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MUTUAL IMPEDANCE OF CIRCULAR PLATE FOR AXIALLY SYMMETRIC FREE VIBRATIONS AT HIGH FREQUENCY OF RADIATING WAVES*

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Mutual normalized impedance of radiation of single axially symmetric normal modes is analysed for a circular clamped plate with a plane rigid baffle. Damping in the plate is ignored and the acoustic waves are assumed to radiate in a homogeneous lossless fluid medium. Selecting an integral representation of the considered acoustic impedance and using the Cauchy theorem of residues, an expression is obtained in an elementary form valid for high frequencies.

1. Introduction

Acoustic power output of a circular plate under forced vibrations was analysed by LEVIN and LEPPINGTON [1] and the present author [5]. Losses in the material and coupled vibrations of surrounding air were both taken into account. The applied mathematical method led to the expressions for a load forcing vibrations, displacements and vibration velocities in the form of known series expansions with respect to the complete set of eigen functions. As a result, the acoustic power of the plate with internal losses and influenced by its surroundings was obtained in the form of double series with rapid convergence. Application of the series to numerical calculations depends on the knowledge of a general term in the series that comprises the normalized specific and mutual impedance of circular plates for the case of axially symmetric free vibrations. For high frequencies of acoustic waves elementary expressions were obtained for free resistance of one mode [1, 3] and mutual resistance of two different modes for the same plate [4].

Elementary formulae for the radiation reactance have up to now been lacking, both for a single mode and for two mutually interacting modes.

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On account of results arrived at in [4] and with the use of LEVIN and LEPPINGTON'S method [1], based on the Cauchy theorem on residues, an elementary formula is derived for a normalized mutual resistance, account being taken of its oscillatory character as dependent on the frequency. A formula for mutual reactance of radiation of two different modes of the same plate is obtained by means of a direct integration of the expression from the papers [1, 5]. Asymptotic formulae for the Bessel functions are used in the presence of sufficiently large values of the interference parameter. It is also shown that the expression for a reactance of a single mode follows from an expression for mutual reactance.

The frequency characteristics of the considered normalized impedance of radiation are also shown diagrammatically.

2. Assumptions

A thin circular plate is thickness h, small with respect to its diameter $2a$, made of homogeneous material of density ρ is immersed in a lossless fluid medium. The plate is clamped at the circumference. For sinusoidal time dependence a normal velocity of the axially symmetric free vibration can be expressed in the form [6]

$$v_n(r)/v_{0n} = J_0(\gamma_n r/a) - \frac{J_0(\gamma_n)}{I_0(\gamma_n)} I_0(\gamma_n r/a), \tag{1}$$

where r – radial coordinate of a point, J_m – Bessel function, I_m – modified Bessel's function (both of order m), γ_n – the n-th root of the frequency equation [2]

$$I_0(\gamma_n) J_1(\gamma_n) = -I_1(\gamma_n) J_0(\gamma_n), \tag{2}$$

that describes the frequency of free vibrations for the n–th mode,

$$f_n = \frac{1}{2\pi a^2} \gamma_n^2 \left(\frac{B}{\rho h}\right)^{1/2} \tag{3}$$

where, in turn, B denotes flexural stiffness of the plate. The constant v_{0n} can be expressed with the use of an amplitude of vibrations for the centre of the plate v'_{0n} in the following manner

$$v'_{0n}/v_{0n} = 1 - J_0(\gamma_n)/I_0(\gamma_n). \tag{4}$$

Mechanical mutual impedance between $(0, n)$ and $(0, s)$ axially symmetric vibration modes of a circular plate with a plane rigid baffle is calculated from the formula, cf. [5]

$$Z_{ns} = \frac{1}{2\sqrt{<|v_n|^2><|v_s|^2>}} \int_\delta p_{ns}\, v_s\, d\delta, \tag{5}$$

where p_{ns} is an acoustic pressure generated by the $(0, n)$ mode of the plate and exerted on the same plate through the $(0, s)$ vibration mode. Mean square of the velocity of $(0, n)$ mode is expressed by

$$<|v|^2> = \frac{1}{2\delta} \int_\delta v_n^2(r)\, d\delta, \tag{6}$$

where $\delta = \pi a^2$.

