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WAVES IN DUCTS DESCRIBED BY MEANS OF POTENTIALS

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The waveguide theory constitutes an important part of classical wave theory and the results are applied in many practical devices. The aim of the paper is to present a coherent approach to the problem of propagation of sound and electromagnetic waves in ducts by means of potentials. The potential concept simplifies the field description and discussion of some basic conditions, posed on mathematical expressions, which come from their physical meaning. These conditions are, for example, the Euler equation for sound waves or the Lorentz gauge condition for electromagnetic field. The analyze of the respected field is carried out by means of velocity potential for sound waves and the Hertz potentials for electromagnetic waves. The acousto-electromagnetic analogies are derived and discussed for infinite circular and rectangular duct. The cut-off frequencies, the waveguide impedance and the power transmitted along the duct is discussed in reference to these analogies.

Keywords: waveguide theory, sound and electromagnetic waves potentials, acousto-electromagnetic analogies, waveguide impedance.

1. Introduction

The fundamental ideas of wave motion are common for all kinds of classical waves and are rooted in electromagnetic elementary forces, manifesting themselves also in such features of material media as viscosity, Young modulus or flexibility which, in turn, determine the elastic waves proprieties.

Longitudinal sound waves propagating in liquids are characterized by fluctuation of pressure, called acoustic pressure and particle velocity (displacement velocity). At first sight they have not much in common with electromagnetic transverse waves described by a pair of electromagnetic field vectors – the electric field intensity $\mathbf{E}$ and the magnetic induction $\mathbf{B}$. Despite that, the unified description of the wave field of both kinds could be obtained by means of potentials.

Analogies play a fundamental role in physics and are governed by the principle: the same mathematical equations have the same solutions, together with additional conditions coming from physical meaning posed on them. The field on which the method of analogies could be successfully applied is the waveguide theory, which constitutes, because of its practical meaning, an important part of classical wave theory. Chapters concerning waveguides could be found in classical textbooks [1, 2], some of which are entirely devoted to waveguides, limiting the discussed subjects strictly to the waveguide theory [3–5].

Nowadays tendency in physics is to unify the study of waves by developing abstract and general features of wave motion of any kind [6, 7]. To meet these expectations the common description of sound and electromagnetic wave propagation in duct is presented. The solution of the wave equation and derivation of the boundary conditions, crucial for obtaining proper solution, will be shortly discussed. The solutions assuming absence of sources are sought.

Waveguides are structure which guide waves and are able to transmit wave energy at long distance. There are different types of waveguides, depending on the type of waves as sound waves, electromagnetic waves (light among them).

In acoustics, ducts of absolutely rigid (hard), absolutely soft and absorbing surface are discussed most often. The model of ideal surface is introduced to simplify the boundary conditions, even thought the obtained solutions provide a meaningful insight into real-duct propagation features. Acoustic rigid ducts are of particular interest, as elements of this kind are met frequently in acoustic devices (heating and ventilation systems, exhaust of jet engines etc.) and are widely considered in scientific papers, for example [8–10]. They are often sources of undesired and harmful noise and thus are subject of active and passive noise control methods [11–13].

Electromagnetic waveguides, transferring power or communication signals, are constructed depending on which portion of the electromagnetic spectrum they are supposed to transmit. They are constructed from conductive or dielectric (for optical frequencies) materials, the last guide optical waves by total internal reflection. In the following ideal conductive ducts, applied as power transferring devices, will be considered.

2. Field potentials in infinite duct

2.1. Acoustic potential in soft and rigid ducts – Dirichlet and Neumann boundary condition

The acoustic field can be considered as the field of acoustic pressure $p(\mathbf{r},t)$ or the acoustic velocity $\mathbf{v}(\mathbf{r},t)$, governed, in the frame of linear theory, by the following formulae [14]

- Euler's equation

$$\nabla p + \rho_0 \frac{\partial \mathbf{v}}{\partial t} = 0, \qquad \rho_0 - \text{mean medium density}, \tag{1}$$

- equation of continuity

$$\nabla \cdot \mathbf{v} + \frac{1}{\rho_0 c^2} \frac{\partial p}{\partial t} = 0, \qquad c - \text{wave velocity}, \tag{2}$$

- state equation

$$\frac{\partial p}{\partial \rho} = c^2, \tag{3}$$

where ∇ means nabla operator.

