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ON STATISTICAL PARAMETERS CHARACTERIZING VIBRATIONS OF DAMPED OSCILLATOR FORCED BY STOCHASTIC IMPULSES

M. JABŁOŃSKI

Jagiellonian University
Faculty of Methematics and Computer Science
Institute of Computer Science
Gołębia 24, 31-007 Kraków, Poland
e-mail: jablonski@softlab.ii.uj.edu.pl

A. OZGA

AGH University of Science and Technology
Faculty of Mechanical Engineering and Robotics
Department of Mechanics and Vibroacoustics
Al. Mickiewicza 30, 30-059 Kraków, Poland
e-mail: aozga@agh.edu.pl

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Our theoretical study aims at finding some statistic parameters characterizing the vibrations of an oscillator with damping and forced by stochastic impulses. We will derive the dependence of these parameters on rigidity and mass of the oscillator and on the stochastic distribution of the impulse magnitude. We will also carry out a numerical simulation verifying the derived mathematical model and interpret the differences between the results obtained in simulation and the mathematical calculations.

This study is the first stage of research aimed at designing a probe that will facilitate measuring parameters determining the quality of a technological process.

Key words: oscillator, stochastic impulses, stochastic process, expectation, variance.

1. Introduction

The work was inspired by attempts at constructing a measuring device that would control the granularity of the medium in a dust pipeline. The device had to signal appearance of big particles in excessive quantity in transported dust. The difficulties that arose then in connection with interpretation of statistical data forced us to search for a mathematical model that would explain its causes.

If mechanical systems such as an oscillator, a string, a membrane etc. are forced by stochastic impulses then parameters of its movement are random variables.

In this study, we will apply a theorem that will allow us to calculate the basic parameters of considered stochastic variables like mathematical expectation and variance for dumped oscillator forced by impulses of the form $f(t) = \sum\limits_{t_i<t} \eta_i \delta_{t_i}$. The proof of the theorem will be given elsewhere.

In our study we will assume that the probability that an impulse will occur in a short time interval is proportional to its length, the moments of impulse occurrences and their magnitude are independent random variables. These assumptions seem quite natural in regard to the actual working conditions of the above mentioned measuring device. The universal theorem presented in this work, though applied for an oscillator, also allows for an analysis of movement of a string, a membrane and of other continuous systems in which another stochastic variable, i.e., location of the impulse impact plays a significant role. Because analysis of continuous systems is very complicated and it demands additional researches we will apply this theorem here to oscillator with dumping. We will receive theoretical formulas for expectation of the deflection of this oscillator and its variance. Further, we will carry out numerical simulation, compare the results of the simulation with theoretical calculations and interpret the differences that may occur.

The obtained results will allow us to suggest possible ways of finding the statistical characteristics of forces influencing the system having statistical characteristics of measurements.

2. Theoretical background

Let $g_i\colon [0,\infty) \to R$, $i = 1, 2, 3, ..., m$ be a sequence of continuous functions, A be a bounded connected Borel subset of R^p for some $p \in N$, $h_i\colon A \to R$, $i = 1, 2, 3, ..., m$ be a sequence of bounded and continuous functions, $\{\tau_i\}_{i=1}^{\infty}$ – a sequence of independent identically distributed (i.i.d.) random variables with exponential distribution $F(x) = 1 - \exp(-\lambda x)$ for $x > 0$ and $F(x) = 0$ for $x < 0$, $\{\eta_i\}_{i=1}^{\infty}$ – a sequence of i.i.d. random variables with finite expectation, $\{\zeta_i\}_{i=1}^{\infty}$ – a sequence of i.i.d. random variables with values in the set A and finally let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of real numbers. Let us put $t_0 = 0$, $t_i = \sum_{j=1}^{i} \tau_j$, $i = 1, 2, 3, ...,$ and

$$\xi(t) = \sum_{n=1}^{m} \alpha_n \sum_{0<t_j<t} \eta_j \, h_n(\zeta_j) \, g_n(t - t_j). \tag{1}$$

Denote by ϕ_ζ and ϕ_η distributions of ζ and η respectively. Let $A_i \subset C$ and $B_i \subset [0,\infty)$, $i = 1, 2, \ldots, m$ be Borel sets and $k(i,j)$, for every fixed i, be an increasing sequence of all natural numbers such that

$$\chi_{A_i}\left(\eta_{k(i,j)}\right) \chi_{B_i}\left(\xi_{k(i,j)}\right) = 1, \tag{2}$$

where $\chi_A(x) = 1$ if $x \in A$ a nd $\chi_A(x) = 0$ if $x \notin A$. Write $t_j^i = t_{k(i,j)}$ and $\tau_j^i = t_j^i - t_{j-1}^i$.

