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GREEN FUNCTION FOR THE PROBLEM OF SOUND RADIATION BY A CIRCULAR SOUND SOURCE LOCATED NEAR TWO-WALL CORNER AND THREE-WALL CORNER

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The construction of the Green functions for the Neumann boundary value problems of the Helmholtz equation at the two-wall corner and the three-wall corner has been described. The Green functions have been expressed in their Fourier representation and have been used for computations of the radiation sound pressure of a flat circular source located in one of the two rigid baffles of the two-wall corner and in one of the three rigid baffles of the three-wall corner.

Key words: sound pressure, Green function, Neumann boundary value problem, Helmholtz equation.

1. Introduction

The Green function represents elementary acoustic pressure exerted by a point source located at $\mathbf{r}_0$ within a region of Ω at a measuring point located at $\mathbf{r}$ within the same region. This quantity is useful for further computations of the acoustic pressure radiated by sources with a continuous normal vibration velocity distribution. The Green function for the one-dimensional Neumann boundary value problem was presented in [1]. The solution was valid only for the linear operator containing second order derivatives of one or two independent variables only. On the other hand, the Fourier representations for the Green function for a free field region Ω bounded by a flat rigid infinite baffle or bounded by a rigid infinite cylinder are well known and were applied to some vibroacoustic problems [2–6]. So far, no application of the Green function has been presented in the literature for a two-wall corner and for a three-wall corner nor for describing vibroacoustic processes generated by some sources with a continuous normal vibration velocity distribution. This paper proposes a construction of such a function that can be applied to vibroacoustic problems.

2. The Green function construction

2.1. Two-wall corner

The Helmholtz equation for the Neumann boundary value problem can be formulated as below [2] at a two-wall corner of a region Ω_1 bounded by the two semi-infinite baffles, described by equations $y = 0$, $z = 0$

$$(\Delta + k^2)G(\mathbf{r}\,|\,\mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0), \tag{1}$$

where $\mathbf{r} = (x, y, z)$ – Cartesian coordinates of the acoustic field point, $\mathbf{r}_0 = (x_0, y_0, z_0)$ – Cartesian coordinates of the source point, $G(\mathbf{r}\,|\,\mathbf{r}_0) \equiv G(x, y, z\,|\,x_0, y_0, z_0) \equiv G(x, y, z\,|\,\mathbf{r}_0)$ – Green function (solution for Eq. (1)), $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ – Laplace operator, $\delta(\mathbf{r} - \mathbf{r}_0) = \delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0)$, $k = \omega/c_0 > 0$ – acoustic wavenumber, ω – circular frequency, c_0 – sound velocity. It is necessary to find the Green function $G(\mathbf{r}\,|\,\mathbf{r}_0)$ for the region under consideration Ω_1, i.e. $-\infty < x < +\infty$, $0 \leq y < +\infty$, $0 \leq z < +\infty$ filled with light fluid. This quantity can be interpreted as the sound pressure amplitude $p(\mathbf{r})$ exerted at the point $\mathbf{r}$ and generated by a source at the point $\mathbf{r}_0$. The following homogeneous boundary conditions are satisfied at the baffles' surfaces and can be formulated as

$$\frac{\partial}{\partial y}G(\mathbf{r}\,|\,\mathbf{r}_0)\Big|_{y=0} = 0, \qquad \frac{\partial}{\partial z}G(\mathbf{r}\,|\,\mathbf{r}_0)\Big|_{z=0} = 0. \tag{2}$$

The Green function considered herein is valid within the region Ω_1 where the operators $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$ possess the continuous spectrum within the limits $-\infty < \xi < +\infty$ and $0 \leq \eta < +\infty$, respectively. Let us formulate the Fourier transform pair (cf., Appendix A)

$$\begin{aligned} G(\mathbf{r}\,|\,\mathbf{r}_0) &= \frac{1}{\pi}\int\limits_{\xi=-\infty}^{+\infty}\int\limits_{\eta=0}^{+\infty} g(\xi, \eta, z\,|\,\mathbf{r}_0)\,\cos\eta y\,\exp(i\xi x)\,\mathrm{d}\xi\mathrm{d}\eta, \\ g(\xi, \eta, z\,|\,\mathbf{r}_0) &= \frac{1}{\pi}\int\limits_{x=-\infty}^{+\infty}\int\limits_{y=0}^{+\infty} G(\mathbf{r}\,|\,\mathbf{r}_0)\,\cos\eta y\,\exp(-i\xi x)\,\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3}$$

