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THE GREEN FUNCTION FOR THE NEUMANN BOUNDARY VALUE PROBLEM AT THE SEMIINFINITE CYLINDER AND THE FLAT INFINITE BAFFLE

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This paper presents the Green function for the Neumann boundary value problem at the rigid semiinfinite cylinder and the flat rigid infinite baffle. The function has been expressed as the combination of the Hankel functions of the first and second kinds, and their derivatives – all n-th order. The acoustic pressure has been presented in the two cases: the time harmonic vibrations of a sector cylinder piston located on the semiinfinite cylinder, and the annular sector piston located on the flat baffle in the vicinity of the semiinfinite cylinder.

Keywords: sound pressure, Green function, Neumann boundary value problem.

1. Introduction

An expression for the acoustic impedance of a radiating system consisiting of a source located on the surface of an infinitely long and rigid circular cylinder has been given by ROBEY [1]. The author used the Green function. The acoustic pressure and the mutual radiation impedance of a rectangular pistons located on the surface of a rigid and infinitely long circular cylinder has been presented by GREENSPON and SHERMAN [2]. The sound pressure distribution located of the source in the vicinity of the baffle and at the flat surface has been obtained using the Neumann boundary conditions in the polar coordinates. The velocity potential of a convex infinitely long pulsating cylinder has been presented by Leppington for the short waves [3]. The impedance of the two pulsating cylindrical rings located on an infinite rigid circular cylinder has been analyzed by RDZANEK [4–8]. So far, the Green function has not been used for any analytical computations of the acoustic pressure of a sector cylinder piston located on the semiinfinite cylinder, and of an annular sector piston located on the flat baffle in the vicinity of the semiinfinite cylinder. This paper deals with this problem.

2. The Green function

The Neumann boundary value problem has been defined in the polar coordinates (r, φ, z) for the region $\Omega = \{a \leq r < \infty;\ 0 \leq z < \infty;\ 0 \leq \varphi \leq 2\pi\}$, consisting of the rigid cylinder of radius a and the rigid plane at $z = 0$ (cf. Fig. 1). The following boundary condition $\frac{\partial}{\partial n} G(\mathbf{r} \,|\, \mathbf{r}_0)\big|_{S_\Omega} = 0$, satisfied at the surface S_Ω bounding the region Ω, can be expressed as

$$\frac{\partial}{\partial r} G(\mathbf{r} \,|\, \mathbf{r}_0)\bigg|_{r=a} = 0, \qquad \frac{\partial}{\partial z} G(\mathbf{r} \,|\, \mathbf{r}_0)\bigg|_{z=0} = 0, \tag{1}$$

where $G(\mathbf{r} \,|\, \mathbf{r}_0)$ is the Green function in the region Ω, $\mathbf{r} = (r, \varphi, z)$ and $\mathbf{r}_0 = (r_0, \varphi_0, z_0)$ are the leading vectors of the field point and of the source point, respectively. The steady state processes have been considered and the time dependence $\exp(-i\omega t)$ has been accepted herein. The following wave equation is satisfied within the whole region Ω

$$(\Delta + k^2)\, G(\mathbf{r} \,|\, \mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0), \tag{2}$$

where $\Delta = (1/r)(\partial/\partial r)\,(r\partial/\partial r) + (1/r^2)(\partial^2/\partial \varphi^2) + (\partial^2/\partial z^2)$ and $\delta(\mathbf{r} - \mathbf{r}_0) = (1/r)\delta(r - r_0)\delta(\varphi - \varphi_0)\delta(z - z_0)$.

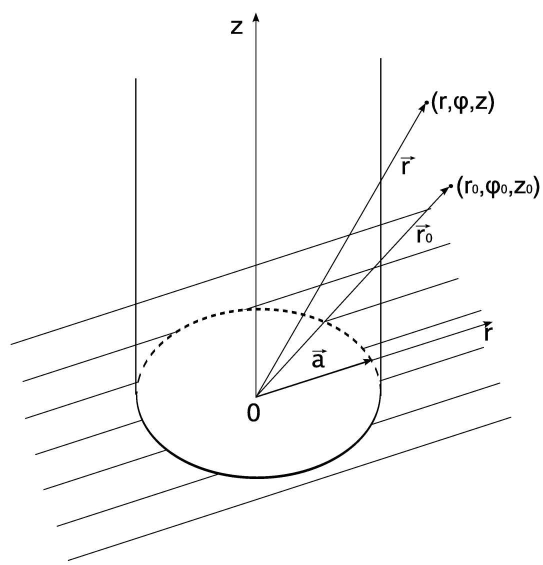

Fig. 1. The semiinfinite rigid cylinder and the flat infinite rigid baffle in the polar coordinates.

