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ACOUSTIC MODELLING OF SURFACE SOURCES
PART III. PISTON MODEL, ERROR OF THE MODEL, AXISYMMETRICAL PROBLEM

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Two piston models of a plane axisymmetric source were compared. In the first of them, the constant vibration velocities of the elements were calculated as an integral mean value of the assumed vibration velocity. However, in the other, as arithmetic mean value of discrete values of vibration velocity. These models were compared for arbitrary discretization and for optimal one. For both models numerical experiments were carried out, first of all, as a function of the number of the discrete values of vibration velocity and the next as a function of nondimensional wave number. Numerical examples pointed out the parameters of such a source for which a "good" piston model may be built based on several discrete values of vibration velocity.

1. Introduction

During the several past years the boundary element method has been applied to various types of acoustic problems [4, 5, 6]. A considerable effort was made to develop the computational techniques in the analysis of radiation and diffraction problems. These problems, solved by the boundary element method, are closely linked to the modelling of the source. The problem of modelling itself (discretization and the methods of calculation of acoustic variables the elements) has no theoretical basis.

It seems that the first papers on modelling the acoustic surface sources problems are [1, 2, 3]; this paper is their continuation.

The first step in modelling a plane source consists is the discretization of its surface and of the vibration velocity. In the simplest model, the surface is replaced by planar elements on which constant vibration velocity is assumed. Then the plane elements of the model vibrate like pistons, and such a model of the source was called the piston model [1].

The piston model of the source with surface of an arbitrary shape and constant vibration velocity was considered in several papers, e.g. [4, 5, 6]. In such a case, only the surface of the source is discretized, and portions of the surface are replaced by planar elements. However, the vibration velocity of the planar elements remains constant.

The piston model of a source with a plane and variable vibration velocity on the surface was considered in papers [1, 2]. In the case of such a source, only the vibration velocity was discretized and it was replaced by constant values on the elements. In papers [1, 2] this constant value was as an integral mean value of the vibration velocity v^m.

The mean value is a rigorous one of an arithmetical mean value v^a of the values of function given in an infinite number of points (I-points) see Refs. [1, 2, 3]. From a practical point of view it means that in order to find the constant vibration velocity in one element of the model, it is necessary to measure the vibration velocity in an infinite number of i-points of the source (what is impossible) and next to calculate the arithmetical mean value.

The first problem of this paper is to define the minimal number of i-points necessary to calculate the v^a which differs from v^m by an assumed values. It also means that the model which has v^a on the elements differs by an assumed value from the model which has v^m on the elements. This difference was defined by analysing the divergence of the directivity functions of both models. Assuming v^m on the elements the model depends only on the discretization and not on the number of i-points. Then the divergence between the model and the source may be interpreted as the discretizing error [3], or the error of optimal piston model ($M_o^m E$) whose elements vibrate as v^m.

To find the influence of i-points on the divergence of the directivity function, an optimal piston model (M_o^m) [3] was considered. In this paper, the constant vibration velocity of the elements were calculated not as v^m but as the arithmetic mean value v^a. Thus, the error of arithmetic mean value ($v^a E$) will be added to the discretizing error. The sum of these errors was called the error of the piston model ($M^a E$); $M_o^m E + v^a E = M^a E$. The errors mentioned above were analyzed for different numbers of i-points as a function of shape vibration velocity, place of vibration surface in the baffle and dimensionless wave number.

The second problem of this paper is to find the optimal piston model in which the constant vibration velocity given on the elements depends on the number of i-points. This problem, for the constant number of elements and i-points on the element means finding the boundaries between the elements i.e. leads to optimal discretization. The divergence between the directivity function of such a model and that of the exact model depends on both the discretization and arithmetic mean vibration velocity. Thus, it is a measure of the error of optimal piston model ($M_o^a E$). Numerical calculations, assuming constant vibration velocity, concerning the second problem were made as a function of nondimensional wave number.

List of main symbols

M	piston model
$M_o(M_r)$	optimal (regular) piston model
$v^m, (v^a)$	integral (arithmetic) mean value vibration velocity

$M^m = M(v^m)$, $(M^a = M(v^a))$ piston model of which elements vibrate at $v^m(v^a)$

$v^a E$ error of arithmetic mean value vibration velocity

$M^m E$, $(M_o^m E)$ error of piston model (optimal piston model) of which elements vibrate at v^m

$M^a E$, $(M_o^a E)$ error of piston model (optimal piston model) of which elements vibrate at v^a.