We insert the Green function from Eq. (3) into Eq. (1). Further, we multiply the equation side by side by factor $(1/\pi)\cos\eta_0 y\,\exp(-i\xi_0 x)$ and integrate along variables x, y within their limits $-\infty \leq x < +\infty$ and $0 \leq y < +\infty$, respectively. We use the following Dirac delta function properties

$$\begin{aligned} \frac{1}{\pi^2}\int\limits_{x=-\infty}^{+\infty}\exp\left[i(\xi - \xi_0)x\right]\mathrm{d}x\int\limits_{y=0}^{+\infty}\cos\eta y\,\cos\eta_0 y\,\mathrm{d}y &= \delta(\xi - \xi_0)\,\delta(\eta - \eta_0), \\ \int\limits_{\xi=-\infty}^{+\infty}\int\limits_{\eta=0}^{+\infty} f(\xi, \eta)\,\delta(\xi - \xi_0)\,\delta(\eta - \eta_0)\,\mathrm{d}\xi\mathrm{d}\eta &= f(\xi_0, \eta_0) \end{aligned} \tag{4}$$

and obtain the following from Eq. (1)

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\gamma^2\right) g(\xi,\eta,z\,|\,\mathbf{r}_0) = -\frac{1}{\pi}\,\exp(-i\xi x_0)\,\cos\eta y_0\,\delta(z-z_0), \tag{5}$$

by substituting ξ_0 and η_0 with ξ and η, respectively, where $\gamma^2 = k^2-\xi^2-\eta^2$. In the case when $z \neq z_0$, we obtain a homogeneous wave equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\gamma^2\right) g(\xi,\eta,z\,|\,\mathbf{r}_0) = 0, \tag{6}$$

instead of Eq. (5) with the following solutions

$$\begin{aligned} g_1(\xi,\eta,z\,|\,\mathbf{r}_0) &= A_1\exp(i\gamma z)+B_1\exp(-i\gamma z) \quad \text{for} \quad z\le z_0,\\ g_2(\xi,\eta,z\,|\,\mathbf{r}_0) &= A_2\exp(i\gamma z)+B_2\exp(-i\gamma z) \quad \text{for} \quad z_0\le z. \end{aligned} \tag{7}$$

These solutions must satisfy "the sharpened Sommerfeld radiation condition" [7], i.e. these solutions must describe the waves propagated along the axis $0z$ for increasing z values, which implies that $B_2 = 0$. The Neumann boundary condition leads to

$$\frac{\mathrm{d}}{\mathrm{d}z} g_1(\xi,\eta,z\,|\,\mathbf{r}_0)\Big|_{z=0} = 0 \tag{8}$$

which results in $A_1 = B_1$. We use the fact that the Green function is continuous for all the values of z as well as for $z{=}z_0$ which implies that $g_1(\xi,\eta,z{=}z_0\,|\,\mathbf{r}_0) = g_2(\xi,\eta,z{=}z_0\,|\,\mathbf{r}_0)$ and

$$\begin{aligned} g_1(\xi,\eta,z\,|\,\mathbf{r}_0) &= A_0\cos\gamma z \quad && \text{for} \quad z\le z_0,\\ g_2(\xi,\eta,z\,|\,\mathbf{r}_0) &= A_0\cos\gamma z_0\,\exp[i\gamma(z-z_0)] \quad && \text{for} \quad z_0\le z, \end{aligned} \tag{9}$$

where it has been denoted that $A_1 = A_0/2$ and $A_2 = A_0\cos\gamma z_0\,\exp(-i\gamma z_0)$. Solutions (9) must satisfy the non-homogeneous equation (5) which is singular for $z = z_0$. The solution derivatives over the variable z show a value jump for $z = z_0$. Inserting solutions (9) into Eq. (5) and integrating over the variable z within its limits $z_0-\epsilon < z < z_0+\epsilon$ covering a singular point $z = z_0$, and further computing the following limit $\epsilon{\to}0$ lead to

$$\begin{aligned} \lim_{\epsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}z} g_2(\xi,\eta,z\,|\,\mathbf{r}_0)\Big|_{z=z_0+\epsilon} &- \lim_{\epsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}z} g_1(\xi,\eta,z\,|\,\mathbf{r}_0)\Big|_{z=z_0-\epsilon}\\ &= -\frac{\exp(-i\xi x_0)}{\pi}\,\cos\eta y_0 \lim_{\epsilon\to 0}\int\limits_{z_0-\epsilon}^{z_0+\epsilon}\delta(z-z_0)\,\mathrm{d}z, \end{aligned}$$