The Green function has been expressed as the Fourier series with respect to the angle variable φ

$$G(\mathbf{r} \,|\, \mathbf{r}_0) = \sum_{n=-\infty}^{+\infty} \exp(in\varphi)\, G_n(r, z \,|\, r_0, z_0)\,. \tag{3}$$

Further it has been inserted into Eq. (2) multiplied side by side by the factor $(1/2\pi)\int_0^{2\pi}$ $\exp(-im\varphi)\,\mathrm{d}\varphi$ and the following equations have been used $\int_0^{2\pi}\exp[i\,(n-m)\varphi]\,\mathrm{d}\varphi = 2\pi\delta_{nm}$ and $\int_0^{2\pi}\delta(\varphi-\varphi_0)\exp(-im\varphi)\,\mathrm{d}\varphi = \exp(-im\varphi_0)$ to give

$$\left(\Delta_{r,z} + k^2 - \frac{n^2}{r^2}\right) G_n(r,z\,|\,r_0,z_0) = -\frac{\delta(r-r_0)}{2\pi r}\,\delta(z-z_0)\,\exp(-in\varphi_0), \qquad (4)$$

where $\Delta_{r,z} = (1/r)(\partial/\partial r)\,(r\partial/\partial r) + \partial^2/\partial z^2$ and δ_{nm} is the Kronecker delta. The following Fourier cosine transforms have been used

$$\begin{aligned} G_n(r,z\,|\,r_0,z_0) &= \sqrt{\frac{2}{\pi}}\int\limits_0^\infty G_n(r\,|\,r_0)\cos\zeta z\,\mathrm{d}\zeta, \\ G_n(r\,|\,r_0) &= \sqrt{\frac{2}{\pi}}\int\limits_0^\infty G_n(r,z\,|\,r_0,z_0)\cos\zeta z\,\mathrm{d}z \end{aligned} \qquad (5)$$

to satisfy the Neumann boundary conditions at the boundary S_Ω for $z=0$ (cf. Eq. (1)). Equation (5) has been inserted into Eq. (4), multiplied side by side by the factor $\sqrt{2/\pi}$ $\int_0^\infty \cos\zeta_0 z\,\mathrm{d}z$, and the following equations $\int_0^\infty \cos\zeta z\cos\zeta z_0\,\mathrm{d}z = (\pi/2)\,\delta(\zeta-\zeta_0)$ and $\int_0^\infty \delta(z-z_0)\cos\zeta z\,\mathrm{d}z = \cos\zeta z_0$ have been used to give

$$\left(\Delta_{r,z} + \gamma^2 - \frac{n^2}{r^2}\right) G_n(r\,|\,r_0) = -\sqrt{\frac{2}{\pi}}\,\frac{\delta(r-r_0)}{2\pi r}\,e^{-in\varphi_0}\,\cos\zeta z_0\,, \qquad (6)$$

where $\Delta_r = (1/r)(\mathrm{d}/\mathrm{d}r)\,(r\mathrm{d}/\mathrm{d}r)$ and $\gamma^2 = k^2-\zeta^2$. The Green function used in Eq. (6) assumes the form of a plane wave propagated in the $0z$ direction

$$\begin{aligned} G_{n1}(r\,|\,r_0) &= A_{n1}H_n^{(1)}(\gamma r) + B_{n1}H_n^{(2)}(\gamma r) \qquad \text{for } a \le r \le r_0 < +\infty, \\ G_{n2}(r\,|\,r_0) &= A_{n2}H_n^{(1)}(\gamma r) + B_{n2}H_n^{(2)}(\gamma r) \qquad \text{for } a \le r_0 \le r < +\infty, \end{aligned} \qquad (7)$$

where $H_n^{(1)}$ and $H_n^{(2)}$ are the Hankel functions of the first and second kinds.

The "sharpened Sommerfeld radiation condition" must be satisfied within the region Ω. Therefore, the acoustic waves can disperse the growing values of the variable r within the region $r \le r_0$, and consequently $B_{n2} = 0$. One of the boundary conditions in Eq. (1) has been expressed as

$$\left.\frac{\mathrm{d}}{\mathrm{d}r}G_n(r\,|\,r_0)\right|_{r=a} = 0 \qquad (8)$$

using some similar computations as in the case of the wave equation. The solution (7) has been inserted into Eq. (8) which gives in $A_{n1} = -B_{n1}H_n^{(2)\prime}(\gamma a)/H_n^{(1)\prime}(\gamma a)$ and

$$\begin{aligned} G_{n1}(r\,|\,r_0) &= B_{n1}\left[H_n^{(1)\prime}(\gamma a)H_n^{(2)}(\gamma r) - H_n^{(2)\prime}(\gamma a)H_n^{(1)}(\gamma r)\right]/H_n^{(1)\prime}(\gamma a), \\ G_{n2}(r\,|\,r_0) &= A_{n2}H_n^{(1)}(\gamma r), \end{aligned} \qquad (9)$$

where the symbol "prime" represents differentiation by the argument.

