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SOUND WAVE PROPAGATION PROBLEMS NEW PERTURBATION METHODOLOGY

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The aim of the paper is to present applications of the new algebraic system theory in acoustic problems. Elements of new types of perturbed ordinary and partial differential equations are discussed. The formulation is applied to the analysis of perturbed wave problems with perturbations in parameters as well as in initial and boundary conditions. Classical perturbation acoustic problems described by differential equations can be solved in the modified algebraic system as easy as usual. Any additional analytical transformations are not required. A novel 2D ray-tracing model of detailed representation of the indoor/outdoor environment is presented. Perturbation ray tracing method is a technique based on geometrical optics with perturbation in parameters which can be an easily applied approximate method for estimating perturbation problems in acoustics. The developed algorithms use the new perturbation methodology where the perturbed images are used to produce 2D-field of illumination zones. It can be easy considered how perturbations (small) of nominal parameter values can change solutions of the considered problems.

Key words: perturbation numbers, perturbed DE, ray tracing.

1. Introduction

Theory of perturbations first appeared in one of the oldest branches of applied mathematics: celestial mechanics. The scope of perturbation theory at the present time is much broader than its applications to celestial mechanics, but the main idea is the same. One can begin with a simple solvable problem, called the unperturbed problem and using the solution of this problem as an approximation we go to the solution of a more complicated problem that differs from the basic one only by some small terms in the equations. Then one looks for a series of succesive approximations to this initial solution, most often in the form of a power series in a small quantity called the perturbation parameter.

First investigations of the perturbed wave equation are given by Keller and are related to perturbed Helmholz equation cf. [2, 3, 10]. There are many papers about these problems, cf. [1, 5, 11–13]. In this paper we shall present a new technique that will be used as the basis for our presentation of wave equation analysis in the situation when all parameters of wave equation are perturbed.

2. Algebraic system of perturbation numbers

DEFINITION 1: *Define a number called further a perturbation number as an ordered pair of real numbers $(x, y) \in R^2$. The set of perturbation numbers is denoted by R_ε. The first element x of the double (x, y) is called a main value and the second y the perturbation value or simply the perturbation* [14, 15]. □

Let $z, z_1, z_2 \in R_\varepsilon$ denote three perturbation numbers and $z = (x, y)$, $z_1 := (x_1, y_1)$, $z_2 := (x_2, y_2)$, x_i, $y_i \in R$, $i = 1,\ 2$. It is called that two perturbation numbers are equal $z_1 \equiv z_2$ iff: $x_1 = x_2$ and $y_1 = y_2$. Further we call x, x_1, x_2 main values of corresponding perturbation numbers.

In the set R_ε we introduce the addition $(+_\varepsilon)$ and multiplication $(\bullet_\varepsilon)$ as follows:

$$z_1 +_\varepsilon z_2 \ := \ (x_1 + x_2,\ y_1 + y_2), \tag{1}$$

$$z_1 \bullet_\varepsilon z_2 \ := \ (x_1 x_2,\ x_1 y_2 + x_2 y_1). \tag{2}$$

Associated with each $z = (x, y)$ is a unique opposite element $-z := (-x, -y)$ such that $z +_\varepsilon (-z) = 0_\varepsilon$.

THEOREM 1: *The set R_ε with addition $(+_\varepsilon)$ and multiplication $(\bullet_\varepsilon)$ defined by Eqs. (1) and (2) with selected neutral addition element $0_\varepsilon := (0, 0)$ and neutral multiplication element $1_\varepsilon := (1, 0)$ is a field. Defined in such way field is called a field of perturbation numbers.* □

Defined in Definition 1 the field R_ε doesn't contain the field of real numbers R. We show that real numbers can be considered as some elements of field R_ε with all classical addition and multiplication formulas and neutral elements of addition and multiplication, cf. [1, 6, 7].

