables a direct application of approximation methods. The above explained calculations result in:

$$Z_{sn} = R_\infty \frac{6}{q^2 - 3q + 3} \left[(1-q)^2 I_{11} + \frac{4q(1-q)}{ka} I_{12} + \frac{4q^2}{(ka)^2} I_{22} \right], \tag{18}$$

$$I_{11} = \frac{1}{2\pi ka} \left[\frac{2\exp[-i(kl - \pi/2)]}{kl} - \frac{\exp[-ik(l-2a)]}{k\sqrt{l(l-2a)}} + \frac{\exp[-ik(l+2a)]}{k\sqrt{l(l+2a)}} \right], \tag{18a}$$

$$I_{12} = \frac{1}{2\pi ka} \left[\frac{\exp(-i[k(l+2a) - \pi/2])}{k\sqrt{l(l+2a)}} + \frac{\exp(-i[k(l-2a) - \pi/2])}{k\sqrt{l(l-2a)}} \right], \tag{18b}$$

$$I_{22} = \frac{1}{2\pi ka} \left[\frac{2\exp[-i(kl - \pi/2)]}{kl} + \frac{\exp[-ik(l-2a)]}{k\sqrt{l(l-2a)}} - \frac{\exp[-ik(l+2a)]}{k\sqrt{l(l+2a)}} \right], \tag{18c}$$

where $R_\infty = 1/3\,\pi\rho c a^2(q^2 - 3q + 3)$ is a normalized factor described as the acoustic resistance when $k \to \infty$ [4].

3. Figures and conclusions

After comparing the mutual impedance frequency characteristics obtained by using integral formulae and their approximate expression (Fig. 1), it is concluded that there is a good agreement of both characteristics above the value of the parameter $kl = 10$ and
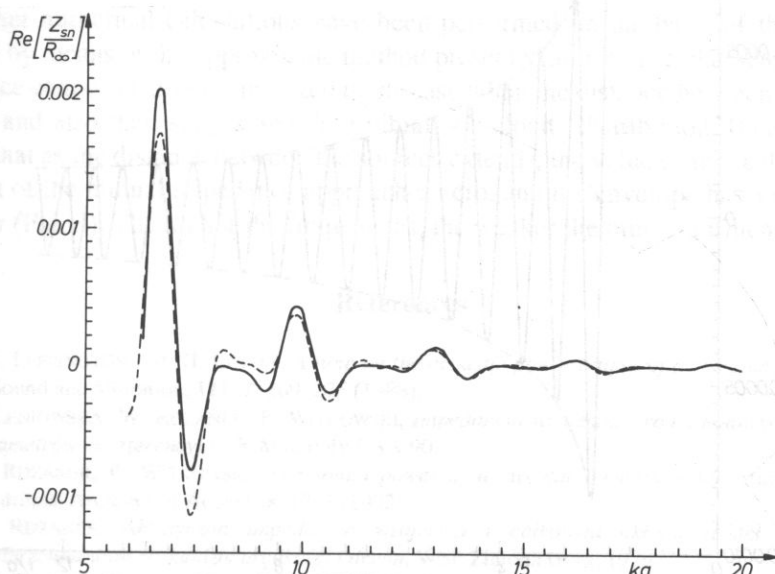