The Green function is continues for all the values of z as well as for $z = z_0$ which implies that $G_{n1}(r \,|\, r_0) = G_{n2}(r \,|\, r_0)$ and gives

$$\begin{aligned} G_{n2}(r \,|\, r_0) \;&=\; B_{n1} \, \frac{H_n^{(1)}(\gamma r)}{H_n^{(1)\prime}(\gamma a) H_n^{(1)}(\gamma r_0)} \\ &\times \Big[H_n^{(1)\prime}(\gamma a) H_n^{(2)}(\gamma r_0) - H_n^{(2)\prime}(\gamma a) H_n^{(1)}(\gamma r_0) \Big]. \end{aligned} \tag{10}$$

Equation (6) has been multiplied side by side by r, integrated within the limits $r \in [r_0 - \epsilon; r_0 + \epsilon]$ and the limit $\epsilon \to 0$ has been computed to give

$$\frac{\mathrm{d}}{\mathrm{d}r} G_{n2}(r \,|\, r_0) \Big|_{r=r_0} - \frac{\mathrm{d}}{\mathrm{d}r} G_{n1}(r \,|\, r_0) \Big|_{r=r_0} = -\sqrt{\frac{2}{\pi}} \, \frac{1}{2\pi r_0} \, e^{-in\varphi_0} \cos \zeta z_0 \tag{11}$$

and consequently

$$\begin{aligned} G_{n1}(r \,|\, r_0) \;&=\; \frac{e^{-in\varphi_0}}{2\sqrt{2\pi}} \cos \zeta z_0 \frac{H_n^{(1)}(\gamma r_0)}{H_n^{(1)\prime}(\gamma a)} \left[J_n'(\gamma a) Y_n(\gamma r) - Y_n'(\gamma a) J_n(\gamma r) \right], \\ G_{n2}(r \,|\, r_0) \;&=\; \frac{e^{-in\varphi_0}}{2\sqrt{2\pi}} \cos \zeta z_0 \frac{H_n^{(1)}(\gamma r)}{H_n^{(1)\prime}(\gamma a)} \left[J_n'(\gamma a) Y_n(\gamma r_0) - Y_n'(\gamma a) J_n(\gamma r_0) \right], \end{aligned} \tag{12}$$

where $J_n(\gamma r)$, $J_n'(\gamma a)$, $Y_n(\gamma r)$, $Y_n'(\gamma a)$ are the Bessel and Neuman n-th order functions and their derivatives.

The solution (12) has been inserted into Eq. $(5)_1$, and then to Eq. (3) and the Green function has been expressed as the Fourier series

$$G_{n1}(\mathbf{r} \,|\, \mathbf{r}_0) = \sum_{n=-\infty}^{+\infty} e^{in(\varphi - \varphi_0)} G_n(r, z \,|\, r_0, z_0) \tag{13}$$

with coefficients

$$\begin{aligned} G_{n1}(r, z \,|\, r_0, z_0) \;&=\; \frac{1}{2\pi} \int_0^{+\infty} \cos \zeta z \cos \zeta z_0 \, \frac{H_n^{(1)}(\gamma r_0)}{H_n^{(1)\prime}(\gamma a)} \\ &\times \left[J_n'(\gamma a) Y_n(\gamma r) - Y_n'(\gamma a) J_n(\gamma r) \right] \mathrm{d}\zeta \qquad \text{for } \; a \le r \le r_0 < +\infty, \\ G_{n2}(r, z \,|\, r_0, z_0) \;&=\; \frac{1}{2\pi} \int_0^{+\infty} \cos \zeta z \cos \zeta z_0 \, \frac{H_n^{(1)}(\gamma r)}{H_n^{(1)\prime}(\gamma a)} \\ &\times \left[J_n'(\gamma a) Y_n(\gamma r_0) - Y_n'(\gamma a) J_n(\gamma r_0) \right] \mathrm{d}\zeta \qquad \text{for } \; a \le r_0 \le r < +\infty. \end{aligned} \tag{14}$$

Since $G_{-n}(r, z \,|\, r_0, z_0) = G_n(r, z \,|\, r_0, z_0)$ the Green function can be expressed in its equivalent form

$$G(\mathbf{r} \,|\, \mathbf{r}_0) = \sum_{n=0}^{+\infty} \varepsilon_n \cos n(\varphi - \varphi_0) G_n(r, z \,|\, r_0, z_0), \tag{15}$$

where $\varepsilon_n = 1$ for $n = 0$ and $\varepsilon_n = 2$ for $n > 0$.

3. Acoustic pressure

From the practical viewpoint the following two cases on time harmonic vibrations are the most interesting: the vibrating sector piston located on the semiinfinite cylinder, and the vibrating annular sector piston located on the flat baffle in the vicinity of the semiinfinite cylinder.