We will say that $\xi(t)$ is decomposable if for every $n \in N$, all Borel sets $A_i \subset A$

and $B_i \subset [0, \infty)$, $i = 1, 2, \dots n$ such that $A_i \times B_i$ are mutually disjoint

$$\bigcup_{i=1}^{\tau} A_i \times B_i = A \times B, \tag{3}$$

τ_j^i are i.i.d random variables with exponential distribution

$$\begin{aligned} F(x) &= 1 - \exp\left(-\lambda \Phi_\xi(A_i) \Phi_\eta(B_i)\right) && \text{for } x > 0 \text{ and} \\ F(x) &= 0 && \text{for } x < 0, \end{aligned} \tag{4}$$

$$\xi_i(t) = \sum_{n=1}^{m} \alpha_n \sum_{0 < t_i^j < t} \eta_{k(i,j)}\, h_n\left(\zeta_{k(i,j)}\right) g_n(t - t_j^i) \tag{5}$$

are independent and

$$\xi(t) = \sum_{i=1}^{n} \xi_i\,. \tag{6}$$

3. Theorem

If the above defined process $\xi(t)$ is decomposable, then

1) characteristic function of $\xi(t)$ is given by

$$\varphi(s) = \exp\left(\lambda t \left(\int\limits_A \int\limits_0^\infty \int\limits_0^1 \exp\left(is \sum_{n=1}^{m} \alpha_n y h_n(z) g(ut)\right) \mathrm{d}u\, \phi_\eta(\mathrm{d}y) \phi_\zeta(\mathrm{d}z) - 1\right)\right); \tag{7}$$

2) the expectation of $\xi(t)$ is

$$E(\xi(t)) = \frac{\varphi'(0)}{i} = \lambda t E(\eta) \sum_{n=1}^{m} \alpha_n E(h_n t(\zeta)) \int\limits_0^1 g_n(tu)\, \mathrm{d}u; \tag{8}$$

3) the variance of $\xi(t)$ is

$$\begin{aligned} D^2(\xi(t)) &= E(\xi^2(t)) - E^2(\xi(t)) = \frac{1}{i^2}\left(\varphi''(0) - (\varphi')^2(0)\right) \\ &= \lambda t E(\eta^2) \sum_{n=1}^{m} \sum_{j=1}^{m} \alpha_n \alpha_j E(h_n(\zeta) h_j(\zeta)) \int\limits_0^1 g_n(tu)\, g_j(tu)\, \mathrm{d}u. \end{aligned} \tag{9}$$

4. Applications

Let us consider the differential equation of the forced harmonic oscillator with damping and with one degree of freedom

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + a^2x + 2b\frac{\mathrm{d}x}{\mathrm{d}t} = f(t), \tag{10}$$

where $0 < b < a$. The solution of the above equation satisfying the following initial conditions

$$x(0) = 0 \tag{11}$$

and

$$\dot{x}(0) = 0 \tag{12}$$

has the form

$$x(t) = \frac{1}{\sqrt{a^2 - b^2}} \int_0^t f(u)\, e^{-b(t-u)} \sin\sqrt{a^2-b^2}(t-u)\,\mathrm{d}u. \tag{13}$$

If η_i is any sequence of real numbers, t_i is any increasing sequence of real numbers and $f(t)$ is given by

$$f(t) = \sum_{t_i<t} \eta_i\, \delta_{t_i}, \tag{14}$$

where δ_{t_i} are δ-Dirac distributions at t_i, then the solution of (10)–(12) takes the following form

$$x(t) = \frac{1}{\sqrt{a^2 - b^2}} \sum_{t_i<t} \eta_i\, e^{-b(t-t_i)} \sin\left(\sqrt{a^2-b^2}(t-t_i)\right). \tag{15}$$

Let us notice that the derivation of the above function is discontinuous at t_i and, consequently, the second derivative of this function does not exist in the normal sense. Fortunately, function (15) can be considered as a solution of (10)–(12) in the distribution sense and it is sufficient for our purposes.