2. Directivity function (DF) of axisymmetric source (AS)

The acoustic field of a circular driving surface placed in an infinite rigid baffle was considered. Assuming an axisymmetric vibrating velocity function, the axisymmetric acoustical field is obtained whose acoustic potential is given by Helmholtz – Rayleigh integral [3]. Directivity function of such an acoustic field normalized with the field of point source placed at the origin of coordinates is determined by formula [3].

$$Q_{0N}(\gamma) = \int_{\rho} v_{0N}(\rho)\, J_0(k\rho \sin\gamma)\, \rho \mathrm{d}\rho, \tag{2.1}$$

where

$$v_{0N}(\rho) = C_1 \sin(\pi\rho) \exp(-C_2\rho), \tag{2.2}$$

$v_{0N}(\rho)$ — asymmetric function of vibration velocity with regard to its zeros, C_1, C_2 — constants, they are so chosen that the function $v_{0N}(\rho)$ should fulfil assumed asymmetry, $J_0(x)$ — Bessel function of first kind and zero order, k — nondimensional wave number, ρ, ϕ, γ — spherical coordinates.

3. Directivity function of the model

3.1. Arithmetic mean value of vibration velocity (v^a)

For AS problem, the discrete data of vibration velocity given in i-points are described only by coordinates ρ_i. Arithmetic mean value of these data on j-element is defined by

$$v_j^a = \frac{1}{I} \sum_{i=1}^{I} v(\rho_i). \tag{3.1}$$

However, an integral mean value is

$$v_j^m = \frac{1}{L_j} \int_{G_{\rho 1}}^{G_{\rho 2}} v_{0n}(\rho)\, \mathrm{d}\rho, \tag{3.2}$$

where $L_j = G_{\rho 2} - G_{\rho 1}$.

The piston model of AS source is composed of sectors of the straight lines vibrating at either v^m or v^a. Directivity function of the model which consists of $j = 1, 2, \ldots, J$ elements is defined by Eq. (2.1). Substituting Eq. (3.1) or into Eq. (2.1) gives

$$Q_{0N}(\gamma)=\sum_{j=1}^{J} v_j \int_{\rho_j} J_0(k\rho \sin\gamma)\,\rho\,\delta\rho \tag{3.3}$$

where

$$v_j=\begin{cases} v_j^m \\ v_j^a \end{cases} \quad Q_{0N}(\gamma)=\begin{cases} Q_{0N}^m(\gamma) \\ Q_{0N}^a(\gamma) \end{cases}. \tag{3.4}$$

4. Modelling errors

As mentioned in the introduction, the modelling errors arise at the stage of discretization (discretization error — M^mE or M_o^mE) or at the stage of calculation of arithmetical mean value vibration velocity on the elements (error of mean value vibration velocity — v^aE). Thus, it was assumed, that if integral mean value v^m on the elements is calculated then the error v^aE is equal to zero.

The deviation between exact directivity function and the model is the result of modelling errors. There are a lot of measures of this deviation. In practice two of them are applied:

- least squares distance: $d_l(...)$
- uniform distance; $d_u(...)$

where $d_l(...)$, $d_u(...)$ are functions of the number of elements (J) and of the boundaries among the elements (G_1, G_2,...). In this paper, the analysis of modelling errors is based on the investigation of $d_l(...)$. A general definition of least squares distance of function $f(x)$ from $g(x)$ is given by formula [3].

$$d_l(f(x),g(x))=d_l(...)=\left\{\int_{x_1}^{x_2}[f(x)-g(x)]^2\,dx\right\}^{1/2}, \tag{4.1}$$

where x_1, x_2 — boundaries of integration.

In Ref. [3] it was pointed out that minimization of $d_l(...)$ belongs to optimalization problems. In the paper minimization of $d_l(...)$ is taken as a condition for obtaining an optimal piston model (M_o^m). The directivity of M_o^m assures the best convergence with the exact directivity in the least squares distance sence.

4.1. Discretizating error (M^mE)

Let the elements of the model vibrate at constant velocities which are calculated as v_j^m, Eq. (3.2); to simplify, the index "j" will be dropped out. If the directivity function of the model Q^m is compared to exact directivity function Q, then

$$M^mE=\rho(Q^m,Q), \tag{4.2}$$

where $Q^m=Q^m(J, G_1,...)$.

