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THE SOUND POWER OF A CIRCULAR MEMBRANE FOR AXIALLY-SYMMETRIC VIBRATIONS

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This paper gives an analysis of the sound power with regard to the influence of a radiated wave on the vibrations of a membrane. The vibrations of the membrane are forces by time-harmonic external pressure. The source is placed in a rigid plannar baffle and radiates into a lossless and homogeneous gas medium. Distributing a velocity in a series of eigenfunctions, we could transform a motion equation into an algebraic system of linear equations. As the final result of the analysis, a relative real power of self and free vibrations for high frequency was derived using an approximate method. The expressions derived here are very useful and convenient for numerical calculations.

1. Introduction

The problem of radiation of surface sources, specified as classical, still exists and is very often considered. It results from the necessity of solving newer and newer theoretical and practical problems as well as from the upgrading of computational methods.

Lately there have been published solutions concerned with the interaction of plates, membranes for axially-symmetric vibrations and also the interaction of two modes of the same source.

Recent studies [1, 4–8] are devoted to sources fixed in a coplanar rigid baffle and radiated into a lossless gas medium. The paper [1] presents an analysis of forced vibrations of a plate with regard to the damping effects caused by an internal friction and the influence of radiated waves through a plate on its vibration. The second part of the paper [1] also shows an approximate method of calculating real power by integration in a complex space using an asymptotic expansion of cylindrical functions. The same results were obtained in the paper [4] applying another approximate method.

With reference to the papers [1, 4], the present one is concerned with the calculation of the acoustic power of a circular membrane set in a planar rigid baffle. The forced vibrations are considered with regard to the influence of a radiated wave on the vibrations of the membrane. The losses inside membrane caused by internal friction were disregarded due to the slender thickness of the membrane. The expressions obtained have a simple mathematical form and can be the basis for further detailed numerical calculations.

2. Damped vibrations of membrane

A circular membrane of a radius a and surface density η, placed in a rigid planar baffle, is surrounded by a gaseous medium with a rest density ρ_0. The membrane is excited to vibrations by an external force $f(r,t) = f_0(r)\exp(-i\omega t)$ for $0 \le r \le a$. The vibrations are modified as a consequence of the interaction of the medium with acoustical pressure p on the membrane surface.

The equation of axially-symmetric vibrations of the circular membrane is as follows:

$$\left(T\nabla^2 - \eta\frac{\partial^2}{\partial t^2}\right)\xi(r,t) = f(r,t) - 2p(r,t), \tag{1}$$

where ξ is the distribution of the transverse vibrations, T, the force stretching the membrane, related to a unit length. Using known formulations for harmonic phenomena between the displacement $\xi_0(r)$ and normal velocity $\xi_0(r) = iv(r)/\omega$ and the acoustical pressure $p_0(r)$ and velocity potential $p_0(r) = \rho_0 i\omega\phi(r)$, Eq. (1) could be presented in a changed form:

$$(k_p^{-2}\nabla^2 + 1)\,v(r) + 2\varepsilon_1 k\phi(r) = -\frac{i}{\eta\omega}f_0(r), \tag{2}$$

where k_p is the wave number defined as $k_p^2 = \eta\omega^2/T$, $\varepsilon_1 = \rho_0/\eta k$, $k = 2\pi/\lambda$.
Let us present the normal velocity in the form of an infinite series of eigenfunctions

$$v(r) = \sum_n c_n\, v_n(r) \tag{3}$$

in which

$$v_n(r) = v_{on}J_0(k_n r),\ 0 \le r \le a \tag{3a}$$

and use the orthonormality property

$$\int_0^a v_n(r)\,v_m(r)\,r\,dr = \delta_{mn} \tag{4}$$

for normalized velocity $v_{on} = \sqrt{2}/aJ_1(k_n a)$ equation; the previous equations (2) turns into an algebraic system of linear equations

$$c_n\left(\frac{k_n^2}{k_p^2} - 1\right) + 2\varepsilon_1 i\sum_m c_m g_{mn} = f_n \tag{5}$$

The quantity f_n expressed as

$$f_n = \frac{1}{\eta\omega}\int_0^a f_0(r)\,v_n(r)\,r\,dr \tag{6}$$

is the coefficient of expansion of the external force into a orthogonal series, whereas g_{mn} is a normalized mutual impedance of axially-symmetric modes for free vibrations [7]:

$$g_{mn} = 2(k_m a)(k_n a)\int_0^{ka - i\infty} \frac{J_0^2(x)x\,dx}{\gamma[x^2 - (k_m a)^2][x^2 - (k_n a)^2]} \tag{7}$$

where $x = ka\sin\theta$, $\gamma = \sqrt{1-(x/ka)^2}$ for $0 \le x < ka$ or $\gamma = i\sqrt{(x/ka)^2 - 1}$ for $ka < x < \infty$. The real part of $g_{nn}\,(m = n)$ can also be interpreted as a relative real power of free vibrations.