If η_i, $i = 1, 2, ...$ are independent and identically distributed random variables with finite expectation and $\tau_i = t_i - t_{i-1}$, $i = 1, 2,...$, are also independent and identically distributed random variables with exponential distribution ($F(u) = 1 - \exp(-\lambda u)$ for $u > 0$ and for some λ and $F(u) = 0$ for $u < 0$) then $x(t)$ is a stochastic process satisfying assumptions of theorem 1 with $m = 1$ and $h_1 = 1$. Applying this theorem we get the following formulas for the expectation and variance of $x(t)$

$$\begin{aligned} E(x(t)) &= \lambda E(\eta)\frac{1}{\sqrt{a^2-b^2}} \int_0^t e^{-bt} \sin\left(\sqrt{a^2-b^2}t\right)\,\mathrm{d}t \\ &= \lambda E(\eta)\frac{e^{-bt}}{\sqrt{a^2-b^2}}\left(b\sin t\sqrt{a^2-b^2} - \sqrt{a^2-b^2}\cos t\sqrt{a^2-b^2}\right) + \frac{\lambda E(\eta)}{a^2}\,, \end{aligned} \tag{16}$$

$$
\begin{aligned}
D^2(x(t)) &= \lambda E(\eta^2)\frac{1}{a^2-b^2}\int\limits_0^t e^{-2bt}\sin^2\left(t\sqrt{a^2-b^2}\right)\mathrm{d}t \\
&= \lambda E(\eta^2)\frac{e^{-2bt}\sin t\sqrt{a^2-b^2}}{4a^2\sqrt{a^2-b^2}}\left(-\frac{2b}{\sqrt{a^2-b^2}}\sin\sqrt{a^2-b^2}t\right. \\
&\qquad \left.-2\cos\sqrt{a^2-b^2}t\right)-\lambda E(\eta^2)\frac{e^{-2bt}}{4ba^2}+\frac{\lambda E(\eta^2)}{4ba^2}\,. \qquad (17)
\end{aligned}
$$

5. Numerical simulation

For numerical simulation we will consider stochastic process $x(t)$ given by (15) for $t \in [0,\ 20\pi/\sqrt{a^2-b^2}]$.

For simplicity of computations we will assume that random variables η_i have only few values. We consider the following cases for distributions of η.

1. $\eta \in \{728,\ 214\}$ and $P(\eta = 728) = 2/3$, $P(\eta = 214) = 1/3$,
2. $\eta \in \{728,\ 42.66\}$, $P(\eta = 728) = 3/4$ and $P(\eta = 42.66) = 1/4$,
3. $\eta \in \{728,\ 385.33\}$, $P(\eta = 728) = 1/2$ and $P(\eta = 385.33) = 1/2$,
4. $\eta \in \{352,\ 240,\ 120,\ 33\}$, $P(\eta = 352) = 2/3$ and $P(\eta = 240) = 1/9$, $P(\eta = 120) = 1/9$, $P(\eta = 33) = 1/9$, $E(\eta^2) = 90\,723.67$,
5. $\eta \in \{330,\ 217,\ 120,\ 33\}$, $P(\eta = 330) = 3/4$ and $P(\eta = 217) = 1/12$, $P(\eta = 120) = 1/12$, $P(\eta = 33) = 1/12$, $E(\eta^2) = 86\,889.83$.

It is easy to calculate that in the first three cases $E(\eta) = 556.67$, in the last two cases $E(\eta) = 278.33$ and $E(\eta^2)$ is equal to 368588.00, 397943.11, 339231.60, 90723.67, 86889.83 respectively. The values of λ are assumed in such a way that $\lambda E(\eta) = 5566.7$ that is $\lambda = 10$ in the first three cases and $\lambda = 20$ in the last two cases ($\lambda E(\eta)$ represents the mass of the medium flowing through the pipe in the unit of the time). To check numerically that theoretical formulas are correct we need a statistical sample. To get a one with n elements we repeat the following procedure n times. First we choose randomly τ_i in accordance with exponential distribution with chosen λ until $t_m > 20\pi/\sqrt{a^2-b^2}$ for the first time. We remember that $t_m = \sum\limits_{i=1}^{m}\tau_i$. After that we choose randomly the values of η_i. We substitute these data into (15) and thus we obtain an element of our statistical sample. Elements of the sample are denoted by $x^k(t)$. Three such elements for the first case are given in the Fig. 1a.

Having n elements of the sample we can write

$$\widetilde{E}_n(x(t)) = \frac{1}{n}\sum_{i=1}^{n} x^i(t) \qquad (18)$$

and

$$\widetilde{D}^2_n(x(t)) = \frac{1}{n}\sum_{i=1}^{n}\left(x^i(t) - E(x(t))\right)^2. \qquad (19)$$

$\widetilde{E}_n(x(t))$ and $\widetilde{D}_n^2(x(t))$ are estimators of the expectation and the variance of $x(t)$ respectively. By the law of large numbers for every fixed $t \in [0, 20\pi/\sqrt{a^2-b^2}]\widetilde{E}_n(x(t))$ and $\widetilde{D}_n^2(x(t))$ are convergent to theoretical expectation $E(x(t))$ and theoretical variance $D^2(x(t))$, respectively, as n tends to infinity.