Because v^m does not depend on the number of i-points, then only discretization leads to M^mE.

The error M^mE may be reduced by

• more refined regular discretization; the number of j-elements (of the same shape and dimension) is increased [1],

• irregular discretization; the number of j-elements is constant but their shape and dimension are changed [2, 3].

Minimization of $d_t(Q^m,Q)$ leads to an optimal piston model M_o^m for which the discretizating error is marked as M_o^mE.

4.2. Error of mean value vibration velocity (v^aE)

Let elements of the model vibrate with the constant velocity calculated as v^a, Eq. (3.1). If the directivity function Q^a of such a model is compared to directivity Q^m (in both models the same discretization is assumed) then

$$v^aE=d_t(Q^a,Q^m) \tag{4.3}$$

where $Q^a=Q^a(J, G_1,...)$.

The error v^aE may be reduced by:

• increase the number of i-points on the element (which are regularly located on it). To investigate the effect of the number of i-points on the error v^aE and, consequently, on M^aE, is the main object of this paper.

• irregular location of i-points on the element assuming the constant number of these points. Here, this possibility will not be investigated.

Note that for constant discretization, i.e., for a constant number and size of the elements, $v^aE(i=\infty)=M^aE$.

4.3. Error of the piston model (M^aE)

Let elements of the model vibrate with a constant velocity v^a If the directivity function Q^a is compared to the exact one, then

$$M^aE=\rho(Q^a,Q). \tag{4.4}$$

It can be seen that the error M^aE depends on the discretization and on the number of i-points to be taken to calculate the constant vibration velocities of the elements.

It should be noted that each of the mentioned errors may be expressed by the other two

$$M^aE=v^aE+M^mE. \tag{4.5}$$

5. Numerical calculations

The error M^aE has already been discussed throughly in paper [3]. A model was found for which $d_t(J=2,G)$ takes the minimum. This model was called the optimal model (M_o^m).

In the first part of numerical calculations, based on M_o^m, two errors are evaluated, i.e. M^aE and v^aE of the model M^a. In order to find the influence of the number of i-points on the model, assuming constant G_o^m of the model M_o^m, the errors M^aE and v^aE of the model M^a were analyzed. The values of these errors are given for several i-points as a function of:

1 — asymmetry of vibration velocity; $\rho_c \in \langle .2,.8\rangle$, $a=0$, $k=2.5$ where ρ_c — place of the extremum of vibration velocity, a, b — inner, outer radius of driving AS surface ($b=a+1$).

2 — place of the driving surface in the baffle; $=3$, $a \in \langle .1, 1\rangle$, $k=2.5$,

3 — nondimensional wave number; $\rho_c=3$, $a=0$, $k \in \langle 1, 10\rangle$.

The results are presented in Figs. 1, 2 and 3 respectively. The Figures consist of three parts part (a) shows the picture of vibration velocity, (b) cross-section of vibration velocity and c error of the model M^aE, (v^aE).

Because of Eq. (4.5) the calculations concern only M^aE error. The error v^aE may be read from Figs. 1–3 as a difference of ordinates M^aE and M^mE. In Figs. 1–3 based on Ref. [3], M^mE is plotted by a solid line. However, error M^aE, for different number of i-points is plotted by dashed lines.

Regardless of the number of i-points assumed on one element, the following conclusions may be noted:

Fig. 1: for $\rho_c \cong 0$ the least error M^aE is obtained. Next, the better model can be obtained for $i=3$ rather than for $i=\infty$ ($\rho_c=0$ gives AS source whose maximum of vibration velocity is at z-axis,

Fig. 2: the error M^aE increases as an annular driving surface with a constant width retreats from the z-axis,

Fig. 3: the error v^aE decreases with the increasing nondimensional wave number k. To verify the last conclusion for a larger range of k, in Fig. 3 the value M^aE was given for $k=20$.

Numerical calculations show, that for $i=15$ the relative error

$$v^aE=\frac{|M^aE(i=15)-M^aE(i=\infty)|}{M^aE(i=\infty)}\cdot 100\% \tag{4.6}$$

is equal to 37.5% for $k=10$ and 34.2% for $k=20$.