The map $j\colon R \to R_\varepsilon$, $j(x) := (x, 0)$ for each $x \in R$, is called the injection of the algebraic system of real numbers R into the algebraic system R_ε. It's the single-valued mapping and preserves corresponding algebraic operations and neutral elements of addition and multiplications. Notice, that since $j(\cdot)$ is the injection then each perturbation number of the form $(a, 0)$, $a \in R$, we can identify with a real number a. We use this notice to simplify a notion for perturbation operations. Denote by ε the perturbation number $(0, 1)$. Assume that the perturbation number $(x, 0)$ is identified with x and $(y, 0)$ with the real y. Then we have

$$(x, y) = (x, 0) +_\varepsilon (0, y) = (x, 0) +_\varepsilon (y, 0) \bullet_\varepsilon (0, 1) = j(x) +_\varepsilon \varepsilon \bullet_\varepsilon j(y) = x +_\varepsilon \varepsilon \bullet_\varepsilon y.$$

From multiplicity formulas it follows that

$$\varepsilon^2 := \varepsilon \bullet_\varepsilon \varepsilon = (0, 1) \bullet_\varepsilon (0, 1) = (0, 0),$$

in simplified notion $\varepsilon^2 = 0$ and in consequence $z = x + \varepsilon y$.

3. Order relation in the set of perturbation numbers

Let $z = x+\varepsilon y$, $z_1 = x_1+\varepsilon y_1$, $z_2 = x_2+\varepsilon y_2$ be arbitrary real perturbation numbers.

In the set of perturbation numbers, similarly as in the sets: R^2, set of complex numbers C^1, R^n, $n > 1$ etc it's not possible to introduce the complete order relation. Following that fact we define the relation of partial order in the strong "$\leq_\varepsilon, \geq_\varepsilon, =_\varepsilon, >_\varepsilon, <_\varepsilon$" and weak matter "$\dot{\leq}_\varepsilon, \dot{\geq}_\varepsilon, \dot{=}_\varepsilon, \dot{>}_\varepsilon, \dot{<}_\varepsilon$".

DEFINITION 2: *For $z_1, z_2 \in R_\varepsilon$, we say that $z_1 \geq_\varepsilon z_2$ if $x_1 \geq x_2$ and $y_1 \geq y_2$.* □

DEFINITION 3: *For $z_1, z_2 \in R_\varepsilon$, we say that $z_1 >_\varepsilon z_2$ if $z_1 \geq_\varepsilon z_2$ and $x_1 > x_2$ and $y_1 > y_2$.* □

In an analogous way we define relations "$\leq_\varepsilon$" and "$<_\varepsilon$".

DEFINITION 4: *For $z_1, z_2 \in R_\varepsilon$, we say that $z_1 \dot{\geq}_\varepsilon z_2$ if $x_1 \geq x_2$ and y_1, y_2 are arbitrary.* □

DEFINITION 5: *For $z_1, z_2 \in R_\varepsilon$, we say that $z_1 \dot{>}_\varepsilon z_2$ if $x_1 > x_2$ and y_1, y_2 are arbitrary.* □

DEFINITION 6: *For $z_1, z_2 \in R_\varepsilon$, we say that $z_1 \dot{=}_\varepsilon z_2$ if $z_1 \dot{\geq}_\varepsilon z_2$ and $z_2 \dot{\geq}_\varepsilon z_1$ (or equivalently $x_1 = x_2$.* □

In the analogous way we define the relation "$\dot{\leq}_\varepsilon$" and "$\dot{<}_\varepsilon$".

Notice that relations between perturbation numbers of the "strong" type as "$\leq_\varepsilon, \geq_\varepsilon, =_\varepsilon, >_\varepsilon, <_\varepsilon$" implies the "weak" relations: "$\dot{\leq}_\varepsilon, \dot{\geq}_\varepsilon, \dot{=}_\varepsilon, \dot{>}_\varepsilon, \dot{<}_\varepsilon$", respectively.

4. Extended ε-functions

Perturbation value functions are defined for perturbation arguments as extensions of classical elementary and trigonometric functions. Properties of ε-functions are analyzed in details, cf. [14–16].

Let $D \subset R_\varepsilon$ be an arbitrary subset. Suppose that we have a rule f_ε which assigns to each element $z \in D$ exactly one element w of R_ε. Then we say that f_ε is an extended function defined on D with values in R_ε. We will denote that function as $f_\varepsilon\colon D \to R_\varepsilon$ or $w = f_\varepsilon(z)$ or simplified $w = \varepsilon - f(z)$.