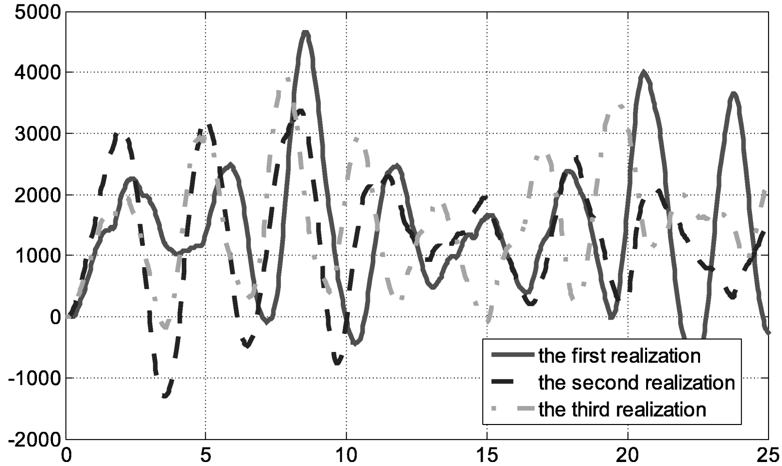

Fig. 1. Three different realizations of the process (15).

If the statistical sample has a large number of elements, for example 100 000, then $\widetilde{E}_n(x(t))$ will be always close to the solution of (10)–(12) with the constant force equal to $\lambda E(\eta)$ whose diagram is given in the Fig. 2 as a theoretical expectation. Diagrams of the estimated variances $\widetilde{D}_n^2(x(t))$ for three statistical samples corresponding to the first distribution of η as well as the diagram of theoretical variance are shown in Fig. 3. We can see high conformity between the simulation and the theoretical results.

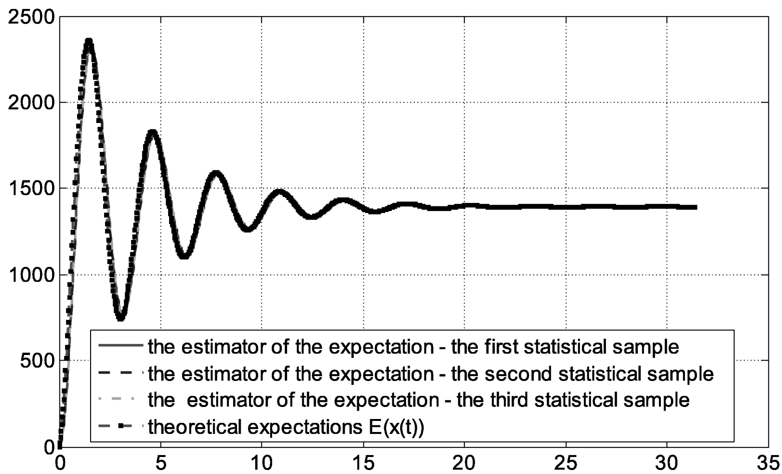

Fig. 2. Estimators of the expectation for the oscillations shown above (Fig. 1).

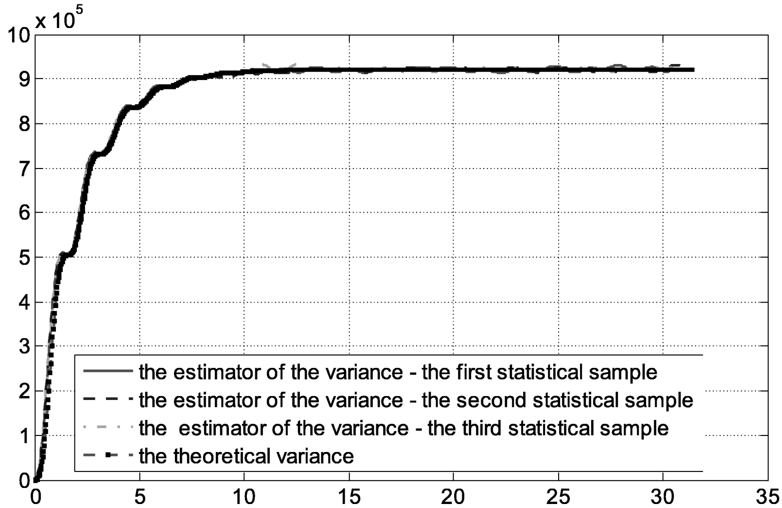

Fig. 3. Estimators of the variance of $x(t)$ for the oscillations shown above (Fig. 1).

