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NEW METHOD FOR THE CALCULATION OF THE TEMPERATURE BEHAVIOUR OF THE PIEZOELECTRIC RESONATOR PARAMETERS

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The derivatives $G_{ijkl}^{(1)}$ and $G_{ijkl}^{(2)}$ of the elastic stiffnesses which described the elastic properties of thermally deformated and with small-amplitude vibrated quartz plates were defined be LEE and YONG in 1985. The values of the derivatives were computed and are given in the paper. The relations for computing the temperature coefficients of frequency suitable for the more precise expression of the temperature behaviour of quartz plates vibrated in thickness modes are published.

1. Introduction

The described, new more precise, method for the calculation of resonant frequency temperature dependence of quartz resonators goes out from the LEE's and YONG's paper [1] and started from the preposition that the linear field equations for small vibrations have to be superposed on the thermally-induced deformations. It is necessary to consider the nonlinear field equations of thermoelasticity when the thermally-induced deformation is calculated.

2. Equations of motion and traction boundary condition

It follows from the solution of the mentioned system of equations derived by LEE and YONG for the thickness vibration of quartz plates that the incremental displacement equations of motions and traction boundary conditions can be considered in the form

$$G_{ijkl}u_{k,ji} = \varrho_0 \ddot{u}_i,$$
$$p_i = n_j G_{ijkl} u_{k,l} \text{ on } S \tag{1}$$

where ϱ_0 is the mass density in natural state, u_i are the displacement components, p_i are the incremental strain components, n_j are components of the unit outward normal to the faces of the plate in natural state and the effective elastic stiffnesses G_{ijkl} and their derivatives $G^{(n)}_{ijkl}$ are given by the relations

$$G_{ijkl} = C_{ijkl} + G^{(1)}_{ijkl}\theta + G^{(2)}_{ijkl}\theta^2,$$

$$G^{(1)}_{ijkl} = C_{sjkl}\alpha^{(1)}_{is} + C_{ijsl}\alpha^{(1)}_{ks} + C_{ijklmn}\alpha^{(1)}_{mn} + C^{(1)}_{ijkl},$$

$$G^{(2)}_{ijkl} = C_{sjtl}\alpha^{(1)}_{is}\alpha^{(1)}_{kt} + C_{sjkl}\alpha^{(2)}_{is} + C_{ijsl}\alpha^{(2)}_{ks} + C^{(1)}_{sjkl}\alpha^{(1)}_{is} + $$

$$+ C^{(1)}_{ijsl}\alpha^{(1)}_{ks} + C_{sjklmn}\alpha^{(1)}_{is}\alpha^{(1)}_{mn} + C_{ijslmn}\alpha^{(1)}_{ks}\alpha^{(1)}_{mn} +$$

$$+ C_{ijklmn}\alpha^{(2)}_{mn} + \frac{1}{2}\tilde{C}^{(2)}_{ijkl}. \tag{2}$$

In the relations (2) $\alpha^{(n)}_{ij}$ are thermal expansion coefficients, C_{ijkl}, C_{ijklmn} are the second- and third-order elastic stiffnesses of the crystal, $C^{(1)}_{ijkl}$, $\tilde{C}^{(2)}_{ijkl}$ are the temperature derivatives of elastic stiffnesses given by LEE and YONG [1] and θ is the temperature change ($\theta = T - T_0$). As

$$G^{(n)}_{ijkl} = G^{(n)}_{klij},$$

Table 1. Calculated values of temperature derivatives $G^{(1)}_{ijkl}$ and $G^{(2)}_{ijkl}$ of effective elastic stiffnesses G_{ijkl}. $G^{(1)}_{ijkl}$ in 10^6 Nm^2 K^{-1}

	..11	..12	..13	..21	..22	..23	..31	..32	..33
11..	−7.879	0.000	0.000	0.000	−10.951	−2.309	0.000	−2.335	−6.598
12..		2.256	−2.309	2.256	0.000	0.000	−2.335	0.000	0.000
13..			−4.462	−2.309	0.000	0.000	−4.489	0.000	0.000
21..				2.256	0.000	0.000	−2.335	0.000	0.000
22..					−7.879	2.324	0.000	2.350	−4.279
23..						−5.285	0.000	−5.312	0.027
31..							4.515	0.000	0.000
32..								−5.338	0.027
33..									−10.714

$G^{(2)}_{ijkl}$ in 10^3N m^{-2} K^{-2}

	..11	..12	..13	..21	..22	..23	..31	..32	..33
11..	−0.379	0.000	0.000	0.000	−0.738	−0.010	0.000	−0.011	−0.310
12..		0.199	−0.010	0.199	0.000	0.000	−0.011	0.000	0.000
13..			−0.069	−0.010	0.000	0.000	−0.057	0.000	0.000
21..				0.199	0.000	0.000	−0.011	0.000	0.000
22..					−0.379	0.011	0.000	0.011	−0.261
23..						−0.092	0.000	−0.074	0.001
31..							−0.044	0.000	0.000
32..								−0.056	0.000
33..									−0.278

but

$$G_{ijkl}^{(n)} \neq G_{jikl}^{(n)} \neq G_{ijkl}^{(n)} \tag{3}$$

generally 45 components of $G_{ijkl}^{(1)}$ and $\tilde{G}_{ijkl}^{(2)}$ must be considered when the resonant frequency temperature dependence is calculated. The computed components $G_{ijkl}^{(1)}$ and $G_{ijkl}^{(2)}$ are given in Table 1. They were calculated from the first and second temperature derivatives of the elastic stiffnesses for alpha quartz at 25°C which were published by LEE and YONG [1].