Referring the mechanical mutual impedance to the specific resistance of fluid medium $\rho_0 c$ and to the area δ of the plate, the normalized mutual impedance between $(0, n)$ and $(0, s)$ modes is obtained [1, 4, 5]:

$$\zeta_{ns} = 4\delta_n^2\delta_s^2 \int_0^{\infty} \frac{x}{\gamma} \left[\frac{a_n \delta_n J_0(\alpha x) - x J_1(\alpha x)}{x^4 - \delta_n^4} \right] \times$$

$$\times \left[\frac{a_s \delta_s J_0(\alpha x) - x J_1(\alpha x)}{x^4 - \delta_s^4} \right] dx. \qquad (7)$$

where $\gamma = (1 - x^2)^{1/2}$ for $0 \le x \le 1$, $\gamma = i(x^2 - 1)^{1/2}$ for $1 \le x < \infty$, $\alpha = k_0 a$, $\delta_n = \gamma_n/\alpha$, $a_n = J_1(\gamma_n)/J_0(\gamma_n)$

Moreover,

$$\zeta_{ns} = \theta_{ns} - i\chi_{ns} \qquad (8)$$

where θ_{ns} is a normalized mutual resistance and χ_{ns} is a normalized mutual reactance.

When $k_0 a/\gamma_n >> 1$ and $k_0 a/\gamma_s >> 1$, the normalized mutual resistance between the $(0, n)$ and $(0, s)$ modes of free vibrating circular plate can be shown in the form

$$\theta_{ns} = h_{ns}\alpha^{-2}, \qquad (9)$$

where

$$h_{ns} = 2(\gamma_n\gamma_s)^2 \frac{a_n\gamma_n - a_s\gamma_s}{\gamma_n^4 - \gamma_s^4} \qquad (9')$$

for $n \ne s$. Although the value of h_{ns} for $n = s$ does exist, the formula (9') is not valid for the case. Suitable formulae were given in [1].

3. Normalized mutual resistance

To obtain more accurate formula for the normalized mutual resistance than (9), the derivation will be based on the real part of its integral representation (7).

Following LEVIN and LEPPINGTON [1], the following function of complex variable is introduced

$$F(z) = a_n\delta_n a_s\delta_s J_0(\alpha z)H_0^{(1)}(\alpha z) - (a_n\delta_n + a_s\delta_s)zJ_0(\alpha z)H_1^{(1)}(\alpha z) +$$

$$+ z^2 J_1(\alpha z)H_1^{(1)}(\alpha z) \qquad (10)$$

such that

$$\mathrm{Re}F(x) = a_n\delta_n a_s\delta_s J_0^2(\alpha x) - (a_n\delta_n + a_s\delta_s)xJ_0(\alpha x)J_1(\alpha x) + x^2 J_1^2(\alpha x), \qquad (11)$$

where x is a real variable.

Further, a contour integral is used,

$$\int_c \frac{zF(z)dz}{\sqrt{1 - z^2}\,(z^4 - \delta_n^4)(z^4 - \delta_s^4)} = 0 \qquad (12)$$

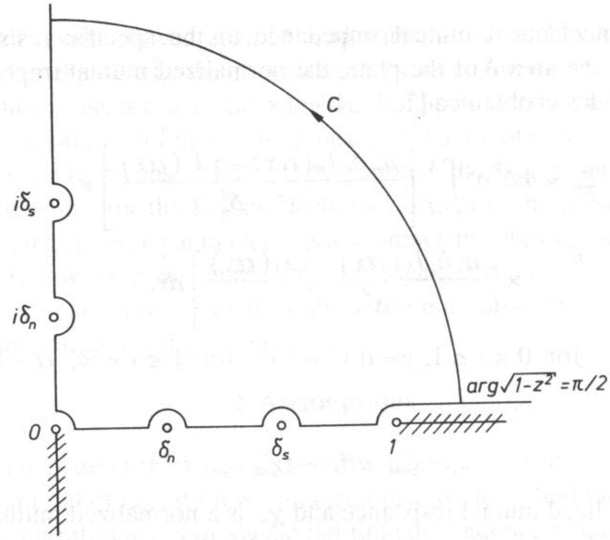

FIG. 1. Integration path for the expression (12), cf. [1].