Euler's equation (1) will become identity if a new quantity $\Phi(\mathbf{r}, t)$, called the velocity potential, is introduced as follows

$$p = \rho_0 \frac{\partial \Phi}{\partial t}, \tag{4}$$

$$\mathbf{v} = -\nabla \Phi. \tag{5}$$

As a result the equation of continuity takes the form of wave equation

$$\triangle \Phi(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \Phi(\mathbf{r}, t)}{\partial t^2} = 0, \tag{6}$$

which, for harmonic excitation of a given angular frequency ω, with time depending factor $e^{-i\omega t}$, turns into Helmholtz equation

$$\triangle \Phi(\mathbf{r}, t) + k^2 \Phi(\mathbf{r}, t) = 0, \qquad k = \omega / c - \text{wave number}, \tag{7}$$

where $\triangle$ is the Laplace operator.

In acoustics, two idealized types of surfaces Σ are considered – the soft surface on which acoustic pressure equals zero $p(\mathbf{r}, t)|_{\Sigma} = 0$, what leads to the boundary condition, known as Dirichlet condition and often referred to as a first-type boundary condition

$$\Phi(\mathbf{r}, t)|_{\Sigma} = 0, \tag{8}$$

and the rigid surface, on which normal component of particle velocity equals zero $v_n|_{\Sigma} = 0$, thus the appropriate boundary condition, called Neumann condition or second-type boundary condition, is given by

$$\left. \frac{\partial \Phi}{\partial n} \right|_{\Sigma} = 0. \tag{9}$$

Assuming that a sound wave of axial wave number γ propagates in an infinite duct stretched along z-axis, its velocity potential

$$\Phi(\mathbf{r}, t) = \Phi(x, y) e^{i(\gamma z - \omega t)}, \tag{10}$$

fulfils the eigenvalue differential equation

$$\left[\triangle(x, y) + \beta^2\right] \Phi(x, y) = 0, \tag{11}$$

where

$$\beta^2 = k^2 - \gamma^2. \tag{12}$$

The solution for a rectangular or circular duct of soft or rigid surface is well known and can be found easily in many textbooks on differential equations, electromagnetism, theoretical acoustics and others. The method usually applied to receive the solution is the variable separation method. As a result the set of eigenvalues β and eigenfunctions

Φ is obtained. Their physical meaning is, respectively, radial wave number of a mode propagating along the duct and its velocity potential.

If one neglects wave absorption, the wave number k is real, what leads to real value of $\gamma = \sqrt{k^2 - \beta^2}$ if $k^2 > \beta^2$, otherwise γ is imaginary and $e^{i(\gamma z - \omega t)} = e^{-|\gamma| z} e^{-i\omega t}$ represents term of solution diminishing with distance exponentially. The limit between propagating and evanescent modes, $k = \beta$, determines the so-called cut-off angular frequency ω_n^{cr}, above which the mode of given radial wave number β_n propagates in a duct without damping

$$\omega_n^{\mathrm{cr}} = \beta_n c. \tag{13}$$

For a duct excited with frequency ω there is a limited set of eigenvalues, representing propagating modes, depending on duct dimensions – radius for cylindrical and length of sides for rectangular duct. The mode corresponding to the smallest eigenvalue β is called the fundamental (principal) mode. For a rigid circular duct it is plane wave with the cut-off frequency equal to zero.

The eigenfunctions corresponding to different eigenvalues are orthogonal, what simplifies expressions for the impedance and power transmitted in the case of multimodal excitation, when more than one mode propagates along the duct and the potential takes the form of the sum of potentials of cut-on modes. The cut-on modes are modes with cut-off frequencies less than the excitation frequency.

2.1.1. Soft and rigid cylindrical ducts

Geometry of the system is presented in Fig. 1 – the duct of radius a is aligned along the z axis. Accounting for the duct symmetry, the solution of the Helmholtz equation is expressed in cylindrical coordinates (ϱ, φ, z) [15]

$$\Phi_m(\varrho, \varphi) = a_m e^{im\varphi} J_m(\beta \varrho), \tag{14}$$

where $J_m(\ )$ stands for Bessel function of order m and a_m is complex amplitude.