Table 1. Boundary G_o^a of M_o^a versus the number of i-points and the shape of vibration velocity (ρ_c)

ρ_c \ "i"	3	5	7	9	11	13	15	31	∞
.0	.61	.57	.56	.56	.56	.56	.55	.55	.55
.2	.41	.44	.72	.67	.64	.62	.61	.57	.55
.3	.37	.41	.43	.85	.84	.83	.82	.78	.75
.5	.45	.35	.29	.24	.21	.19	.17	.11	.07
.7	.65	.54	.45	.41	.39	.37	.36	.32	.29
.8	.76	.68	.59	.53	.51	.49	.48	.45	.43

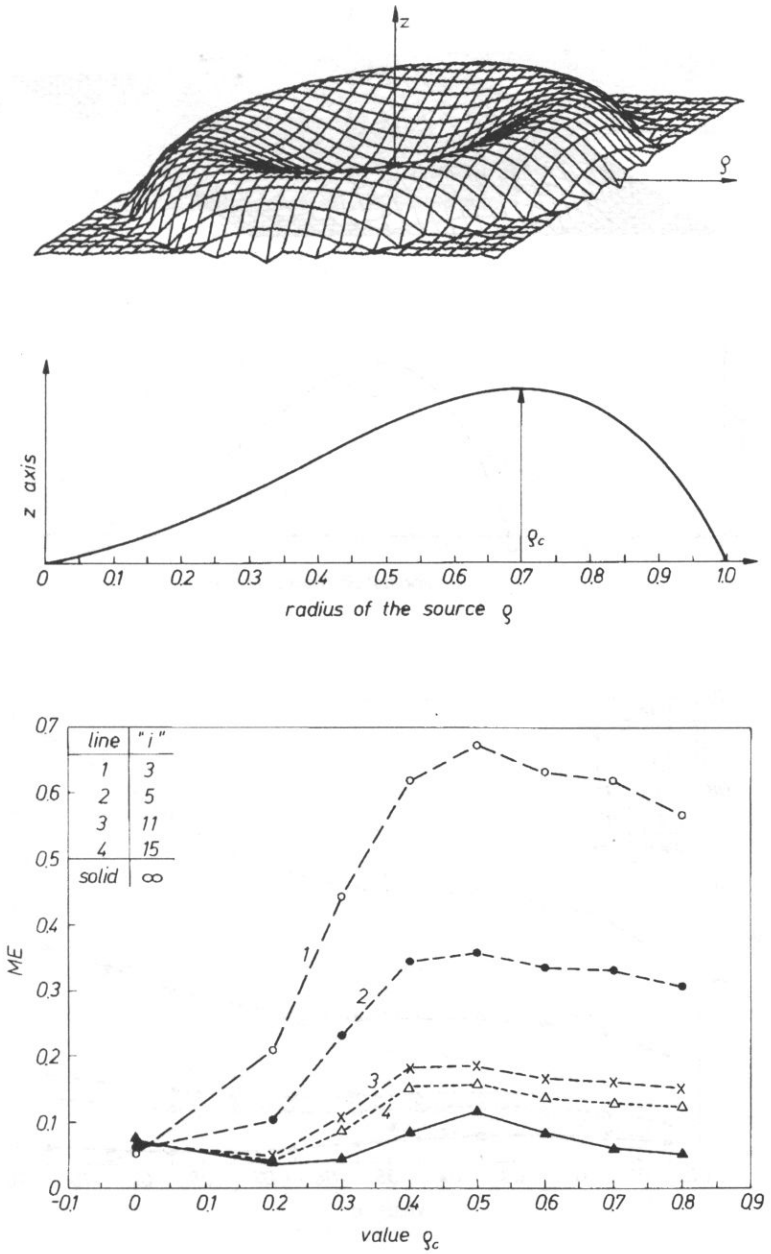

Fig. 1. (a) Picture of the driving surface, (b) Cross-section of the driving surface, (c) Model error (ME) versus the shape of vibration velocity (ρ_c).