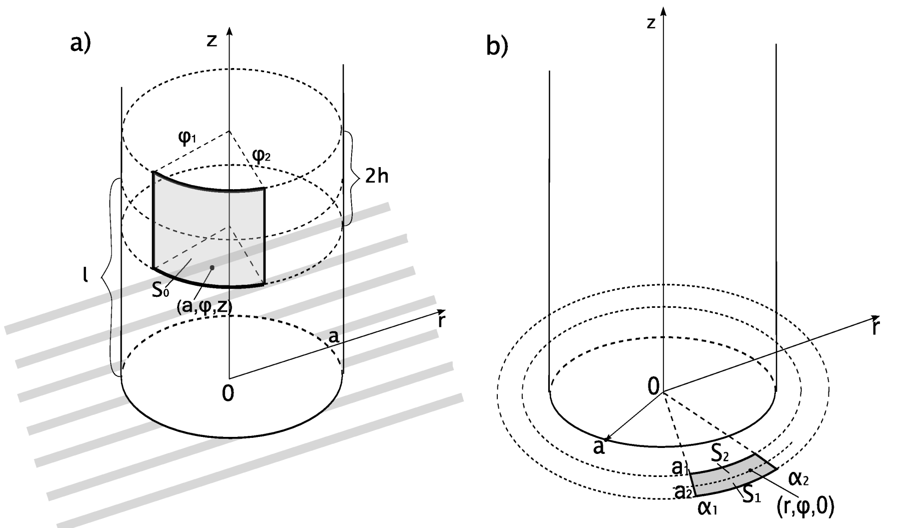

Fig. 2. The acoustic source located: a) on the surface of the semiinfinite cylinder and b) on the surface of the flat baffle.

The relation between the acoustic pressure $p(\mathbf{r},t) = p(\mathbf{r})e^{-i\omega t}$ and the acoustic potential $\phi(\mathbf{r},t) = \phi(\mathbf{r})e^{-i\omega t}$

$$
\begin{aligned}
p(\mathbf{r},t) &= \varrho_0 \frac{\partial}{\partial t}\phi(\mathbf{r},t) = -i\omega\rho_0\phi(\mathbf{r},t), \\
p(\mathbf{r}) &= -i\omega\rho_0\phi(\mathbf{r}), \qquad \phi(\mathbf{r}) = \int\limits_{S_0} v_N(\mathbf{r}_0)\, G(\mathbf{r}\,|\,\mathbf{r}_0)\, \mathrm{d}S_0
\end{aligned}
\tag{16}
$$

has been used where $v_N(\mathbf{r}_0)$ is the normal component of the vibration velocity of the source, S_0 is its surface.

In the case when the source has been placed on the semiinfinite cylinder, the sound pressure is (cf. Fig. 2a)

$$
p_0(\mathbf{r}) = -i\omega\varrho_0 \int\limits_{S_0} v_N(\mathbf{r}_0) G_2(\mathbf{r}\,|\,\mathbf{r}_0)\, \mathrm{d}S_0 \tag{17}
$$

for $a \leq r$.

In the case when the source is located on the flat baffle in the vicinity of the semiinfinite cylinder, the sound pressure is (cf. Fig. 2b)

$$p_{1,2}(\mathbf{r}) = -i\omega\varrho_0 \begin{cases} \displaystyle\int_{S_1} v_N(\mathbf{r}_0)\, G_1(\mathbf{r}\,|\,\mathbf{r}_0)\, \mathrm{d}S_1 & \text{for } r < a_1\,, \\ \displaystyle\int_{S_1} v_N(\mathbf{r}_0)\, G_1(\mathbf{r}\,|\,\mathbf{r}_0)\, \mathrm{d}S_1 & \\ \qquad + \displaystyle\int_{S_2} v_N(\mathbf{r}_0)\, G_2(\mathbf{r}\,|\,\mathbf{r}_0)\, \mathrm{d}S_2 & \text{for } a_1 \le r < a_2\,, \\ \displaystyle\int_{S_2} v_N(\mathbf{r}_0)\, G_2(\mathbf{r}\,|\,\mathbf{r}_0)\, \mathrm{d}S_2 & \text{for } a_2 \le r. \end{cases} \tag{18}$$

It is worth noticing that the following Wronskian $H_n^{(2)\prime}(x)H_n^{(1)}(x) - H_n^{(1)\prime}(x)H_n^{(2)}(x) = -4i/\pi x$ [9] can be useful while using Eq. (17).

4. Concluding remarks

The Green function for the semiinfinite cylinder and the flat infinite baffle have been presented in its Fourier representations. It is useful for some further computations of the acoustic pressure radiated by some vibrating source situated on the baffle. The results presented herein can also be useful for some computations of the active and reactive acoustic power radiated by such source or by a system of such sorces.

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