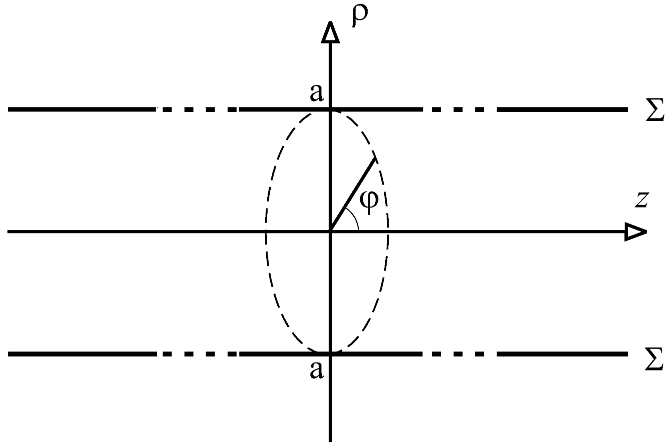

Fig. 1. Geometry of an infinite cylindrical duct in (ϱ, φ, z) coordinates.

For a rigid duct the Neumann boundary condition

$$\left.\frac{\partial \Phi}{\partial n}\right|_{\Sigma} = -\left.\frac{\partial \Phi}{\partial \varrho}\right|_{\varrho=a} = 0, \tag{15}$$

leads to

$$\left.\frac{\mathrm{d}J_m(\beta\varrho)}{\mathrm{d}\varrho}\right|_{\varrho=a} = J'_m(\beta a) = 0. \tag{16}$$

Denoting $J'_m(\mu_{mn}) = 0$, where μ_{mn} is the n-th root of $J'_m(\ )$, the radial wave numbers and the cut-off frequencies are:

$$\beta_{mn} = \frac{\mu_{mn}}{a}\,, \qquad \omega^{\mathrm{cr}}_{mn} = \frac{\mu_{mn}c}{a}\,, \tag{17}$$

thus the corresponding velocity potential, which is finite on the duct axis [14] is given by

$$\Phi_{mn}(\varrho,\varphi,z,t) = a_{mn}e^{im\varphi}J_m\left(\frac{\mu_{mn}\varrho}{a}\right)e^{i(\gamma_{mn}z-\omega t},\tag{18}$$

where $\gamma_{mn} = \sqrt{k^2 - \left(\frac{\mu_{mn}}{a}\right)^2}$ and a_{mn} is the complex amplitude. The (m,n) subscripts determine the asimuthal (circumferential) and radial order of a particular mode. In a soft duct of radius a the boundary condition of Dirichlet type $\widetilde{\Phi}|_{\Sigma} = 0$ leads to the equation $J_m(\tilde{\beta}a) = 0$ and so, if ν_{mn} denotes the n-th zero of $J_m(\ )$, the radial wave numbers and the cut-off frequencies are given by

$$\tilde{\beta}_{mn} = \frac{\nu_{mn}}{a}\,, \qquad \widetilde{\omega}^{\mathrm{cr}}_{mn} = \frac{\nu_{mn}c}{a}\,. \tag{19}$$

Hereafter quantities concerning soft duct are distinguished by tilde sign. Finally, the velocity potential of mode (m,n) takes the form

$$\widetilde{\Phi}_{mn}(\varrho,\varphi,z,t) = a_{mn}e^{im\varphi}J_m\left(\frac{\nu_{mn}\varrho}{a}\right)e^{i(\tilde{\gamma}_{mn}z-\omega t)},\tag{20}$$

where the axial wave number

$$\tilde{\gamma}_{mn} = \sqrt{k^2 - \left(\frac{\nu_{mn}}{a}\right)^2}\,. \tag{21}$$

Modes for which $m = 0$ are axis-symmetrical and called radial modes, their velocity potential, considered on the duct cross-section, takes zero value at circles. The modes with $m \neq 0$ are called circumferential and their potential is zero also on m diameters. Figure 2 presents schematically some of these modes. Some authors numerate symmetrical radial modes starting from $n = 0$, what depicts last figure.

The hard duct transfers wave of any frequency and the principal mode $(0,\,0)$ is the plane wave, contrary to the soft duct for which the cut-off frequency of the fundamental mode $(0,\,1)$ is given by $\widetilde{\omega}_{\min} = \widetilde{\omega}^{\mathrm{cr}}_{01} = 2.405\,c/a$. Thus the soft duct does not transfer any wave below that frequency.