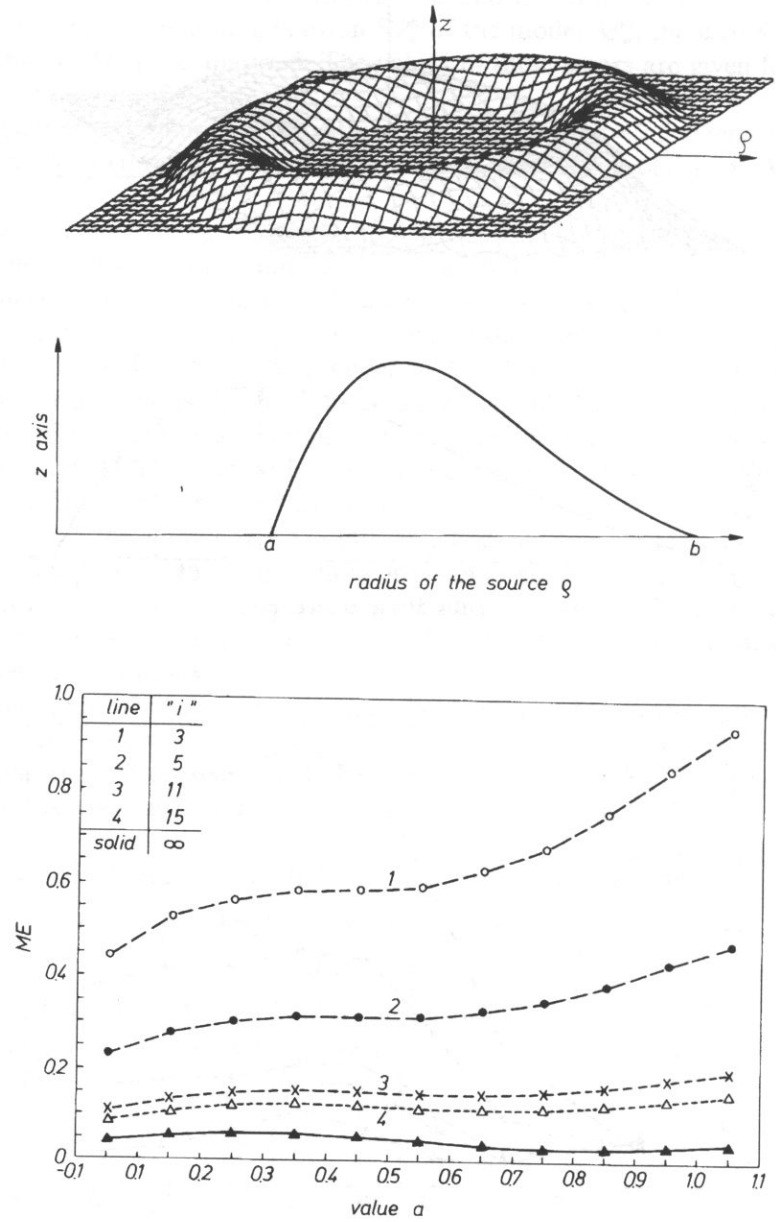

Fig. 2. (a) Picture of the driving surface, (b) Cross-section of the driving surface, (c) Model error (ME) versus a place (value „a") of the driving surface in the baffle.