In order to calculate the acoustical power of a circular membrane let us use the definition [4]:

$$N = \frac{1}{2}\int_\sigma p(r)\,v(r)\,d\sigma \tag{8}$$

which, in the case of axially-symetric velocity (3), leads to

$$N = \pi\rho_0 c_0 k^2 \sum_m \sum_n c_m c_n^* \int_0^{\pi/2 - i\infty} W_m(\vartheta) W_n^*(\vartheta)\sin\vartheta\,d\vartheta. \tag{9}$$

When we regard the value of the characteristic function $W_m(\vartheta)$ [4] and the relation (7), the acoustical power has the following form [6]:

$$N = \pi\rho_0 c_0 \sum_m \sum_n c_m c_n^* g_{mn}. \tag{10}$$

It is possible to reach another form of the formula describing acoustical power. Let us multiply Eq. (5) by c_n^* and the sum by n, then employ Eq. (10). The formula for the acoustical power of forced vibration takes the form of single series:

$$N = \frac{i\rho_0 c_0 \pi}{2\varepsilon_1}\sum_n \left[c_n^2\left(\frac{k_n^2}{k_p^2} - 1\right) - c_n^* f_n \right], \tag{11}$$

where ε_1 determines the influence of the wave radiated by a membrane on its vibration.

If we assume that the density of the gaseous medium is much smaller than that of the membrane, ε_1 approaches zero and then we get

$$c_n = f_n\left(\frac{k_n^2}{k_p^2} - 1\right)^{-1}. \tag{12}$$

3. The real power for high frequency wave radiation

Considering the linear phenomena sinusoidally dependent on time, the axially-symmetric vibration of a circular membrane can be described by Eq. (3a). For that distribution of velocity, the characteristic function [4] $W_n(\vartheta)$ for the $(0, n)$ modal is as follows:

$$W_n(\vartheta) = v_{on} k_n a J_1(k_n a) \frac{J_0(ka \sin\vartheta)}{k_n^2 - k^2 \sin^2\vartheta}. \tag{13}$$

Basing on the relation (13) and expression [4]

$$N_n = \rho_0 c_0 \pi k^2 \int_0^{\pi/2} W_n^2(\vartheta) \sin\vartheta d\vartheta, \tag{14}$$

the real power is expressed by the integral formula

$$N_n = \pi \rho_0 c_0 k^2 v_{on}^2 a^4 (k_n a)^2 J_1^2(k_n a) \int_0^{\pi/2} \frac{J_0^2(ka \sin\vartheta) \sin\vartheta d\vartheta}{[(k_n a)^2 - (ka \sin\vartheta)^2]^2} \tag{15}$$

Now, adopting the notations $\zeta = \sin\vartheta$, $ka = \beta$, $\delta_n = k_n a / ka$, the new version of the expression (15) becomes

$$\sigma_n = \frac{N_n}{N_0} = 2\delta_n^2 \int_0^1 \frac{J_0^2(\beta\zeta)\zeta d\zeta}{\sqrt{1-\zeta^2}(\zeta^2 - \delta_n^2)^2}. \tag{16}$$

The factor

$$N_0 = 1/2 \pi \rho_0 c_0 v_{on}^2 a^2 J_1^2(k_n a) \tag{17}$$

specifies the radiated power for the n-th axi-symmetric modal velocity profile at vanishing small wavelengths, viz. $k \to \infty$ [4]. The coefficient δ_n means a relative real power. Let us introduce the function of a complex variable

$$F(z) = J_0(\beta z) H_0^{(1)}(\beta z), \tag{18}$$

selected such that

$$\operatorname{Re} F(\zeta) = J_0^2(\beta\zeta), \tag{19}$$

and consider the complex integral

$$\int_C \frac{F(z) z dz}{\sqrt{1-z^2}(z^2 - \delta_n^2)^2} \tag{20}$$

instead of Eq. (16).

The contour C (Fig. 1) bypasses the singular point of second order at $z = \delta_n$, the branch point at $z = 0$ and branch cut, between $z = 1$ and $z = \infty$. The integral (20) is equal to zero. This is the consequence of its single-valued and regular integrans within C. The contour C consists of several parts; this can be written symbolically as

$$\fint_0^1 + \frac{1}{2}\int_{r_\varepsilon} + \int_1^\infty + \int_{R_\infty} + \int_\infty^0 = 0 \tag{21}$$

The contribution of two integrals vanishes: from the large circular R_∞ when the radius grows indefinitely and also the real part of the integral along an imaginary axis, what results from the relation $\operatorname{Re} F(i\tau) = 0$, τ is real.