calculated along the path C, Fig. 1. Assuming δ_n, $\delta_s < 1$, the integrand stays unique and regular inside C. As a consequence, the frequency range is limited because $\delta_n = \gamma_n/\alpha$, $\delta_s = \gamma_s/\alpha$, $\alpha = k_0 a$ and the formula (3) is valid. Employing the Cauchy theorem on residues, the integral (12) can be shown symbolically as

$$\fint_0^1 + \int_1^\infty + \int_{C_R} + \fint_\infty^0 = \pi i \sum_{j-1}^{4} \mathrm{Rez}\,(z_j), \tag{13}$$

where for $z_j = \delta_n$, $i\delta_n$, δ_s, $i\delta_s$ singular points exist with first order poles and the integrals $\fint_0^1 \fint_\infty^0$ are interpreted in the eigenvalue tense. The latter vanish on integration along a large circle when its radius tends to infinity. They also vanish on integration along small circles around the bifurcation points ($z = 0$, $z = 1$) when their radii shrink to zero. The following auxiliary functions are introduced to determine residua at singular points as first order poles:

$$F_1(z) = \frac{zF(z)}{\sqrt{1-z^2}(z+\delta_n)(z^2+\delta_n^2)(z^4-\delta_s^4)}, \quad z = \delta_n,$$

$$F_2(z) = \frac{zF(z)}{\sqrt{1-z^2}(z+i\delta_n)(z^2-\delta_n^2)(z^4-\delta_s^4)}, \quad z = i\delta_n,$$

$$F_3(z) = \frac{zF(z)}{\sqrt{1-z^2}(z+\delta_s)(z^2+\delta_s^2)(z^4-\delta_n^4)}, \quad z = \delta_s,$$

$$F_4(z) = \frac{zF(z)}{\sqrt{1-z^2}(z+i\delta_s)(z^2-\delta_s^2)(z^4-\delta_n^4)}, \quad z = i\delta_s. \tag{14}$$

Accounting for $\operatorname{Re} F(iy) = 0$ for real values of y, the expression (13) yields

$$\operatorname{Re} \fint_0^1 \frac{xF(x)\,dx}{\sqrt{1-x^2}\,(x^4-\delta_n^4)(x^4-\delta_s^4)} = \int_1^\infty \frac{x \operatorname{Im} F(x)\,dx}{\sqrt{x^2-1}\,(x^4-\delta_n^4)(x^4-\delta_s^4)} +$$
$$+ \operatorname{Re}(\pi i\,[\,F_1(\delta_n) + F_2(i\delta_n) + F_3(\delta_s) + F_4(i\delta_s)\,]). \tag{15}$$

It immediately finishes $\operatorname{Im} F(\delta_n) = \operatorname{Im} F(i\delta_n) = \frac{2}{\pi\alpha}\,\alpha_s\delta_s$, $\operatorname{Im} F(\delta_s) = \operatorname{Im} F(i\delta_s) = \frac{2}{\pi\alpha}\,\alpha_n\delta_n$.