Below we present diagrams (Fig. 4) of estimations of the variances corresponding to the considered distributions of η when $\lambda E(\eta)$ is of the same value, but variances $D^2(\eta)$ are different.

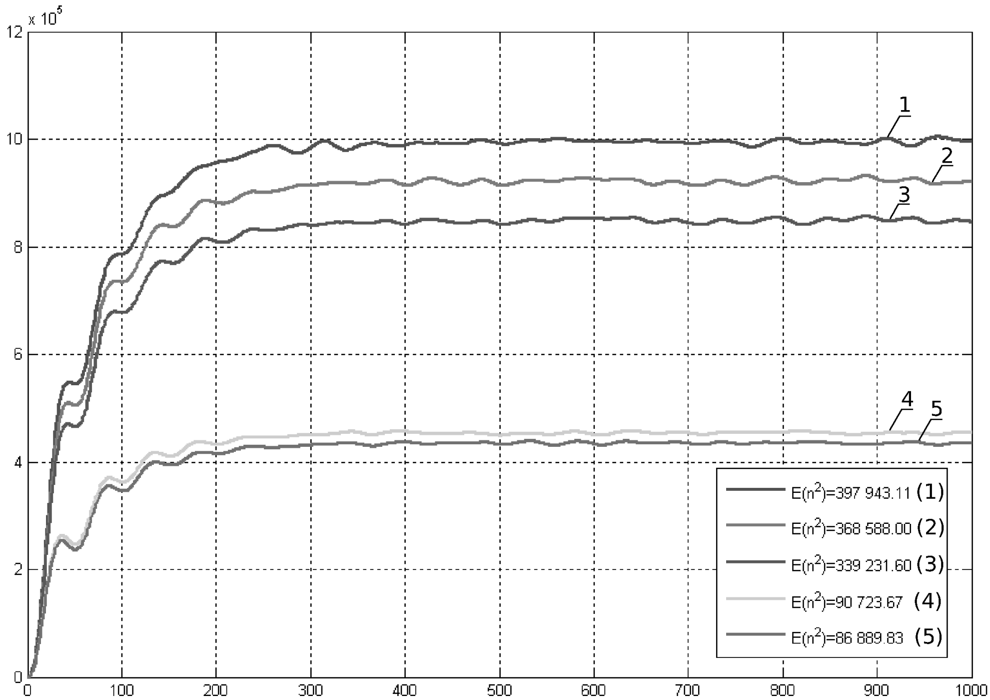

Fig. 4. Estimators of the variance of $x(t)$ when value $\lambda E(\eta)$ is of the same, but variances $D^2(\eta)$ are different.

In the real technical system appearance of too large particles with too large probability at the same mean value of transported mass in a unit of time is an unwanted effect.

Our results suggests to us how to detect this fact. The greater is the variance the greater is the probability of appearance of too large particles.

6. Conclusions

The first stage of computer simulation was to test whether the derived formulas for parameters characterizing the stochastically forced vibrations of damped oscillator were derived correctly. This stage was also meant to assess the size of the statistical sample, so that the estimators $\widetilde{E}_n(x(t))$ and $\widetilde{D}^2_n(x(t))$ could well approximate the theoretical solution.

The second stage of simulation was to visualize the dependence between the distribution of particle sizes with the same mean statistical value multiplied by the strike rate λ, but with different variances.

If λ increases, the maximum $E(\eta^2)$ at constant flow $\lambda E(\eta) = \mathrm{const}$ must decrease, and thus, on the basis of the diagrams we may conclude about $D^2(x(t))$ that the less the $D^2(x(t))$, the closer the size of a falling particle to the mean value and the lesser the probability of a large particle strike.

In technological conditions the mean value of distribution of particle sizes multiplied by the mean strike rate is constant, at least in certain time intervals. The value of variance that we will be able to measure with the methods similar to those used in simulation will inform us about irregularities in the technological process.

Actually, application of an oscillator in construction of a technological apparatus is little probable, it will have to be substituted by a continuous model. Mathematical models of such models are much more complex as regards calculations. In particular, the mean value and the variance of $x(t)$ depend on the countable numbers of harmonic vibrations. This additional difficulty requires further consideration. The presented results for an oscillator justify the need for studying these systems.

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