and

$$\begin{aligned} &\lim_{\epsilon\to 0} A_0\cos\gamma z_0\exp(-i\gamma z_0)\,(i\gamma)\exp[i\gamma\,(z_0+\epsilon)]\\ &\qquad - \lim_{\epsilon\to 0} A_0\,(-\gamma)\sin\gamma\,(z_0-\epsilon) = -\frac{1}{\pi}\,\exp(-i\xi x_0)\,\cos\eta y_0, \end{aligned}$$

which implies that $A_0 = (i/\pi\gamma)\cos\eta y_0 \exp(-i\xi x_0)\exp(i\gamma z_0)$. The Green function from Eq. (3) has been expressed by the solutions (9) and assumes the form of

$$G(\mathbf{r}\,|\,\mathbf{r}_0) = \frac{i}{\pi^2}\int\limits_{\xi=-\infty}^{+\infty}\int\limits_{\eta=0}^{+\infty} I(\gamma, z\,|\,z_0)\,\exp[i\xi(x-x_0)]\,\cos\eta y_0\cos\eta y\,\frac{\mathrm{d}\xi\mathrm{d}\eta}{\gamma}, \tag{10}$$

where

$$\begin{aligned} I(\gamma, z\,|\,z_0) &= \begin{cases} \cos\gamma z\exp(i\gamma z_0) & \text{for} \quad 0 \le z \le z_0 < +\infty \\ \cos\gamma z_0\exp(i\gamma z) & \text{for} \quad 0 \le z_0 \le z < +\infty \end{cases} \\ &= \cos\gamma z\exp(i\gamma z_0)\,H(z_0-z) + \cos\gamma z_0\exp(i\gamma z)\,H(z-z_0) \end{aligned} \tag{11}$$

and

$$H(z-z_0) = \begin{cases} 1, & z > z_0, \\ 1/2, & z = z_0, \\ 0, & z < z_0, \end{cases} \tag{12}$$

is the Heaviside function.

In the specific case when the field point as well as the source point are located on the plane $z = 0$, the Green function assumes the form of

$$G(x, y, 0\,|\,x_0, y_0, 0) = \frac{1}{\pi^2}\int\limits_{\xi=-\infty}^{+\infty}\int\limits_{\eta=0}^{+\infty} \exp[i\xi(x-x_0)]\,\cos\eta y_0\,\cos\eta y\,\frac{\mathrm{d}\xi\mathrm{d}\eta}{\gamma} \tag{13}$$

valid within the region Ω_1 limited by the two-wall corner.

2.2. *Three-wall corner*

The Green function for the Helmholtz equation (1) satisfies the boundary conditions expressed below within the region of the three-wall corner Ω_2 limited by the rigid infinite baffles x=0, y=0, z=0 of the Neumann boundary value problem

$$\frac{\partial}{\partial x}G(\mathbf{r}\,|\,\mathbf{r}_0)\Big|_{x=0} = 0, \qquad \frac{\partial}{\partial y}G(\mathbf{r}\,|\,\mathbf{r}_0)\Big|_{y=0} = 0, \qquad \frac{\partial}{\partial z}G(\mathbf{r}\,|\,\mathbf{r}_0)\Big|_{z=0} = 0. \tag{14}$$

This function deals with the region Ω_2, i.e. $0 \le x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$. In this case, the operator $\partial^2/\partial x^2$ is defined within the semi-infinite range $0 \le x < \infty$ with a continuous spectrum $0 \le \xi < \infty$. We must use the following Fourier transform pair (cf. Appendix A) instead of the transform pair from Eq. (3)

$$\begin{aligned} G(\mathbf{r}\,|\,\mathbf{r}_0) &= \frac{2}{\pi}\int\limits_{\xi=0}^{+\infty}\int\limits_{\eta=0}^{+\infty} g(\xi,\eta,z\,|\,\mathbf{r}_0)\,\cos\xi x\,\cos\eta y\,\mathrm{d}\xi\mathrm{d}\eta,\\ g(\xi,\eta,z\,|\,\mathbf{r}_0) &= \frac{2}{\pi}\int\limits_{\xi=0}^{+\infty}\int\limits_{\eta=0}^{+\infty} G(\mathbf{r}\,|\,\mathbf{r}_0)\,\cos\xi x\,\cos\eta y\,\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{15}$$