The second integral in (15) can be calculated with the use of known asymptotic relationships:

$$J_1(\alpha x)N_1(\alpha x) \sim -J_0(\alpha x)N_0(\alpha x) \sim (\pi\alpha x)^{-1}\cos 2\alpha x,$$
$$J_0(\alpha x)N_1(\alpha x) \sim -(\pi\alpha x)^{-1}(1+\sin 2\alpha x), \tag{16}$$

for $\alpha \to \infty\; x > 1$. Since the "non-oscillating" part of the integral is equal to

$$\frac{4}{\alpha}\int_1^\infty \frac{x\,dx}{\sqrt{1-x^2}\,(x^4-\delta_n^4)(x^4-\delta_s^4)} = \frac{1}{\delta_n^4-\delta_s^4}\left[\frac{1}{\delta_n^2}\left(\frac{1}{\sqrt{1-\delta_n^2}} - \frac{1}{\sqrt{1+\delta_n^2}}\right) + \right.$$
$$\left. - \frac{1}{\delta_s^2}\left(\frac{1}{\sqrt{1-\delta_s^2}} - \frac{1}{\sqrt{1+\delta_s^2}}\right)\right], \tag{17}$$

the "oscillating" part can be dealt with by means of asymptotic method

$$2\sqrt{\alpha/\pi}\,(1-\delta_n^4)(1-\delta_s^4)\int_1^\infty [\,(x^2 - a_n\delta_n a_s\delta_s)\cos 2\alpha x +$$
$$+ (a_n\delta_n + a_s\delta_s)x\sin 2\alpha x\,]\frac{dx}{\sqrt{x^2-1}\,(x^4-\delta_n^4)(x^4-\delta_s^4)} = \tag{18}$$
$$= (1 - a_n\delta_n a_s\delta_s)\cos(2\alpha + \pi/4) + (a_n\delta_n + a_s\delta_s)\sin(2\alpha + \pi/4)$$

Accounting for

$$\operatorname{Re}\Big(\pi i\Big[\,F_1(\delta_n) + F_2(i\delta_n) + F_3(\delta_s) + F_4(i\delta_s)\,\Big]\Big) =$$
$$= \frac{1}{2\alpha}\,\frac{1}{\delta_n^4-\delta_s^4}\left[\frac{a_n\delta_n}{\delta_s^2}\left(\frac{1}{\sqrt{1-\delta_s^2}} - \frac{1}{\sqrt{1+\delta_s^2}}\right) + \right.$$
$$\left. - \frac{a_s\delta_s}{\delta_n^2}\left(\frac{1}{\sqrt{1-\delta_n^2}} - \frac{1}{\sqrt{1+\delta_n^2}}\right)\right], \tag{19}$$

the following formula for the normalized mutual resistance is finally arrived at:

$$\theta_{ns} = \alpha^{-1}\frac{a_n\delta_n - a_s\delta_s}{\delta_n^4 - \delta_s^4}\left[\delta_n^2\left(\frac{1}{\sqrt{1-\delta_s^2}} - \frac{1}{\sqrt{1+\delta_s^2}}\right) + \right.$$

$$+ \delta_s^2\left(\frac{1}{\sqrt{1-\delta_n^2}} - \frac{1}{\sqrt{1+\delta_n^2}}\right) + 2\pi^{-1/2}\alpha^{-3/2}\frac{\delta_n^2\delta_s^2}{(1-\delta_n^4)(1-\delta_s^4)}\left[\left(1 - \right.\right.$$

$$\left.\left. + \quad a_n\delta_n\alpha_s\delta_s\right)\cos(2\alpha+\pi/4) + (\alpha_n\delta_n + \alpha_s\delta_s)\sin(2\alpha+\pi/4)\right] \tag{20}$$

within an accuracy of $o(\delta_n^2\delta_s^2\alpha^{-3/2})$.

When δ_n^2, $\delta_s^2 << 1$, an approximate formula $(1 \pm t^2)^{-1/2} \approx 1 \mp \frac{1}{2}t^2$ for $t << 1$ is used leading to the expression (20) in the form

$$\theta_{ns} = 2(\gamma_n\gamma_s)^2\alpha^{-2}\frac{\alpha_n\gamma_n - \alpha_s\gamma_s}{\gamma_n^4 - \gamma_s^4} + 2\pi^{-1/2}(\gamma_n\gamma_s)^2\alpha^{-11/2}\cos(2\alpha+\pi/4). \tag{20'}$$

Its first term is identical to (9) which comes from the paper [4].