Denoting N_c number of roots such that $0 \leq \mu_{ml} < ka$ or number of roots such that $0 < \nu_{ml} < ka$, N_c gives number of modes which can propagate in hard or soft duct

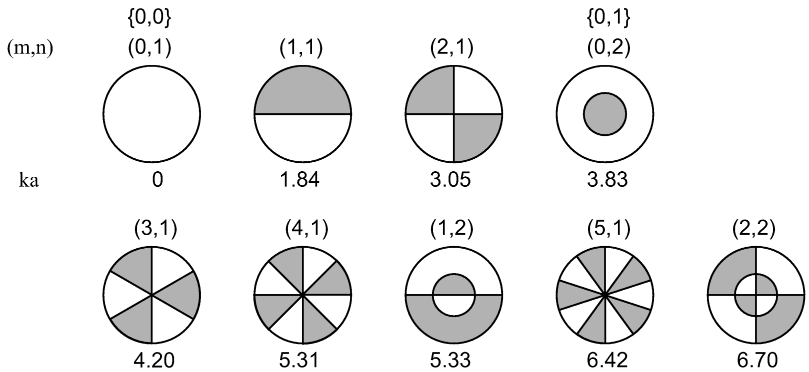

Fig. 2. Schematic presentation of radial and circumferential modes in acoustically hard cylindrical duct, also H-type modes in conducting duct.

without damping. From the above one sees that the dimensionless diffraction parameter ka is a suitable quantity for describing modes in duct of a given radius a. In literature, the diffraction parameter is also called Helmholtz parameter or duct reduced frequency $ka = \omega_{\mathrm{red}} = \omega a/c$.

In case of multimodal excitation [16], when apart from the principal mode some higher modes can propagate, the velocity potential is a summ of potentials expressed by (18) or (20)

$$\Phi(\varrho, \varphi, z, t) = \sum \Phi_{mn}(\varrho, \varphi, z, t), \tag{22}$$

summation covering all pairs of indices (m, n), corresponding to real values of axial wave numbers.

To summ up

- in a rigid cylindrical duct the plane wave, with the cut-off frequency equal to zero, is the principal mode,
- in a soft cylindrical duct the principal mode is the mode with $m = 0$, $n = 1$, thus only waves of angular frequencies ω greater than $2.405\, c/a$ are transferred along the duct.

2.1.2. Soft and rigid rectangular ducts

Consider a rectangular duct aligned along the z axis which walls are described in Cartesian coordinates as follows: $x = 0$, $0 \le y \le b$, $x = a$, $0 \le y \le b$, $y = 0$, $0 \le x \le a$, $y = b$, $0 \le x \le a$ and assume $a \ge b$.

Solutions of Helmholtz equation (7) obtained by means of variables separation method, with appropriate boundary condition – Neumann type (9) for rigid and Dirichlet type (8) for soft duct, are given by

$$\Phi_{mn}(x, y, z, t) \;=\; a_{mn} \cos \frac{m\pi x}{a}\, \cos \frac{n\pi y}{b} e^{i(\gamma_{mn} z - \omega t)}, \tag{23}$$

$$\widetilde{\Phi}_{mn}(x, y, z, t) \;=\; a_{mn} \sin \frac{m\pi x}{a}\, \sin \frac{n\pi y}{b} e^{i(\tilde{\gamma}_{mn} z - \omega t)}, \tag{24}$$

where

$$\beta_{mn} = \tilde{\beta}_{mn} \; = \; \pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}\,, \tag{25}$$

$$\gamma_{mn} = \tilde{\gamma}_{mn} \; = \; \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}\,. \tag{26}$$

For rectangular duct, rigid or soft, the axial wave numbers are the same, while velocity potentials are products of two even (cosine) or odd (sine) functions.

2.2. *Electromagnetic waves Hertz potentials in conducting duct*

Consider an infinite conducting duct filled with isotropic and homogenous medium transferring electromagnetic waves. To find the field inside the duct one can solve the set of Maxwell's equations [1]. As it is not an easy way to obtain the solution, scalar ϕ and vectorial $\mathbf{A}$ potentials of electromagnetic field were introduced, such that [1]

$$\mathbf{B} \; = \; \boldsymbol{\nabla} \times \mathbf{A}\,, \tag{27}$$

$$\mathbf{E} \; = \; -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}\,, \tag{28}$$

leading to the gauge Lorentz condition [1]

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0, \tag{29}$$

which ensures the $\mathbf{A}$ potential fulfils the wave equation.

