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AN APPLICATION OF GREEN'S FUNCTION FOR ACOUSTIC RADIATION OF A SOURCE LOCATED NEAR THE TWO WALL CORNER

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This paper focuses on the problem of sound radiation of a time harmonically vibrating rectangular piston embedded into one of two baffles configured spatially as the two wall corner. The sound radiation pressure and sound radiation power, active and reactive, have been presented as their Fourier representations using the Green's function. The directivity pattern has been expressed in terms of some elementary functions whereas the radiation efficiency has been expressed as a low frequency approximation. The elementary formulas obtained make it possible to clearly interpret the influence of the baffles of two wall corner on the sound radiation of the piston.

Key words: modal and total radiation efficiency, sound radiation, sound pressure, sound power.

1. Introduction

Theoretical research on some mutual interactions and sound radiation of some vibrating piston sources located within the plane of rigid baffle are well known [1–4]. Some sample experimental investigations indicate that the baffles play an essential role in the sound power radiated by some flat sources [5]. Especially important acoustic processes appear for some low frequency radiated waves which specially refers to the active sound power.

The main aim of this paper is to analyze the influence of the two wall corner on the radiation efficiency of a rectangular piston. Green's function in its Fourier representation has been used to express the sound pressure radiated, active and reactive sound power and radiation efficiency as well as some elementary formulations have been obtained that are valid for the directivity pattern. Some elementary asymptotic formulas, describing the radiation efficiency of the source within the low frequency range, have been presented. The formulas show how the quantity depends on the distance of the source's center from the two wall corned and how it depends on vibration frequency.

2. Sound radiation pressure and sound radiation power

The following time dependence has been assumed $\exp(-i\omega t)$. The sound pressure radiated by a vibrating surface source into the two wall corner $-\infty < x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$, i.e. the region bounded by two rigid orthogonal half planes $y = z = 0$ (cf., Fig. 1). The sound source is located in the half-plane $z = 0$. The radiated sound pressure amplitude has been formulated using Green's function [6]

$$\begin{aligned} p(x, y, 0) &= ik\varrho_0 c_0 \phi(x, y, 0) \\ &= -ik\varrho_0 c_0 \int\limits_{S'} G(x, y, 0 \,|\, x', y', 0) \left.\frac{\partial \phi}{\partial n'}\right|_{z'=0} \mathrm{d}S', \end{aligned} \tag{1}$$

given that

$$\left.\frac{\partial \phi}{\partial n'}\right|_{z'=0} = -\left.\frac{\partial \phi}{\partial z'}\right|_{z'=0} = \begin{cases} v(x', y') & \text{for } x', y' \in S' \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where $G(x, y, z \,|\, x', y', z')$ is Green's function of the Helmholtz equation that satisfies the homogeneous Neumann boundary value conditions for $y = z = 0$ formulated for the region of $-\infty < x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$. Green's function has been expressed as follows [7]

$$\begin{aligned} G(x, y, z \,|\, x', y', z') &= \frac{i}{\pi^2} \int\limits_{\xi=-\infty}^{+\infty} \int\limits_{\eta=0}^{+\infty} \cos \eta y' \cos \eta y \\ &\times \exp[i\xi(x - x')] \begin{Bmatrix} \cos \gamma z \, \exp(i\gamma z') & \text{for } z \le z < z' < \infty \\ \cos \gamma z' \exp(i\gamma z) & \text{for } z \le z' < z < \infty \end{Bmatrix} \frac{\mathrm{d}\xi \mathrm{d}\eta}{\gamma}, \end{aligned} \tag{3}$$

where $\gamma^2 = k^2 - \xi^2 - \eta^2$.

It has been assumed that a rectangular piston, vibrating time harmonically with the normal components of vibration velocity $v(t) = v_0 \exp(-i\omega t)$ where $v_0 = \mathrm{const}$, is the sound source. The piston is embedded into the rigid half-plane $z = 0$ (cf., Fig. 1).