It should be emphasized that:

– the formulae (20) and (20') are valid for $n \neq s$ only and the limiting case $n = s$ is not possible,

– the formula for the resistance of a single mode can be found in [1] where the same mathematical procedure was used,

– for $n = s$ the terms of (20) and (20') containing trigonometric functions (characterizing oscillatory character of variations in radiation resistance) are of the same form as those relevant for free resistance equations (20) and (21) in [1].

4. Normalized mutual reactance

The starting point to calculate the mutual reactance of a circular plate radiating acoustic waves with the help of two axially symmetric modes $(0, n)$ and $(0, s)$ is an imaginary part of the integral formula (7)

$$\chi_{ns} = 4\delta_n^2\delta_s^2\int_1^{\infty}\left[\frac{\alpha_n\delta_n J_0(\alpha x) - xJ_1(\alpha x)}{x^4 - \delta_n^4}\right]\times$$

$$\times\left[\frac{\alpha_s\delta_s J_0(\alpha x) - xJ_1(\alpha x)}{x^4 - \delta_s^4}\right]\frac{x\,dx}{\sqrt{x^2-1}}. \tag{21}$$

Integration of (21) will be made for sufficiently large interference parameter $\alpha = k_0 a >> 1$ i.e. for $\alpha \to \infty$. The asymptotic formulae

$$J_0(\alpha x)J_1(\alpha x) \sim -(\pi\alpha x)^{-1}\cos(2\alpha x),$$

$$J_0^2(\alpha x) \sim (\pi\alpha x)^{-1}(1 + \sin 2\alpha x),$$

$$J_1^2(\alpha x) \sim (\pi\alpha x)^{-1}(1 - \sin 2\alpha x), \tag{22}$$

are used for $\alpha \to \infty$ $(x > 1)$ to be put into the integrand of (21). "Non-oscillating" part of the integral is calculated with the use of the formulae

$$\int_1^\infty \frac{dx}{\sqrt{x^2-1}\,(x^2-t^2)} = \frac{\arcsin t}{t\sqrt{1-t^2}},$$

$$\int_1^\infty \frac{dx}{\sqrt{x^2-1}\,(x^2+t^2)} = \frac{\mathrm{Arsh}\, t}{t\sqrt{1+t^2}}, \tag{23}$$

whereas the "oscillating" part is arrived at with the use of asymptotic method. The result is

$$\chi_{ns} = \frac{2}{\pi\alpha(\delta_n^4 - \delta_s^4)} \Bigg[\alpha_n \alpha_s \Bigg(\delta_s^3 \Bigg(\frac{\arcsin \delta_n}{\sqrt{1-\delta_n^2}} - \frac{\mathrm{Arsh}\,\delta_n}{\sqrt{1+\delta_n^2}} \Bigg)$$

$$- \delta_n^3 \Bigg(\frac{\arcsin \delta_s}{\sqrt{1-\delta_s^2}} - \frac{\mathrm{Arsh}\,\delta_s}{\sqrt{1+\delta_s^2}} \Bigg) \Bigg) + \delta_n \delta_s^2 \Bigg(\frac{\arcsin \delta_n}{\sqrt{1-\delta_n^2}} +$$

$$+ \frac{\mathrm{Arsh}\,\delta_n}{\sqrt{1+\delta_n^2}} \Bigg) - \delta_s \delta_n^2 \Bigg(\frac{\arcsin \delta_s}{\sqrt{1-\delta_s^2}} + \frac{\mathrm{Arsh}\,\delta_s}{\sqrt{1+\delta_s^2}} \Bigg) \Bigg] +$$

$$+ \frac{2\delta_n^2 \delta_s^2}{\pi^{1/2} \alpha^{3/2} (1-\delta_n^4)(1-\delta_s^4)} \Big[\Big(\alpha_n \delta_n \alpha_s \delta_s - 1 \Big) \sin(2\alpha + \pi/4) \ +$$

$$+ (\alpha_n \delta_n + \alpha_s \delta_s) \cos(2\alpha + \pi/4) \Big] \tag{24}$$

with the accuracy of $o(\delta_n^2 \delta_s^2 \alpha^{-3/2})$.