3. Resonant frequency temperature dependence of thickness vibrations of quartz plates

Let $2b$ be the thickness and n_i the components of the unit outward normal to the face of quartz plate in normal state. For traction-free face conditions the traction p_i on the surface of the plate is zero.

The solution for harmonic, antisymmetric thickness vibrations

$$u_k = A_k \sin \xi n_p X_p e^{j\omega t} \tag{4}$$

satisfies Eqs. (1) for $p_i = 0$ at $X_j = \pm b$ (X_j are crystallographical axes of quartz) provided that

$$(Q_{ik} - \lambda \delta_{ik}) A_k = 0$$

$$\xi = \frac{n\pi}{2b}, \qquad n = 1, 3, 5 \tag{5}$$

where

$$Q_{ik} = Q_{ki} = G_{ijkl} n_j n_l$$

$$\lambda = \varrho_0 \frac{\omega^2}{\xi^2} = \varrho_0 \left(\frac{2b\omega}{n\pi} \right)^2 . \tag{6}$$

The amplitude A_k and the eigenvalue λ are function of the change θ of temperature ($\theta = T - T_0$)

$$A_k = A_k^{(0)} + A_k^{(1)} \theta + A_k^{(2)} \theta^2$$

$$\lambda = \lambda^{(0)} + \lambda^{(1)} \theta + \lambda^{(2)} \theta^2 . \tag{7}$$

By inserting Eqs. (7) into Eqs (5) and comparing coefficients with the same powers of θ, can be obtained the system of Eqs.

$$[Q_{ik}^{(0)} - \lambda^{(0)} \delta_{ik}] A_k^{(0)} = 0, \tag{8}$$

$$[Q_{ik}^{(0)} - \lambda^{(0)} \delta_{ik}] A_k^{(1)} + [Q_{ik}^{(1)} - \lambda^{(1)} \delta_{ik}] A_k^{(0)} = 0, \tag{9}$$

$$[Q_{ik}^{(0)} - \lambda^{(0)} \delta_{ik}] A_k^{(2)} + [Q_{ik}^{(1)} - \lambda^{(1)} \delta_{ik}] A_k^{(1)} + [Q_{ik}^{(2)} - \lambda^{(2)} \delta_{ik}] A_k^{(0)} = 0, \tag{10}$$

where

$$Q_{ik}^{(m)} = G_{ijkl}^{(m)} n_j n_l \text{ for } m = 0, 1, 2. \tag{11}$$

The relations for solutions of $A_k^{(n)}$ and $\lambda^{(n)}$ were derived by LEE and YONG in [1] and here there will be given only the results of the derivation.

The zero order eigenvalue $\lambda^{(0)}$ can be derived from the relation

$$[Q_{ik}^{(0)} - \lambda^{(0)} \delta_{ik}] = 0. \tag{12}$$

The zero order amplitude $A_k^{(0)}$ can be normalized by the relation

$$A_k^{(0)} A_k^{(0)} = 1 \tag{13}$$

and calculated from the any two of Eqs. (8).

The first order eigenvalue $\lambda^{(1)}$ can be calculated from the relation

$$\lambda^{(1)} = A_i^{(1)} Q_{ik}^{(1)} A_k^{(0)}. \tag{14}$$

The first order amplitude $A_k^{(1)}$ is orthogonal to $A_k^{(0)}$

$$A_k^{(1)} A_k^{(0)} = 0 \tag{15}$$

and $A_k^{(1)}$ can be calculated from Eq. (9).

Finaly the second order eigenvalue $\lambda^{(2)}$ can be calculated from the relation

$$\lambda^{(2)} = A_i^{(0)} Q_{ik}^{(2)} A_k^{(0)} + A_i^{(0)} Q_{ik}^{(1)} A_k^{(1)}. \tag{16}$$

From second Eq. (6) follows

$$f = \frac{n}{2b} v \left[\frac{1}{\varrho_0} \lambda^{(0)} + \lambda^{(1)} \theta + \lambda^{(2)} \theta^2 \right] \tag{17}$$

and

$$f_0 = \frac{n}{2b} v \left(\frac{1}{\varrho_0} \lambda^{(0)} \right). \tag{18}$$

After substituting the first and second derivatives of the Eq. (17) with respect to temperature θ into the Bechmann's definition of the temperature coefficient of frequency

$$Tf^{(n)} = \frac{1}{n! f_0} \frac{\partial^{(n)} f}{\partial T^n} \bigg|_{T_0} \tag{19}$$

the relations for the first end second order of frequency temperature coefficients can be obtained

$$Tf^{(1)} = \frac{\lambda^{(1)}}{2\lambda^{(0)}},$$

$$Tf^{(2)} = \frac{1}{2\lambda^{(0)}} [\lambda^{(2)} - \lambda^{(0)} (Tf^{(1)})^2]. \tag{20}$$

4. Conclusion

The values of the derivatives $G_{ijkl}^{(1)}$ and $G_{ijkl}^{(2)}$ of the elastic stiffnesses G_{ijkl} convenient for the expression of the elastic properties of thermally deformated and with small-amplitude vibrated quartz plates are given in the paper. The derivatives were calculated from the temperature derivatives of elastic stiffnesses $C_{pq}^{(1)}$ and $\tilde{C}_{pq}^{(2)}$ for alpha quartz published by LEE and YONG [1]. The procedure for the more precise computation of the temperature coefficients of frequency of the thickness modes of vibrations of quartz plates is described. The piezoelectric properties of the plates were neglected in the published relations for the calculation of the temperature coefficients of frequency.

References

[1] P. S. Y. LEE, Y. K. YONG, *Temperature derivative of elastic stiffness derived from the frequency–temperature behaviour of quartz plates*, J. Appl. Phys., **56**, 5, 1514–1521 (1984).