We apply the following Dirac delta function integral properties

$$\begin{aligned} \frac{4}{\pi^2}\int\limits_{x=0}^{+\infty}\cos\xi x\,\cos\xi_0 x\,\mathrm{d}x\int\limits_{y=0}^{+\infty}\cos\eta y\,\cos\eta_0 y\,\mathrm{d}y &= \delta(\xi-\xi_0)\,\delta(\eta-\eta_0),\\ \int\limits_{\xi=0}^{+\infty}\int\limits_{\eta=0}^{+\infty} f(\xi,\eta)\,\delta(\xi-\xi_0)\,\delta(\eta-\eta_0)\,\mathrm{d}\xi\mathrm{d}\eta &= f(\xi_0,\eta_0) \end{aligned} \tag{16}$$

and substitute the non-homogeneous wave equation (1) with

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\gamma^2\right) g(\xi,\eta,z\,|\,\mathbf{r}_0) = -\frac{2}{\pi}\,\cos\xi x_0\,\cos\eta y_0\,\delta(z-z_0). \tag{17}$$

We rearrange the Green function to formulate it below in a similar way as in Eq. (11) in the case of the two-wall corner

$$G(\mathbf{r}\,|\,\mathbf{r}_0) = \frac{4i}{\pi^2}\int\limits_{\xi=0}^{+\infty}\int\limits_{\eta=0}^{+\infty} I(\gamma,z\,|\,z_0)\,\cos\xi x\,\cos\xi x_0\,\cos\eta y\,\cos\eta y_0\,\frac{\mathrm{d}\xi\mathrm{d}\eta}{\gamma}, \tag{18}$$

where function $I(\cdot)$ has been defined in Eq. (11). In the case when the field point as well as the source point are located on the plane $z = 0$, the Green function assumes the form of

$$G(x,y,0\,|\,x_0,y_0,0) = \frac{4i}{\pi^2}\int\limits_{\xi=0}^{+\infty}\int\limits_{\eta=0}^{+\infty}\cos\xi x\,\cos\xi x_0\,\cos\eta y\,\cos\eta y_0\,\frac{\mathrm{d}\xi\mathrm{d}\eta}{\gamma} \tag{19}$$

which deals with the region of the three-wall corner Ω_2.

3. Acoustic pressure

The Green function shown here represents an elementary acoustic pressure. It can be used, e.g., for computations of the acoustic vibration velocity potential amplitude

$$\phi(\mathbf{r}) = \int\limits_{S_0} v_n(\mathbf{r}_0)\,G(\mathbf{r}\,|\,\mathbf{r}_0)\,\mathrm{d}S_0, \tag{20}$$

where S_0 – surface of sound source, and v_n – normal component of vibration velocity of an acoustic particle being in direct contact with the surface S_0. The acoustic pressure depends on the acoustic potential as follows $p(\mathbf{r}) = \varrho_0\, \partial\phi(\mathbf{r})/\partial t$. The time dependence has been assumed as $\exp(-i\omega t)$ which implies that $p(\mathbf{r}) = -i\omega\varrho_0\phi(\mathbf{r})$, and leads to

$$p(\mathbf{r}) = -i\omega\varrho_0 \int\limits_{S_0} v_n(\mathbf{r}_0)\, G(\mathbf{r}\,|\,\mathbf{r}_0)\, \mathrm{d}S_0, \tag{21}$$

where ϱ_0 – light fluid density in the rest state. Equation (21) represents acoustic pressure exerted at the point $\mathbf{r}$ by a source with a continuous surface harmonic vibration velocity distribution.

Let us assume that a flat source is located at the baffle surface, i.e. for $z = 0$. Then, in the case of the two-wall corner region Ω_1 we use Eq. (10) to formulate the acoustic pressure amplitude as

$$p_1(\mathbf{r}) = \frac{\omega\varrho_0}{\pi^2} \int\limits_{\xi=-\infty}^{+\infty} \int\limits_{\eta=0}^{+\infty} \exp(i\xi x)\, \cos\eta y\, \exp(i\gamma z)\, M_1(\xi,\eta)\, \frac{\mathrm{d}\xi\mathrm{d}\eta}{\gamma}\,, \tag{22}$$

where $M_1(\xi,\eta) = \int\limits_{S_0} v_n(\mathbf{r}_0)\, \exp(-i\xi x_0)\, \cos\eta y_0\, \mathrm{d}S_0$. In the case of the three-wall corner region Ω_2 we use Eq. (18) and formulate the acoustic pressure amplitude in the form of

$$p_2(\mathbf{r}) = \frac{4\omega\varrho_0}{\pi^2} \int\limits_{\xi=0}^{+\infty} \int\limits_{\eta=0}^{+\infty} \cos\xi x\, \cos\eta y\, \exp(i\gamma z)\, M_2(\xi,\eta)\, \frac{\mathrm{d}\xi\mathrm{d}\eta}{\gamma}, \tag{23}$$

where $M_2(\xi,\eta) = \int\limits_{S_0} v_n(\mathbf{r}_0)\, \cos\xi x_0\, \cos\eta y_0\, \mathrm{d}S_0$. The acoustic pressure amplitudes $p_1, p_2 \in \mathbb{C}$ radiated by the source located on the plane $z = 0$ are valid for any point of the regions Ω_1 and Ω_2, respectively.