Introducing vectorial potential $\mathbf{\Pi}^E$, called Hertz electric vector or Hertz electric potential, such that [1]

$$\begin{aligned} \text{(i)} \quad \mathbf{A} \; &= \; \frac{1}{c^2}\frac{\partial \mathbf{\Pi}^E}{\partial t}\,, \\ \text{(ii)} \quad \phi \; &= \; -\nabla \cdot \mathbf{\Pi}^E, \end{aligned} \tag{30}$$

the Lorentz gauge condition turns into identity, while the Hertz electric potential itself satisfies the wave equation.

Considering Maxwell's equations without sources (electric currents or charges) one can introduce dual potentials [1], called also antipotentials ϕ_*, A_*, for which

$$\mathbf{D} \; = \; -\nabla \times \mathbf{A}_*, \tag{31}$$

$$\mathbf{H} \; = \; -\nabla\phi_* - \frac{\partial \mathbf{A}_*}{\partial t}. \tag{32}$$

The Lorentz gauge condition

$$\nabla \cdot \mathbf{A}_* + \frac{1}{c^2}\frac{\partial \phi_*}{\partial t} = 0, \tag{33}$$

turns into identity, if a new quantity $\mathbf{\Pi}^H$, called the magnetic Hertz potential or the magnetic Hertz vector, is introduced in a following way

$$\begin{aligned} &\text{(i)} \quad \mathbf{A}_* = \frac{1}{c^2}\frac{\partial \mathbf{\Pi}^H}{\partial t}, \\ &\text{(ii)} \quad \phi_* = -\nabla \cdot \mathbf{\Pi}^H. \end{aligned} \tag{34}$$

The magnetic Hertz potential fulfills the wave equation.

The Hertz potentials, electric $\mathbf{\Pi}^E$ and magnetic $\mathbf{\Pi}^H$, are adequate for expressing the electromagnetic field in limited area, such as ducts. It results from the following features of the electromagnetic field in duct [19]:

- the electromagnetic wave field in duct is not transversal, but has also longitudinal component, along the duct axis,
- any wave in duct could be expressed as a superposition of TM (Transversal Magnetic) and TE (Transversal Electric) waves,
- TM waves, called also E-waves, with longitudinal component of the electric field $\mathbf{E}$ nonequal zero are described by the electric Hertz potential $\mathbf{\Pi}^E = [0,\, 0,\, \Pi^E]$,
- TE waves, called also H-waves, with longitudinal component of the magnetic field $\mathbf{H}$ nonequal zero are described by the magnetic Hertz potential $\mathbf{\Pi}^H = [0,\, 0,\, \Pi^H]$.

According to (27), (28) and (30), the electromagnetic field vectors of the E-wave (TM) are equal to

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{\Pi}^E}{\partial t^2} + \nabla(\nabla \cdot \mathbf{\Pi}^E), \tag{35}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{c^2}\nabla \times \frac{\partial \mathbf{\Pi}^E}{\partial t}, \tag{36}$$

while the adequate boundary condition for conducting surface, derived from the Maxwell equations [1], takes the form

$$\Pi^E|_{\Sigma} = 0, \tag{37}$$

and is identical with the boundary condition in a soft acoustic duct (8).

The electromagnetic field vectors **D**, **H** of the H-wave (TE), expressed by means of the magnetic Hertz potential $\mathbf{\Pi}^{\mathbf{H}}$, according to (31), (32) and (34), are given by

$$\mathbf{D} = -\nabla \times \mathbf{A}_* = -\frac{1}{c^2}\nabla \times \frac{\partial \mathbf{\Pi}^H}{\partial t}, \tag{38}$$

$$\mathbf{H} = -\nabla\phi_* - \frac{\partial \mathbf{A}_*}{\partial t} = \nabla\left(\nabla \cdot \mathbf{\Pi}^H\right) - \frac{1}{c^2}\frac{\partial^2 \mathbf{\Pi}^H}{\partial t^2}, \tag{39}$$

with the boundary condition on conducting surface

$$\left.\frac{\partial \Pi^H}{\partial n}\right|_{\Sigma} = 0, \tag{40}$$

in which one can recognize the boundary condition for acoustic waves in a hard duct (9).

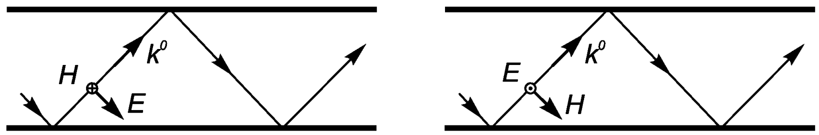

Fig. 3. Electromagnetic waves in cylindrical duct: TM (Transversal Magnetic) waves, called also E-waves and TE (Transversal Electric), called also H-waves.