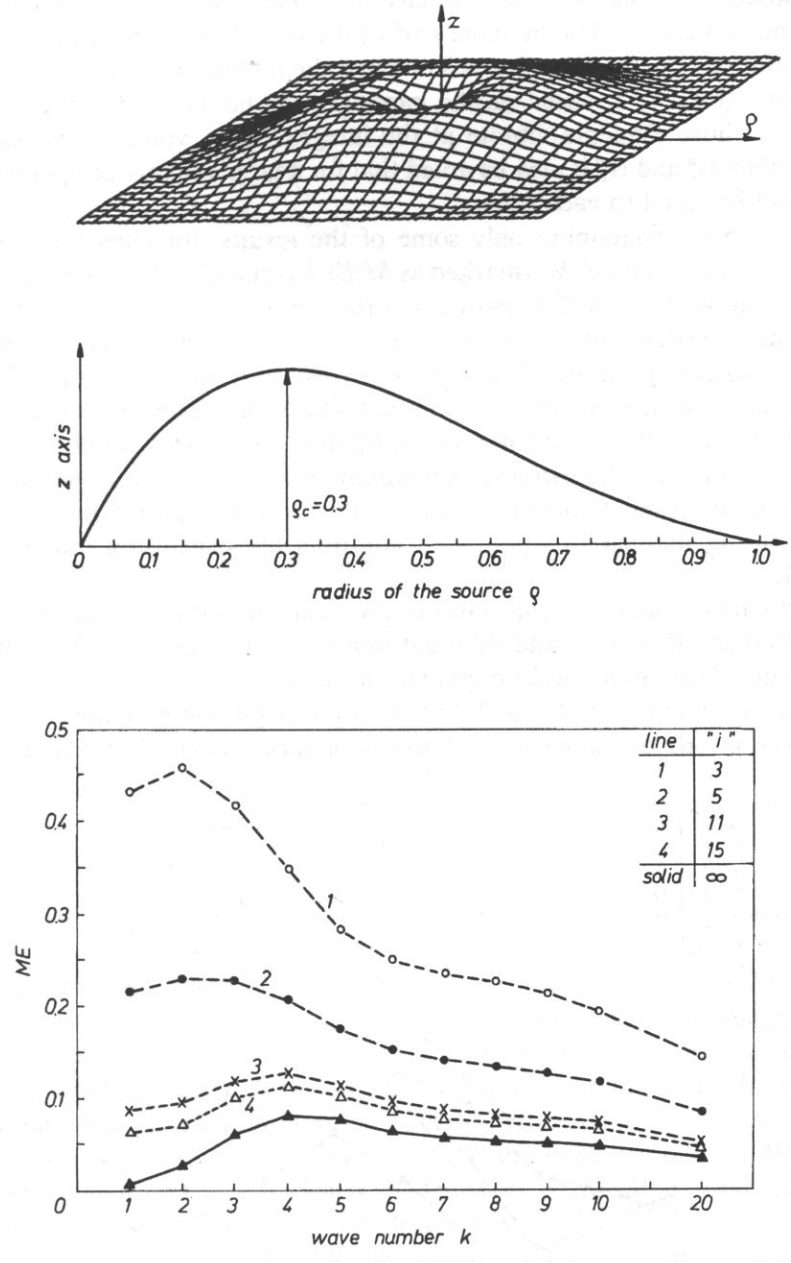

Fig. 3. (a) Picture of the driving surface, (b) Cross-section of the driving surface, (c) Model error (ME) versus the nondimensional wave number (k).

The second part of the numerical calculations is concerned with an optimal two-piston model M_o^a. The vibration velocity of its elements was calculated as an arithmetical mean value v^a. The boundary G^a of the model M_o^a is presented in Table 1 as a function of assymetry vibration velocity and the number of i-points, $k=2.5$. In the last column of Table 1, based on the paper [3], the boundary G_o^m of the model M_o^m was given whose elements vibrate at the integral mean vibration velocity v^m. Comparing values G_o^a and G_o^m it may be noted that for a large number of i-points these boundaries will be equal to each other.

Based on Table 1 containing only some of the results, for chosen number of i-points, the error of the model M_o^a (marked as M_o^aE) was calculated as a function of ρ_c. The results are shown in Fig. 4. Comparing the ordinates in Figs. 1 and 4 it is clear that, for the same number of i-points, the error M_o^a (Fig. 4) is considerably smaller than M^aE (Fig. 1). Furthermore, examining Fig. 4, it is interesting to note that for $\rho_c \cong 0$ the best results assures a smaller number of i-points identical conclusion arises from Fig. 1.

Figure 5 shows the effect of the number of i-points on the M_o^aE for chosen values ρ_c. This figure shows that the increase in the number of i-points decreases the error M_o^aE only in the case $\rho_c \neq 0$. Otherwise, for $\rho_c=0$ the error M_o^aE also slightly increases with the increasing number of i-points. It confirms the conclusion implied from Figs. 1 and 4.

In the last part of numerical calculations, the boundary G_o^a of the model M_o^a was found for different values of k and different number of i-points ($\rho_c=.3$, $a=0$ were assumed). Some of the results were presented in Table 2.