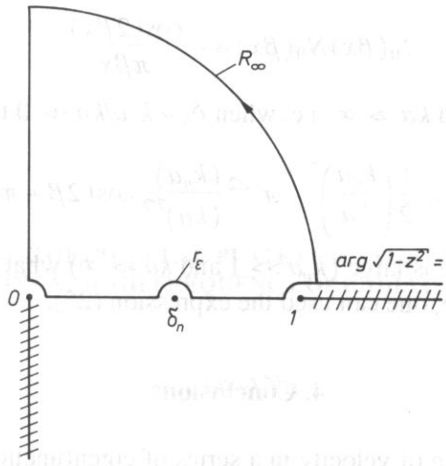

FIG. 1. The integration contour C for pattern (20).

Then there only remains

$$\int_0^1 \frac{F(x)x\,dx}{\sqrt{1-x^2}\,(x^2-\delta_n^2)^2} - \pi i F'(\delta_n) + \int_1^\infty \frac{F(x)x\,dx}{-i\sqrt{x^2-1}\,(x^2-\delta_n^2)^2} + \int_\infty^0 \frac{F(i\tau)i\tau i\,d\tau}{\sqrt{1+\tau^2}\,(\tau^2+\delta_n^2)^2} = 0, \tag{22}$$

where

$$F(z) = \frac{F(z)z}{\sqrt{1-z^2}\,(z+\delta_n)^2}. \tag{23}$$

Taking the real part of the expression (22), we get a value of the integral (16):

$$\int_0^1 \frac{J_0^2(\beta x)x\,dx}{\sqrt{1-x^2}\,(x^2-\delta_n^2)^2} = \mathrm{Re}\left[\pi i F'(\delta_n)\right] + \int_1^\infty \frac{J_0(\beta x)N_0(\beta x)x\,dx}{\sqrt{x^2-1}\,(x^2-\delta_n^2)^2}, \tag{24}$$

because

$$\mathrm{Re}\int_0^1 \frac{F(x)x\,dx}{\sqrt{1-x^2}\,(x^2-\delta_n^2)^2} = \int_0^1 \frac{J_0^2(\beta x)x\,dx}{\sqrt{1-x^2}\,(x^2-\delta_n^2)^2} \tag{24a}$$

Finally, the normalized real power radiated by the membrane is given as

$$\sigma_n = \frac{1}{(1-\delta_n^2)^{1/2}} - \frac{\delta_n^2}{(1-\delta_n^2)^2}\,\frac{1}{\pi^{1/2}\beta^{3/2}}\cos(2\beta+\pi/4). \tag{25}$$

In order to determine the last integral in Eq. (24), the method of a constant phase was used. Besides, the cylindrical functions were presented in an asymptotic form, right in the high frequency ($ka \Rightarrow \infty$) [3]:

$$J_0(\beta x)N_0(\beta x) \cong -\frac{\cos(2\beta x)}{\pi\beta x} \tag{26}$$

For the case of fixed n and $ka \Rightarrow \infty$, i.e. when $\delta_n = k_n a/ka \Rightarrow 0$ the result is

$$\sigma_n = 1 + \frac{1}{2}\left(\frac{k_n a}{ka}\right)^2 - \pi^{-1/2}\frac{(k_n a)^2}{(ka)^{7/2}}\cos(2\beta + \pi/4). \tag{27}$$

When the mode number n is large ($k_n a >> 1$ and $ka \Rightarrow \infty$) what means that δ_n is of the order of unity; in this case, one can used the expression (25).

4. Conclusions

By using a distribution of velocity in a series of eigenfunctions, the acoustic power radiated by an excited membrane with regard to the influence of a radiated wave on the vibrations of this membrane has been derived. The solutions obtained have the form of series (10), (11). Because of the high rate of convergence of series, the first solution (10) is especially useful for numerical calculations. It results from the character of changes of mutual impedance, the values of which are strong decreasing together with an increase of the mode numbers m and n. The second shape of the solution (11), expressed by a single series, is simpler but slowly convergent. It is the consequence of the small difference between the values of the $c_n^2(k_n^2/k_p^2 - 1)$ and $c_n^* f_n$ dependent on summation.

The real part of normalized self impedance obtained for high frequency consists of a polynomial and oscillatory term. This solution is generalized of the pattern received in [8].

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