Thus, the solutions obtained just far for hard and soft ducts are, in parallel, solutions for electromagnetic waves of both types. The electric Hertz potential Π^E_{mn}, corresponding to the wave E_{mn} has the form of acoustic potential $\widetilde{\Phi}_{mn}$ of the mode (m,n) in a soft duct, while the magnetic Hertz potential Π^H_{mn} corresponding to the wave H_{mn} has the form of acoustic potential Φ_{mn} of the mode (m,n) in a hard duct, what allows to transpose the solutions obtained for acoustic wave modes onto electromagnetic ones.

The derived acousto-electromagnetic analogies are as follows

$$\Pi^E_{mn} \leftrightarrow \widetilde{\Phi}_{mn}, \qquad \omega^{E,\mathrm{cr}}_{mn} = \widetilde{\omega}^{\mathrm{cr}}_{mn}, \tag{41}$$

$$\Pi^H_{mn} \leftrightarrow \Phi_{mn}, \qquad \omega^{H,\mathrm{cr}}_{mn} = \omega^{\mathrm{cr}}_{mn}, \qquad \text{exept for } \ \omega^{\mathrm{cr}}_{00}\,. \tag{42}$$

It should be emphasized that the analogy between description of acoustic and electromagnetic waves in infinite ducts occurs on a level of potentials – acoustic velocity potential in soft or rigid duct and Hertz potential of electric or magnetic type in conducting

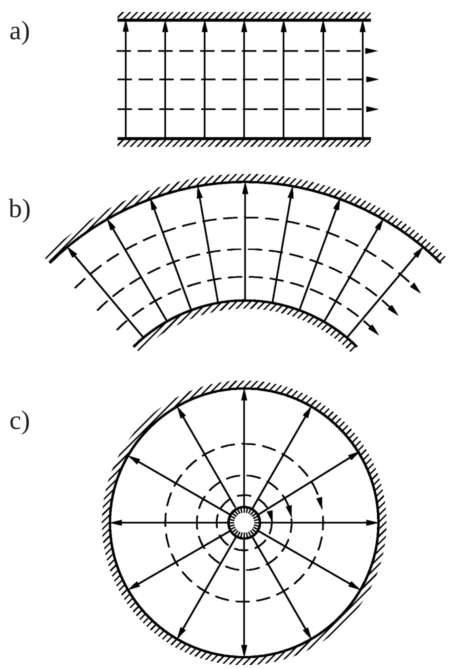

Fig. 4. Structures, in which the electromagnetic plane wave can propagate: parallel planes (a), pair of cylinders (b), transmission line (c).

duct. Waves of both types differ substantially – the acoustic wave is described by a scalar function of acoustic pressure p, while the electromagnetic wave is defined by field intensity vectors $\mathbf{E}$, $\mathbf{H}$. The most important difference is, that contrary to the plane acoustic wave in a hard duct, the electromagnetic plane wave do not propagates in a conducting duct (the corresponding mode potential gives zero value of electromagnetic field).

In cylindrical duct the H_{11} mode is the principal mode between all waves (TM and TE) and has the lowest cut-off frequency $\omega_{\min}^{\mathrm{cr}} = 1.84c/a$, so waves for which $\lambda/2a < 1.71$ are transfered along the duct. Between TM (E-type) waves the E_{01} mode has the lowest cut-off frequency $\omega_{\min}^{E,\mathrm{cr}} = 2.41c/a$, what gives $\lambda/2a < 1.31$.

In rectangular duct the E_{0n} and the E_{n0} waves do not exist (compare Eq. (24)), the fundamental mode of all waves is the H_{10} mode (assuming $a > b$), with critical frequency $\omega_{\min}^{\mathrm{cr}} = \pi c/a$ and critical wavelength $\lambda^{\mathrm{cr}} = 2a$. The lowest critical frequency between E-waves has the E_{11} wave.

The electric Hertz potential is connected with polarization [1]. For electric dipole of momentum $\mathbf{p}$ it is equal

$$\mathbf{\Pi}^E(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\,\mathbf{p}\,\frac{e^{i(kR-\omega t)}}{R}\,, \tag{43}$$

where R means distance between the field point and the center of a dipole.