In the last two columns of Table 2, based on paper [3], the boundary G_o^m of the model M_o^m was given. A comparison of the boundaries at the last two columns

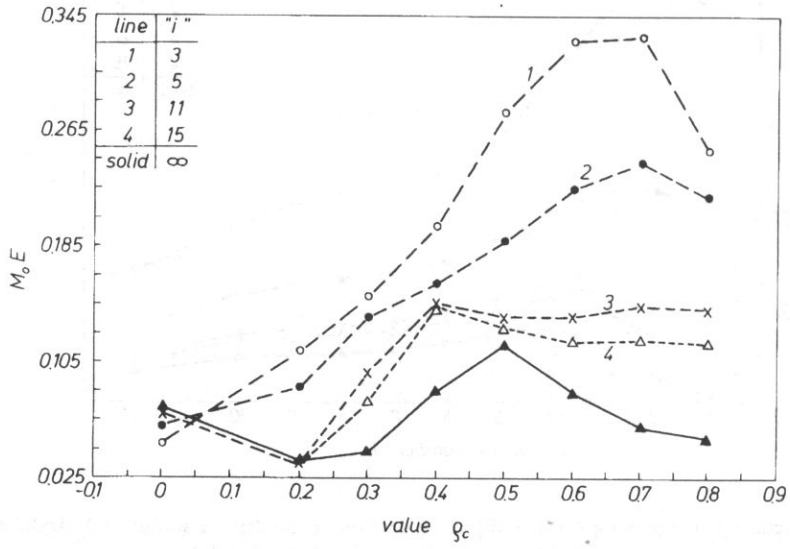

Fig. 4. Error of the optimal piston model (M_oE) versus the shape of vibration velocity (ρ_c)

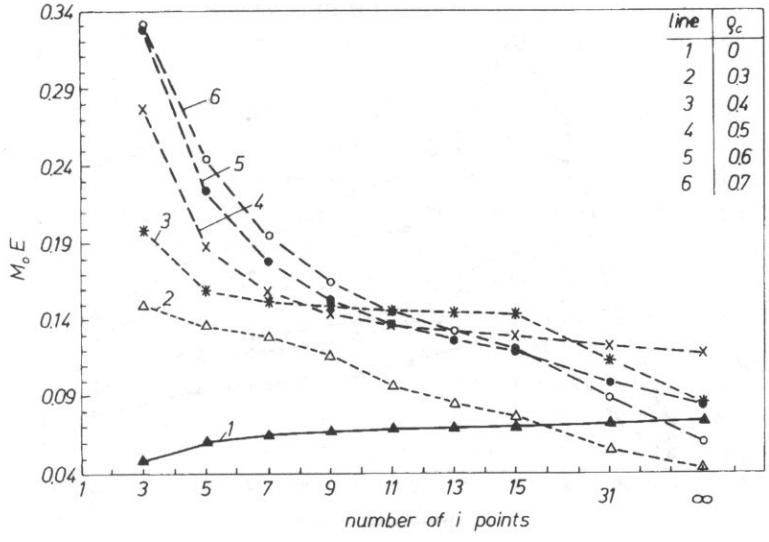

Fig. 5. Error of the optimal piston model (M_oE) versus the number of i-points

Table 2. Boundary G_o^a of M_o^a versus the number of i-points and the nondimensional wave number (k)

k \ "i"	3	5	7	9	11	13	15	31	∞
1	.44	.46	.47	.48	.86	.84	.82	.76	.71
2	.39	.41	.43	.44	.88	.86	.84	.79	.75
5	.41	.56	.63	.65	.68	.67	.67	.67	.67
9	.43	.49	.50	.52	.69	.69	.69	.69	.69
10	.43	.50	.53	.67	.68	.68	.68	.68	.68

indicates that, for lower values of k, the covergence between G_o^a and G_o^m is rather weak. However for higher values of k, several i-points on the element lead to $G_o^a = G_o^m$.

For boundaries G_o^a and chosen values of i-points, given in Table 2, the error $M_o^a(v_o^a E)$ is plotted in Fig. 6 as a function of k; note the jump of k from 10 to 20. To point out better advantages of an optimal discretization, the errors M^aE (depicted in Fig. 3) and M_o^a (Fig. 6) were compared. Examining the ordinates in the two figures, for particular k and i, it is clearly evident that $M_o^aE < M^aE$. Both M_o^aE and M^aE decrease if k increases, but only M_o^a decreases if $k<4$.

In the last numerical example, for several values of k, the error M_o^a, (v_o^a E) was investigated as a function of the number of i-points. Figure 7 shows the results; note the jump of i-points from 15 to 31 and from 31 to ∞. Examination of the data shows that the increase in the number of i-points ensures the decrease in the error M_o^aE but the decrease is little for a big number of i-points.