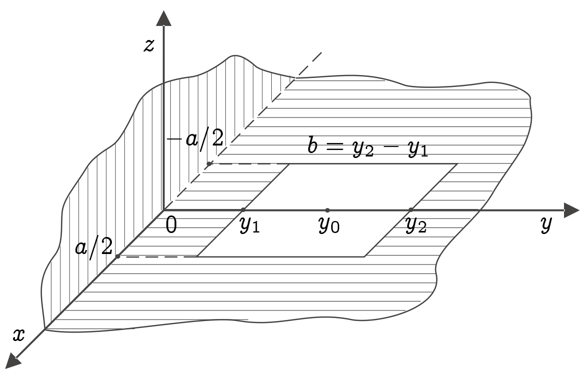

Fig. 1. The sound source located near the two wall corner.

After integrating Green's function in Eq. (1) along the rectangular piston surface, the following expression for the radiated sound pressure amplitude at the plane of the piston has been obtained in its Fourier representation

$$p(x,y,0) = \varrho_0 c_0 k v_0 \frac{ab}{\pi^2} \int\limits_{\xi=-\infty}^{+\infty} \int\limits_{\eta_1=0}^{+\infty} \cos(\eta y_0) \frac{\sin(\xi a/2)}{\xi a/2} \times \frac{\sin(\eta b/2)}{\eta b/2} \cos\eta y \, \exp(i\xi x) \frac{\mathrm{d}\xi \mathrm{d}\eta}{\gamma}, \tag{4}$$

whereas the time averaged radiation sound power

$$N = \frac{1}{2} \int\limits_S p(x,y)\, v^*(x,y)\, \mathrm{d}S, \tag{5}$$

has been expressed in its Fourier representation

$$N = \varrho_0 c_0 (abv_0)^2 \frac{k}{2\pi^2} \int\limits_{\xi_1=-\infty}^{+\infty} \int\limits_{\eta_1=0}^{+\infty} \frac{\sin^2(\xi a/2)}{(\xi a/2)^2} \frac{\sin^2(\eta\, b/2)}{(\eta\, b/2)^2} \cos^2(\eta y_0) \frac{\mathrm{d}\xi \mathrm{d}\eta}{\gamma}, \tag{6}$$

where v^* is the conjugate value for the piston vibration velocity amplitude v, $N = N' - iN''$, N', N'' are the real and imaginary components of radiation sound power, respectively. The change of variables introducing the polar coordinates $0 \le \theta < \frac{\pi}{2} - i\infty$, $0 \le \varphi \le \pi$ in the plane of complex variable $\theta = \theta' + i\theta''$ and using transformations

$$\xi = k \sin\theta \cos\varphi, \qquad \eta = k \sin\theta \sin\varphi, \tag{7}$$

where $\gamma = k\cos\theta$, $\mathrm{d}\xi\mathrm{d}\eta = k^2 \sin\theta\, \cos\theta\, \mathrm{d}\theta\mathrm{d}\varphi$ has provided

$$\frac{N}{N^{(\infty)}} = \frac{k^2 ab}{\pi^2} \int\limits_0^{\pi/2-i\infty} \int\limits_0^{\pi} M^2(\theta,\varphi) \sin\theta \, \mathrm{d}\theta\mathrm{d}\varphi, \tag{8}$$

instead of Eq. (6) where

$$N^{(\infty)} = \lim_{k\to\infty} N(k) = \frac{\varrho_0 c_0}{2} abv_0^2, \tag{9}$$

whereas

$$M(\theta,\varphi) = \frac{1}{ab} \int\limits_{-a/2}^{a/2} \exp(ikx \sin\theta\, \cos\varphi)\, \mathrm{d}x \int\limits_{y_1}^{y_2} \cos(ky \sin\theta \sin\vartheta) \mathrm{d}y = \frac{\sin\left(\frac{1}{2} ka \sin\theta \cos\varphi\right)}{\frac{1}{2} ka \sin\theta\cos\varphi} \frac{\sin\left(\frac{1}{2} kb \sin\theta \sin\varphi\right)}{\frac{1}{2} kb \sin\theta \sin\varphi} \cos(ky_0 \sin\theta \sin\varphi) \tag{10}$$

is the characteristic function that is also useful to express the corresponding directivity pattern

$$R(\theta', \varphi) = |M(\theta', \varphi)|. \tag{11}$$

Equation (8) has been expressed as a sum of two terms $N = N_1 + N_2$. The first term