For δ_n^2, $\delta_0^2 << 1$ the expression (24) takes the form

$$\chi_{ns} = \frac{32}{15\pi} (\gamma_n \gamma_s)^2 \alpha^{-5} - 2\pi^{-1/2} (\gamma_n \gamma_s)^2 \alpha^{-11/2} \sin(2\alpha + \pi/4). \tag{24'}$$

The expression (24) was simplified to become (24') after taking account of the first three terms of expansions with respect to $x = \delta_n \delta_s$ for the root function and for arc sin x, Arsh x.

In the limiting case $n = s$ the formula (24) yields a normalized reactance of circular plate radiating with the help of the axially symmetric vibration mode $(0, n)$

$$\chi_{nn} = (\pi\alpha)^{-1} \Big[(a_n^2(-3 + 4\delta_n^2) - 1 + 2\delta_n^2) \frac{\arcsin \delta_n}{2\delta_n (1-\delta_n^2)^{3/2}} +$$

$$+ \Big(\alpha_n^2 (3 + 4\delta_n^2) - 1 - 2\delta_n^2 \Big) \frac{\mathrm{Arsh}\,\delta_n}{2\delta_n (1+\delta_n^2)^{3/2}} + \frac{1 + a_n^2 \delta_n^2}{1 - \delta_n^4} \Big] +$$

$$+ 2\delta_n^4 \pi^{-1/2} \alpha^{-3/2} (1 - \delta_n^4)^{-2} \big[(\alpha_n^2 \delta_n^2 - 1) \sin(2\alpha + \pi/4) +$$

$$+ 2\alpha_n \delta_n \cos(2\alpha + \pi/4) \big] \tag{25}$$

with the accuracy of $o(\delta_n^4 \alpha^{-3/2})$.

5. Final remarks

The theoretical analysis of the radiations of a thin circular plate fournished elementary formulae for normalized impedance of axially symmetric modes of free vibrations. They can be applied only for sufficiently short acoustic wave length as compared with the diameter of the plate.

Suitable terms, characterizing the "oscillatory" character of variations in both the real part (20), (20') and in the imaginary part (24) of the normalized mutual impedance of circular plate, were determined, Fig. 2.

When the interference parameter $k_0 a$ tends to infinity, the expression (20') takes the form (9), given in [4].

The normalized mutual reactance (24) of two vibration modes $(0, n)$ and $(0, s)$ in the limit $n = s$ assumes the form (25) which corresponds to the normalized specific case is possible in the case of the real part of the mutual impedance.