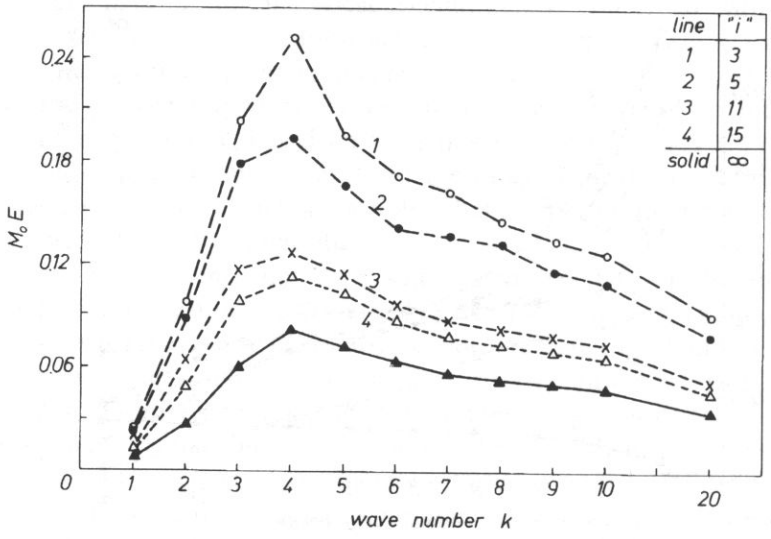

Fig. 6. Error of the optimal piston model (M_0E) versus the nondimensional wave number (k)

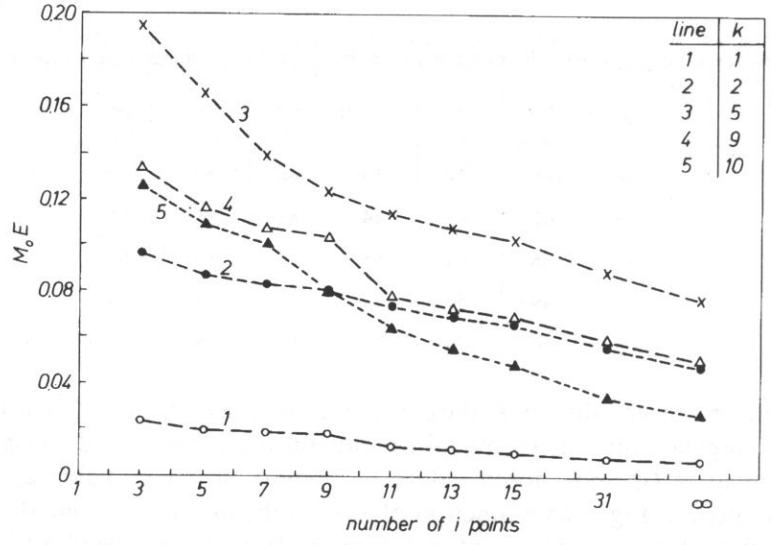

Fig. 7. Error of the optimal piston model (M_0E) versus the number of i-points

Conclusions

From the example performed in the first part of the numerical calculations it follows that, for arbitrary discretization, the error of vibration velocity v^aE decreases if the number of i-points increases (Figs. 1, 2, 3). This conclusion is not right if the

maximum of vibration velocity is in the axis of the source ($\rho_c = 0$). In this case, the smaller number of i-points on the element rather than greater one may be used to build a better model; Fig. 4.

In the second part of the numerical calculation, first of all the boundaries G_o^a of the model M_o^a were looked for. Considerations show that G_o^a depends on the number of i-points on the element, nondimensional wave number k and on the shape of the vibration velocity function. It was proved, quite similarly as in the first part, that $v_o^a E$ decreases if the number of i-points increases except the case when $\rho_c = 0$; Fig. 5.

Furthermore, in the third part of the numerical calculation, it was pointed out (Fig. 6) that, for low and high values of nondimensional wave number, several i-points on the element are necessary to build a good piston model.

The numerical examples show that, for axisymmetric vibration velocity of the plane source and for high wave number k, the good piston model may be build based on several (but not arbitrary) known values of vibration velocity in i-points. The last conclusion is confirmed by the fact that in all the investigated cases an optimal piston model assured better results than the regular one.

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