To illustrate how we can construct generalizations of usual real functions we use a simple function. We discuss now an extension of a simple exponential function $\exp(x)$, $x \in R$. With polynomials and rational functions it is one of the simplest elementary functions. How can we understand the notion $\exp(z)$, where $z = x + \varepsilon y \in R_\varepsilon$?

Notice that we can expand $\exp(x)$ into a classical series

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{x^k}{k!}\,, \qquad x \in R, \tag{3}$$

which is convergent for all $x \in R$. Define the new function $\exp_\varepsilon(z)$, for $z = x + \varepsilon y \in R_\varepsilon$ as

$$\exp_\varepsilon(z) := 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{z^k}{k!}\,, \qquad z \in R_\varepsilon\,. \tag{4}$$

Following equations (3) and (4) we write

$$\exp_\varepsilon(z) := 1 + \frac{x + \varepsilon y}{1!} + \frac{x^2 + \varepsilon 2xy}{2!} + ... = (1 + \varepsilon y)\exp(x), \qquad z \in R_\varepsilon\,. \tag{5}$$

We can prove the generalized convergence of the sequence (4) for every $z \in R_\varepsilon$. We have

$$j(\exp(x)) = (\exp(x), 0) = \exp_\varepsilon(x),$$

which proves that the new function $\exp_\varepsilon(\cdot)$ is the extension of real function $\exp(x)$.

In the similar way one can define more complicated multidimensional functions.

5. Differentiation of perturbation functions

Let the function $f_\varepsilon(t)$, $t \in D \subset R_\varepsilon$ be a perturbation function of a real perturbation variable.

DEFINITION 7: *If in the point $z_0 \in D \subset R_\varepsilon$ there exists a limit (in the ε – convergence sense) of difference quotient*

$$\frac{f_\varepsilon(z_0 + \Delta z) - f_\varepsilon(z_0)}{\Delta z}\,, \tag{6}$$

for $\Delta z = \Delta x + \varepsilon \Delta y \xrightarrow{\varepsilon} 0_\varepsilon$ then one says that the ε-function $f_\varepsilon(z)$ is ε-differentiable (or simply differentiable) in z_0. The limit is called ε-differential of the function f_ε in z_0 and is denoted

$$f'_\varepsilon(z_0) := \varepsilon - \lim_{\Delta z \to 0_\varepsilon} \frac{f_\varepsilon(z_0 + \Delta z) - f_\varepsilon(z_0)}{\Delta z}\,. \qquad \square$$

We can generalize these definitions to higher order differentials [15–16].

6. Wave equation with perturbation components

New mathematical formalism is applied to classical perturbation differential problems arising in theoretical mechanics and physics [15, 16]. Dynamic perturbation problems of a simple vibration problem is given. The advantages of the methodology is presented in analytical calculations and in special numerical procedures dedicated to linear wave equation with perturbations in parameters as well as in initial conditions and boundary conditions. Consider the following perturbed wave equation with mixed-type variables

$$\frac{\partial^2}{\partial x^2} u_\varepsilon(t, x) - \frac{1}{a_\varepsilon^2}\frac{\partial^2}{\partial t^2} u_\varepsilon(t, x) = 0_\varepsilon\,, \tag{7}$$

where $t \in [0, \infty[$, $x \in [0, l_\varepsilon]$, $a_\varepsilon = a_0 + \varepsilon a_1$, $l_\varepsilon = l_0 + \varepsilon l_1 \in R_\varepsilon$, perturbed initial conditions of Cauchy type

$$\begin{aligned} u_\varepsilon(x,0) &= f_\varepsilon(x), \\ \frac{\partial}{\partial t} u_\varepsilon(x,0) &= g_\varepsilon(x), \end{aligned} \qquad x \in (0, l_\varepsilon), \tag{8}$$

and perturbed boundary conditions

$$\begin{aligned} u_\varepsilon(0_\varepsilon, t) &\equiv 0_\varepsilon, \\ u_\varepsilon(l_\varepsilon, t) &\equiv 0_\varepsilon, \end{aligned} \qquad t \in (0, \infty)\,. \tag{9}$$

7. Ray acoustic emission algorithm with perturbations

This paper investigates the feasibility of predicting the perturbations of Acoustic Emission (ε-AE) signals travelling within a indoor/outdoor environment. Such ε-AE can occur due to external stimulation or internal events. The attenuation of these signals is affected not only by perturbed material properties but also by the perturbed geometry of the object. For example, wave propagation is complex because of intricate shape with perturbed variations and discontinuities in thickness and surface curvature. In contrast to much of the reported literature that models the transmission of sound in rooms and buildings, this paper reports the development of a perturbation ray firing procedure to model the transmission of rays both across the surface and through the interior of a complex perturbed shapes, cf. [9].