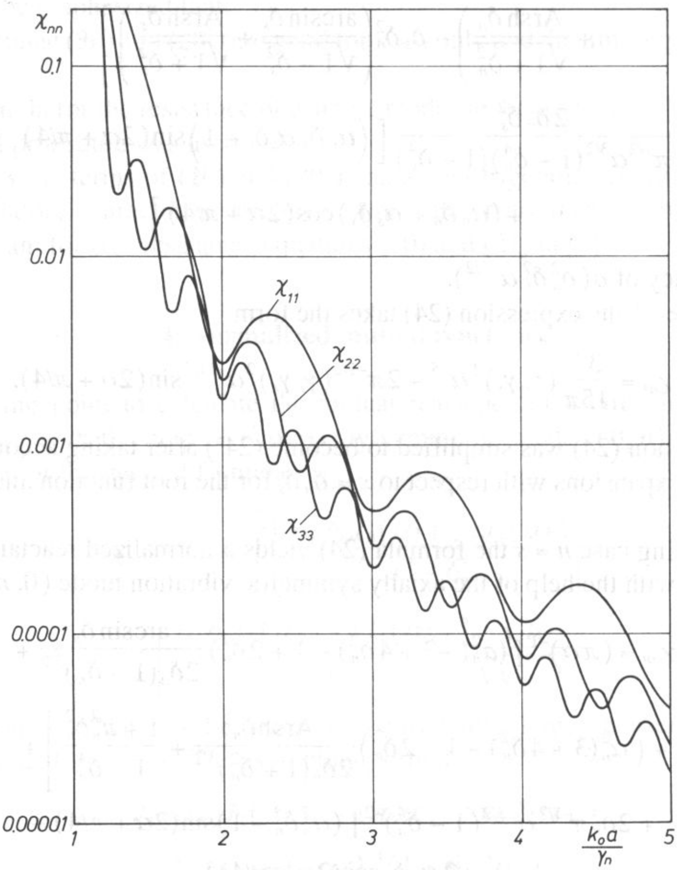

FIG. 2. Normalized reactance of a circular plate (25) vs. parameter $k_0 a/\gamma_n$ for the first three axially symmetric vibration modes.

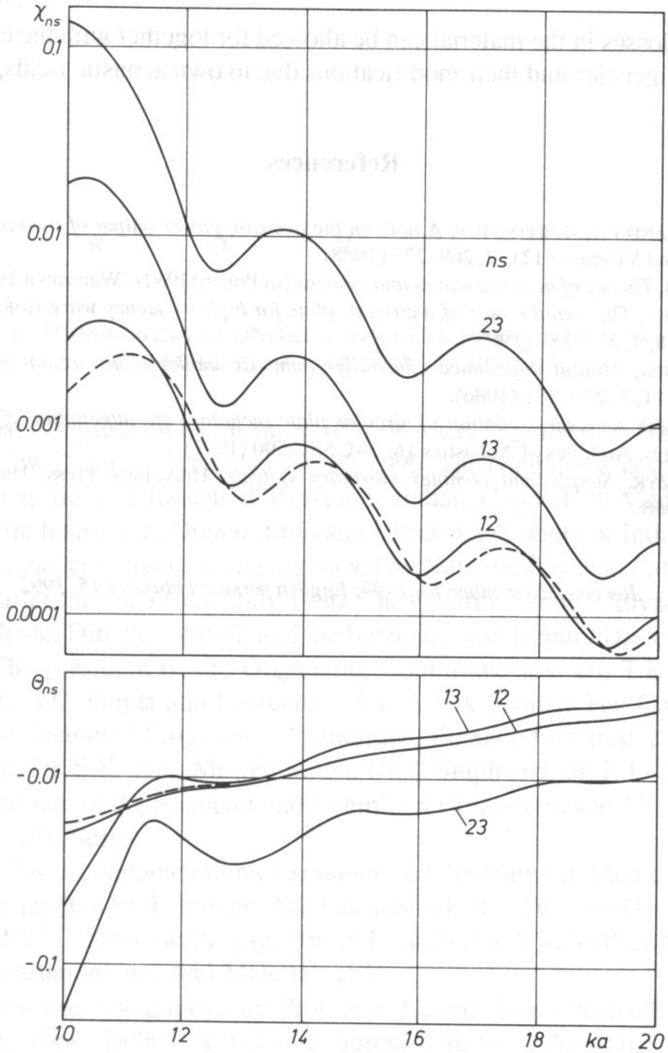

FIG. 3. Normalized mutual impedance of a circular plate (20), (24), vs. parameter k_0a for axially symmetric vibration modes. Curves obtained from the formula (7) are dashed.

Exceptionally simple form of the approximate formula (25) for the normalized reactance can be onset for computations under less rigorous constraints than $k_0a >> \gamma_n$. For instance, at $k_0a > 3\gamma_n$, the normalized reactance (25) of the circular plate is, for a number of initial vibration modes, determined to within an accuracy of several per cent, Fig. 3.

When $k_0a << \gamma_1\gamma_s$ or when great accuracy of results is required, computer-sided numerical integration of the formula (7) can be used.

The obtained simple expressions for the normalized mutual impedance of axially symmetric vibration modes for a circular plate can be employed to analyse more complex situations, e.g. losses in the material can be allowed for together with the effects of vibration enforcing agencies and their modifications due to own acoustic fields, cf. [1, 5].

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