$$\frac{N_1}{N^{(\infty)}} = \frac{k^2 ab}{2\pi^2} \int\limits_0^{\pi/2-i\infty} \int\limits_0^{\pi} \left\{ \frac{\sin\left(\frac{1}{2}ka\sin\theta\cos\varphi\right)}{\frac{1}{2}ka\sin\theta\cos\varphi} \times \frac{\sin\left(\frac{1}{2}kb\sin\theta\sin\varphi\right)}{\frac{1}{2}kb\sin\theta\sin\varphi} \right\}^2 \sin\theta\,\mathrm{d}\theta\, d\varphi \tag{12}$$

describes the radiated sound power (active and reactive) of a vibrating piston with only one baffle (cf., Ref. [3]). The second term

$$\frac{N_2}{N^{(\infty)}} = \frac{k^2 ab}{2\pi^2} \int\limits_0^{\pi/2-i\infty} \int\limits_0^{\pi} \left\{ \frac{\sin\left(\frac{1}{2}ka\sin\theta\cos\varphi\right)}{\frac{1}{2}ka\sin\theta\cos\varphi} \times \frac{\sin\left(\frac{1}{2}kb\sin\theta\sin\varphi\right)}{\frac{1}{2}kb\sin\theta\sin\varphi} \right\}^2 \cos(2ky_0\sin\theta\sin\varphi)\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \tag{13}$$

represents the additional sound power radiated by the piston as a consequence of acoustic interaction between the piston and the baffle orthogonal to the piston plane.

3. Radiation efficiency

The radiation efficiency has been defined using Eqs. (12) and (13) as follows

$$\sigma = \frac{N'}{N_\infty} = \frac{k^2 ab}{2\pi^2} \int\limits_0^{\pi/2} \sin\theta\,\mathrm{d}\theta \int\limits_0^{\pi} \left\{ \frac{\sin\left(\frac{1}{2}ka\sin\theta\cos\varphi\right)}{\frac{1}{2}ka\sin\theta\cos\varphi} \right\}^2 \times \left\{ \frac{\sin\left(\frac{1}{2}kb\sin\theta\sin\varphi\right)}{\frac{1}{2}kb\sin\theta\sin\varphi} \right\}^2 [1 + \cos(2ky_0\sin\theta\sin\varphi)]\,\mathrm{d}\varphi. \tag{14}$$

In the specific case when $ka/2$, $kb/2 < 1$ the following expansion series has been used $\sin^2 x \sin^2 y/(xy)^2 \simeq 1 - \frac{1}{3}(x^2 + y^2)$ which leads to

$$\sigma = \frac{k^2 ab}{2\pi^2} \int_0^{\pi/2} \sin\theta \mathrm{d}\theta \int_0^{\pi} [1 - (k^2/12)\sin^2\theta(a^2\cos^2\vartheta + b^2\sin^2\vartheta)] \times [1 + \cos(2ky_0 \sin\theta \sin\varphi)]\,\mathrm{d}\varphi. \tag{15}$$

During the integration along variable φ the following equation has been used [8]

$$\int_0^{\pi} \cos(z\sin\varphi)\cos(2n\varphi)\,\mathrm{d}\varphi = \pi J_{2n}(z), \quad n = 0, 1, 2, \ldots, \tag{16}$$

leading to

$$\sigma = \frac{k^2 ab}{2\pi} \int_0^{\pi/2} F(\theta)\sin\theta\,\mathrm{d}\theta, \tag{17}$$

where

$$F(\theta) = \left\{1 - \frac{(ka)^2 + (kb)^2}{24}\sin^2\theta\right\}[1 + J_0(2ky_0\sin\theta)] + \frac{(kb)^2 - (ka)^2}{24}\sin^2\theta\, J_2(2ky_0\sin\theta). \tag{18}$$