4. Concluding remarks

The construction of the Green function for the two-wall corner as well as for the three-wall corner have been proposed and presented in their Fourier representations. These formulas have not been presented earlier for the Neumann boundary value problems as described herein. They are useful for some further computations of the acoustic pressure radiated by flat sound sources with continuous normal vibration velocity distributions.

Appendix A. Fourier transforms

The Fourier transform from Eq. (3) has been formulated using the Fourier transform pair $\mathcal{F}_x, \mathcal{F}_\xi^{-1}$ for the variables x, ξ within their infinite limits $(-\infty, +\infty)$

$$\begin{aligned} G(\mathbf{r}\,|\,\mathbf{r}_0) &= \mathcal{F}_\xi^{-1} g(\xi, y, z\,|\,\mathbf{r}_0) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} g(\xi, y, z\,|\,\mathbf{r}_0)\, \exp(i\xi x)\, \mathrm{d}\xi, \\ g(\xi, y, z\,|\,\mathbf{r}_0) &= \mathcal{F}_x\, G(x, y, z\,|\,\mathbf{r}_0) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} G(\mathbf{r}\,|\,\mathbf{r}_0)\, \exp(-i\xi x)\, \mathrm{d}x, \end{aligned} \tag{A1}$$

and the Fourier transform pair $\mathcal{C}_y, \mathcal{C}_\eta^{-1}$ for the variables y, η within their semi-infinite limits $[0, +\infty)$

$$\begin{aligned} G(\mathbf{r}\,|\,\mathbf{r}_0) &= \mathcal{C}_\eta^{-1} g(x, \eta, z\,|\,\mathbf{r}_0) = \sqrt{\frac{2}{\pi}} \int\limits_{0}^{+\infty} g(x, \eta, z\,|\,\mathbf{r}_0)\, \cos\eta y\, \mathrm{d}\eta, \\ g(x, \eta, z\,|\,\mathbf{r}_0) &= \mathcal{C}_y\, G(\mathbf{r}\,|\,\mathbf{r}_0) = \sqrt{\frac{2}{\pi}} \int\limits_{0}^{+\infty} G(\mathbf{r}\,|\,\mathbf{r}_0)\, \cos\eta y\, \mathrm{d}y. \end{aligned} \tag{A2}$$

Equations (3) have been obtained by composing the following transforms

$$\begin{aligned} G(\mathbf{r}\,|\,\mathbf{r}_0) &= \mathcal{F}_\xi^{-1} \mathcal{C}_\eta^{-1}\, g(\xi, \eta, z\,|\,\mathbf{r}_0), \\ g(\xi, \eta, z\,|\,\mathbf{r}_0) &= \mathcal{F}_x \mathcal{C}_y\, G(\mathbf{r}\,|\,\mathbf{r}_0). \end{aligned} \tag{A3}$$

In the case of the three-wall corner, the pair of Fourier transforms (15) has been obtained by composing the following transforms (A2)

$$\begin{aligned} G(\mathbf{r}\,|\,\mathbf{r}_0) &= \mathcal{C}_\xi^{-1} \mathcal{C}_\eta^{-1} g(\xi, \eta, z\,|\,\mathbf{r}_0), \\ g(\xi, \eta, z\,|\,\mathbf{r}_0) &= \mathcal{C}_x \mathcal{C}_y\, G(\mathbf{r}\,|\,\mathbf{r}_0). \end{aligned} \tag{A4}$$

The Dirac delta function integral properties (4) and (16) have been obtained using the formulas $f(\xi, \eta) \equiv \mathcal{F}_x \mathcal{C}_y\, \mathcal{F}_\xi^{-1} \mathcal{C}_\eta^{-1} f(\xi, \eta)$ and $f(\xi, \eta) \equiv \mathcal{C}_x \mathcal{C}_y\, \mathcal{C}_\xi^{-1} \mathcal{C}_\eta^{-1} f(\xi, \eta)$, respectively.

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