The magnetic Hertz potential $\mathbf{\Pi}^H$ is connected with magnetization of a medium [1]. For the magnetic dipole of momentum $\mathbf{m}$ it is equal to

$$\mathbf{\Pi}^H(\mathbf{r},t) = \frac{\mu_0}{4\pi}\,\mathbf{m}\,\frac{e^{i(kR-\omega t)}}{R}\,. \tag{44}$$

3. Further acousto-electromagnetic analogies

Below some other analogies, useful in depicting fields of both kinds in ducts are developed. They refer to decomposition of duct wave into plane waves, impedance concept and power transmission.

3.1. Field potential as a superposition of plane waves. Group and phase velocities

In describing some physical phenomena, as for instance the wave reflection at a boundry between two media, it is convenient to depict potential of a circumferential mode as a summ of plane waves and apply formulae governing reflection of a plane wave.

If β_{mn} means the radial wave number in a circular duct, the (m,n) mode potential takes the form

$$\left.\begin{array}{c} \mathit{\Pi}(x,y) \\ \mathit{\Phi}(x,y) \end{array}\right\} = \int_0^{2\pi} f(\alpha)\, e^{i\beta_{mn}(x\cos\alpha + y\sin\alpha)}\,\mathrm{d}\alpha, \tag{45}$$

where

$$f(\alpha) = 2\pi i^m\, e^{im\alpha}, \tag{46}$$

or, in the cylindrical co-ordinates $x = \varrho\cos\varphi$, $y = \varrho\sin\varphi$,

$$\left.\begin{array}{c} \Pi(\varrho,\varphi) \\ \Phi(\varrho,\varphi) \end{array}\right\} = 2\pi i^m \int_0^{2\pi} e^{im\alpha}\, e^{i\beta_{mn}\varrho\cos(\alpha-\varphi)}\,\mathrm{d}\alpha\,. \tag{47}$$

Application of the integral form of the Bessel function [20]

$$J_m(x) = \frac{i^{-m}}{2\pi}\int_0^{2\pi} e^{im\alpha}\, e^{ix\cos\alpha}\,\mathrm{d}\alpha\,, \tag{48}$$

leads to expression for the (m,n) mode potential

$$\left.\begin{array}{c} \Pi(\varrho,\varphi) \\ \Phi(\varrho,\varphi) \end{array}\right\} = e^{im\varphi}\, J_m(\beta_{mn}\,\varrho). \tag{49}$$

The wave of longitudinal wave number γ_{mn} and radial wave number β_{mn} is represented by an infinite summ of plane waves, what is pictured in Fig. 5. In rectangular duct, decomposition of mode (m,n) into a summ of plane waves is even easier and is based on expressing the sine or cosine function by means of Euler formula $e^{ix} = \cos x + i\sin x$. In general, the mode is a summ of four plane waves [18], reduced for H_{m0} and H_{0n} modes to two (E_{m0} and E_{0n} waves do not exist in rectangular duct).

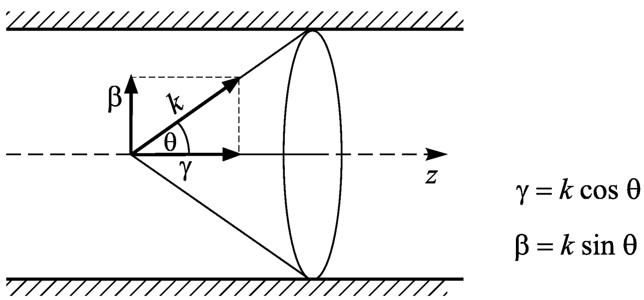

Fig. 5. Directions of propagation of plane waves constituting the cylindrical wave – wave vectors are displayed on a cone.

The wave phase velocity c_φ and group velocity c_g could be denoted by the cut-off angular frequencies ω^{cr}

$$c_\varphi = \frac{c}{\sqrt{1-\left(\dfrac{\omega^{\mathrm{cr}}}{\omega}\right)^2}}\,, \tag{50}$$

$$c_g = c\cdot\sqrt{1-\left(\frac{\omega^{\mathrm{cr}}}{\omega}\right)^2}\,, \tag{51}$$

their product is equal to c^2.

3.2. Wave impedance

There are some others similarities between the equations describing the propagation of sound and electromagnetic waves in ducts. Here the impedance concept is to be developed.

In mechanical systems (for example waves on a rod) impedance is a ratio of force to particle velocity, in electrical circuits – electric impedance is a ratio of voltage to current.