Sonine's integral has also been used (cf., Refs. [8, 9])

$$\int_0^{\pi/2} J_\nu(z\sin\theta)\cos^{2\mu+1}\theta\,\sin^{\nu+1}\theta\,\mathrm{d}\theta = \frac{2^\mu \Gamma(\mu+1)}{z^{\mu+1}} J_{\nu+\mu+1}(z) \tag{19}$$

given that $\mathrm{Re}\,\nu$, $\mathrm{Re}\,\mu > -1$. Further, the Bessel functions $J_{n+1/2}$ for $n = 0, 1, 2, 3, \ldots$ have been expressed by some trigonometric functions giving

$$\sigma = \sigma_0\left[1 + \frac{\sin u_0}{u_0} + \mathcal{E}(k, a, b, y_0)\right], \tag{20}$$

where it has been denoted $\sigma_0 = k^2 ab/2\pi$, $u_0 = 2ky_0$, and

$$\mathrm{cs}\,x \equiv \frac{1}{x}\left(\frac{\sin x}{x} - \cos x\right), \tag{21}$$

$$\mathcal{E}(k, a, b, y_0) = \frac{(ka)^2 + (kb)^2}{12}\left(\frac{\mathrm{cs}\,u_0 - \sin u_0}{2u_0} - \frac{1}{3}\right) + \frac{(kb)^2 + (ka)^2}{24}\,\frac{3\mathrm{cs}\,u_0 - \sin u_0}{u_0}. \tag{22}$$

In Eq. (22), the following integral values have been used $\int_0^{\pi/2} J_0(x\sin\theta)\,\sin\theta\,\mathrm{d}\theta = \sin x/x$, $\int_0^{\pi/2} J_0(x\sin\theta)\,\sin^{-3}\theta\,\mathrm{d}\theta = (\sin x - \mathrm{cs}\,x)/x$, $\int_0^{\pi/2} J_2(x\sin\theta)\,\sin^{-3}\theta\,\mathrm{d}\theta = (3\mathrm{cs}\,x - \sin x)/x$. In Eq. (20), quantity $\mathcal{E}(k,a,b,y_0)$ represents a correction for all the reflected waves. It has been determined by neglecting all the terms containing $(ka)^{4n}$, $(kb)^{4n}$, $(k^2ab)^{2n}$ for $n \geq 1$. An increase in the range of k, where Eq. (20) is valid, can be reached by including some more successive approximation terms from the expansion series of function $f(x,y) = (\sin x/x)^2(\sin y/y)^2$, where $x = (ka/2)\sin\theta\cos\varphi$, $y = (kb/2)\sin\theta\sin\varphi$ in Eq. (14). Quantity

$$\sigma_0 = \frac{k^2ab}{2\pi} = \frac{2\pi S}{\lambda^2}, \tag{23}$$

represents the radiation efficiency of a vibrating rectangular piston embedded into a flat rigid baffle in the case of a single baffle where $S = ab$ and $k = 2\pi/\lambda$. On the other hand, increasing the distance between the two wall corner and the piston (i.e., length y_0) results in decreasing down to zero in amplitude of oscillating term $\sin u_0/u_0$, i.e., the influence of the two wall corner on radiation efficiency of the piston vanishes.

4. Concluding remarks

A theoretical analysis of the influence of the two wall corner on the acoustic field generated by some time harmonic vibrations of a rectangular piston source has been performed. The sound pressure radiated, the complex radiation sound power and the radiation efficiency of the source have been expressed as their Fourier exponent-cosine representations. For this purpose, Green's function for the Helmholtz equation, for the Neumann boundary condition and "the sharpened Sommerfeld radiation condition" satisfied, has been used. The directivity pattern has been expressed in its elementary form (11) including the influence of the two wall corner on the sound pressure radiated. The two terms of complex radiation sound power (12) and (13) as well as the radiation efficiency (14) have been expressed in their integral forms. Equation (12) is enough if only one baffle appears whereas the two wall corner requires the sum of two terms, i.e., Eqs. (12) and (13). The second term represents the correction for all the reflected waves. In the specific case when the linear sizes of the source are small enough as compared with the wavelengths radiated then the radiation efficiency can be approximated with Eq. (20) where the term representing the reflected waves has been separated and mainly depends on the distance y_0 between the source and the two wall corner, and on the acoustic wavenumber k. All the computations presented in this paper are valid for the acoustic radiation of a flat piston. However, the method presented herein can easily be used for such sources as circular plates and rectangular plates. The influence of the air column on source's vibrations and sound radiation can also be included using presented results.

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