There is a strong analogy between the physical propagation of sound and light, both reflect of boundaries at incident angles that determine the paths taken by the energy. The technique employed by the ε-RayAE algorithm exploits this analogy through classical steps reminiscent of rendering:

- Generate a source of plain perturbed vector field (ε-PVF) to epresent external/ internal ε-AE. Filter ε-PVF, keeping only the vectors (i.e. a ray representing the direction (ordered in weak-sense) the sound will travel along) that lie on the surface, or fully inside the model. Each of these vectors is the start of an ε-acoustic energy path (ε-AEP).
- For each ε-AEP, generate a series of weak edge segments that represent perturbed reflection/transmission of the ε-AE ray. Essentially, this maps the propagation of the ε-AE wave through and/or over the body. Each edge segment in an ε-AEP is referred to as an ε-AES.
- Create a representation of the sensor/sensors.
- For each sensor location test all ε-AEPs and record the number of times an single ε-AES of the ε-AEP intersects the sensor.

8. Example

Consider a room with internal dimensions 10×5, suppose an isotropic 30° sector-directional sound source placed in the left bottom corner. In the room there are 3 protective acoustic screens and an internal construction element, see Fig. 1. All dimensions, locations and absorbent coefficients of walls, screens and internal construction can be perturbed. The values of perturbed sound pressure level in any point of the room can be calculated in the sense of previous methodology. The main values of randomly perturbed absorbent coefficients of all walls, screens and internal construction are known as deterministic values. All numerical values are dimensionless.

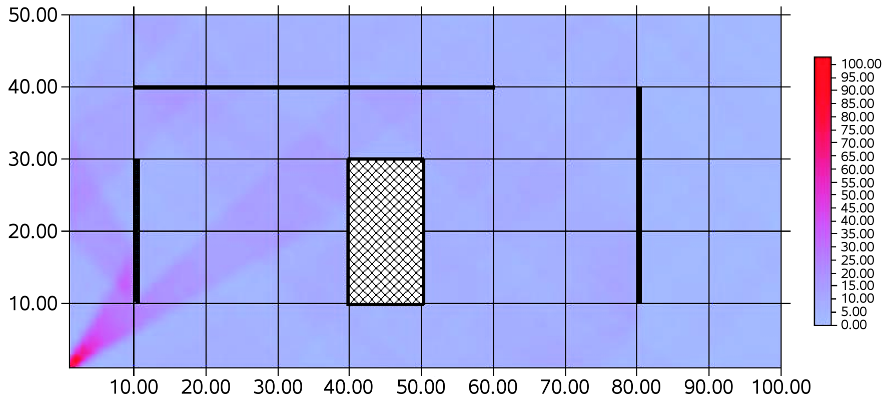

Fig. 1. The main value of perturbed sound pressure level for absorbent coefficients $= 0.1$.

9. Conclusions

Calculations with use of a new perturbation differential calculus to applications are mathematically equivalent with I-order approximations in classical perturbation methods.

Advantages of the presented algebraic system are as follows:

- we can omit all complex analytical calculations typical for expanding approximated values of solutions in infinite series. It works for expanding unknown values – solutions as well as for perturbed coefficients of the problem;
- we get a great simplification of all calculations in mathematical analysis area which appear in analytical formulation and analysis of the problem;
- most of known classical results of the theory of differential equations can be simply adapted for the new system of calculations without any serious difficulties.

With the modified algebraic system we get a set of very simple and useful mathematical tools which can be easy used in analytical and computational parts of analysis of complex perturbation differential problems.

Examples of applications for perturbation formulation in classical problems of vibrations described by wave equations with perturbed coefficients and initial and boundary conditions are given [15, 16].

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