FIG. 1. Normalized mutual resistance depending on the parameter ka. The solid line denotes a curve which has been obtained by numerical integration of formulae (6a, b, c); the dashed line denotes a curve which has been calculated using approximation expressions (7a, b, c).
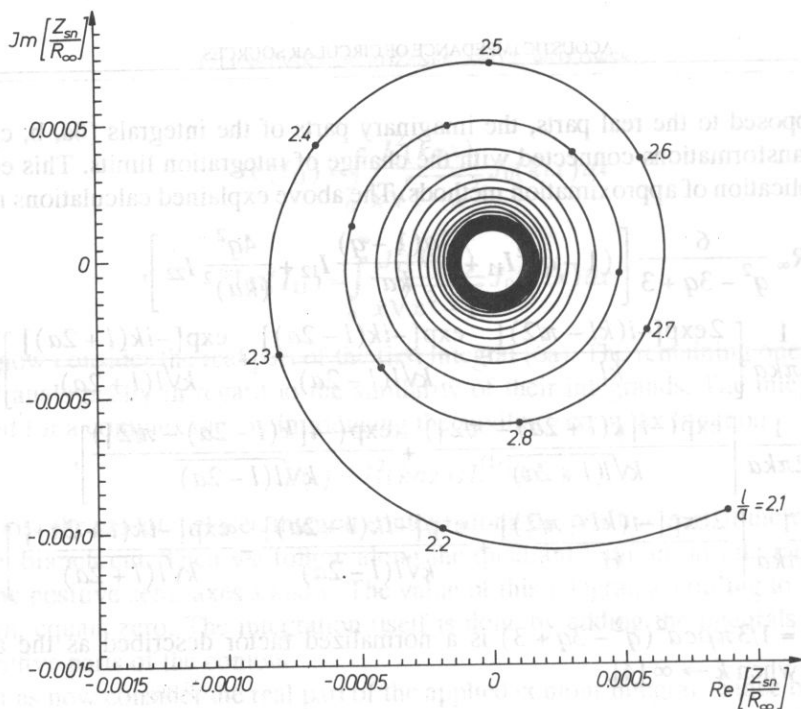
FIG. 2. The normalized mutual impedance as a function of l/a for $ka = 10$, $q = 1$.
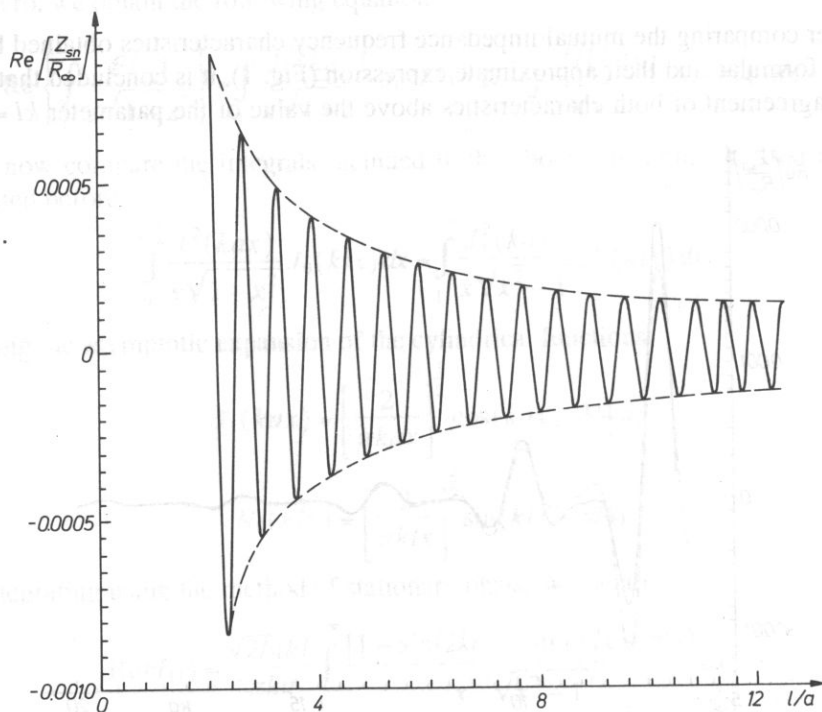

FIG. 3. The normalized mutual resistance as a function of l/a for $ka = 10$ and $q = 1$. The dashed curve denotes the envelope of extreme values of resistance.
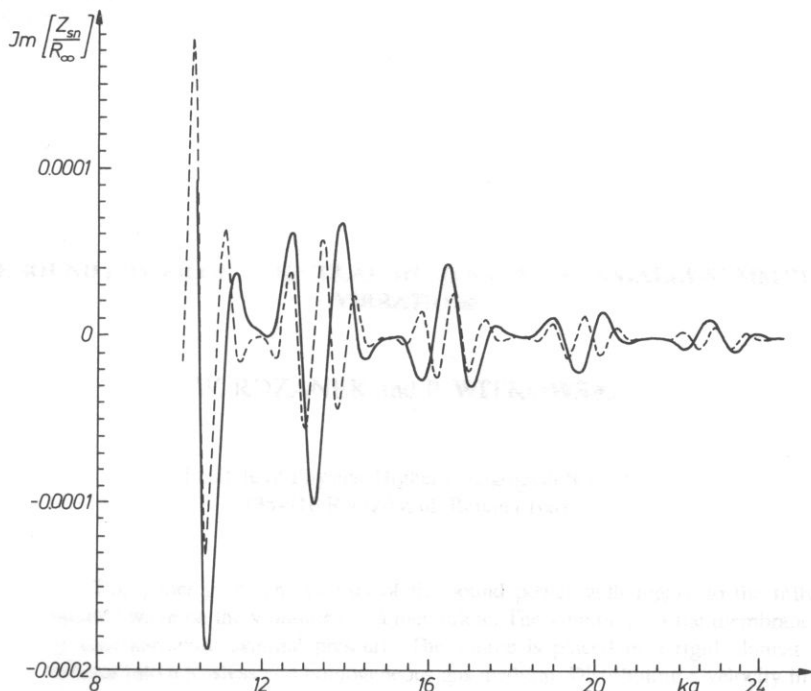
FIG. 4. Normalized mutual reactance as a function ka, $q = 1$. $---l/\alpha = 5$, ——— $l/\alpha = 8$.

that the relative error does not exceed 1%.

Further numerical calculations have been performed on the basis of the formulae obtained by means of the approximate method presented above. The character of mutual impedance changes has been analyzed in the case when the distance between the sources changes and also their shapes and their vibration velocity distribution. It can be easily noticed that as the distance between the sources extends, the value of the real and imaginary part of the mutual impedance approaches zero, and the envelope has a exponential character (Fig. 3). The higher the frequencies, the weaker the mutual influences.

References

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