For sound waves the acoustic wave impedance is the quotient of acoustic pressure and particle velocity [14]

$$Z^{\mathrm{ak}} = \frac{p}{\mathrm{v}} \, . \tag{52}$$

The value obtained for a plane wave propagating in a free space is called the characteristic impedance of the medium and for the sound wave it turns to be $Z_c^{\mathrm{ak}} = \rho_0 c$, where ρ_0 is the medium density.

If a real or hypothetic surface is introduced, then the acoustic impedance referred to the surface is expressed as

$$Z^{\mathrm{ak}} = \frac{p}{\mathrm{v}_n} \, , \tag{53}$$

where v_n is the particle velocity component, normal to the surface.

For acoustic waves in ducts, apart from the plane wave, the acoustic pressure and velocity are functions of displacement and thus it seems that to compute the impedance one should apply the general definition of the impedance, as ratio of the power $\mathcal{P}$ transmitted through the duct cross-section to product of the mean square of particle velocity $\langle \mathrm{v}_n^2 \rangle$ and the duct surface S

$$Z = \frac{\mathcal{P}}{S\langle \mathrm{v}_n^2 \rangle} \, , \tag{54}$$

where

$$\langle \mathrm{v}_n^2 \rangle = \frac{1}{S} \left\langle \int\limits_S \mathrm{v}_n \mathrm{v}_n^* \, \mathrm{d}s \right\rangle_T \, , \tag{55}$$

where $\langle \; \rangle_T$ denotes mean value versus time.

Anyhow, for a single mode (m, n) propagating along the duct, the acoustic pressure is proportional to the acoustic velocity v_n in every point of the duct cross-section, thus impedance of the hard and soft waveguide can be calculated according to (53)

$$Z^{\mathrm{ak}} = Z_c^{\mathrm{ak}} \frac{1}{\sqrt{1 - \left(\dfrac{\omega^{\mathrm{cr}}}{\omega} \right)^2}} \, , \tag{56}$$

where ω^{cr} is given by (17) for hard and by (19) for soft duct.

For electromagnetic waves the wave impedance is defined as a quotient of two normal to the direction of propagation, components of the electric and magnetic field [6]

$$Z^{\mathrm{EM}} = \frac{E_\perp}{H_\perp} = \frac{|E_\perp| e^{i \arg E_\perp}}{|H_\perp| e^{i \arg H_\perp}}\,, \tag{57}$$

In homogenous, isotropic, nondissipative and nonconductive unlimited medium the electromagnetic plane wave impedance, called the characteristic impedance of a medium Z_c^{EM} is expressed by dielectric ε and magnetic μ constants

$$Z_c^{\mathrm{EM}} = \sqrt{\mu/\varepsilon} = 1/\varepsilon c. \tag{58}$$

Calculating, along with equations (35), (36), (38) and (39), the adequate components of the electric $\mathbf{E}$ and the magnetic $\mathbf{H}$ field intensity, the waveguide impedance of the H-wave and the E-wave turns to be

$$Z^H = \frac{Z_c^{\mathrm{EM}}}{\sqrt{1-\left(\dfrac{\omega^{H,\mathrm{cr}}}{\omega}\right)^2}}\,, \tag{59}$$

$$Z^E = Z_c^{\mathrm{EM}} \sqrt{1-\left(\dfrac{\omega^{E,cr}}{\omega}\right)^2}, \tag{60}$$

where, according to (41), (42) $\omega^{H,\mathrm{cr}} = \omega^{\mathrm{cr}}$, $\omega^{E,\mathrm{cr}} = \widetilde{\omega}^{\mathrm{cr}}$ are cut-off frequencies.

To summ up, discussion of the waveguide impedance leads to the following conclusions

- for the TE and sound waves the waveguide impedance is a quotient of the characteristic impedance of a medium Z_c and a term $\sqrt{1-(\omega^{\mathrm{cr}}/\omega)^2}$, while for the TM waves it is a product of these two terms, where ω^{cr} is the cut-off angular frequency of a given type of wave,
- if $\omega \to \omega^{\mathrm{cr}}$ and $\omega > \omega^{\mathrm{cr}}$ the TM and sound waves impedance tends to infinity, the TE waves impedance goes to zero,
- for $\omega^{\mathrm{cr}} > \omega$ the impedance of a nondissipative medium becomes imaginary (reactance).

In the preceding discussion the waveguide has been assumed to be infinitely long. If the waveguide is finite and is attached at $z = l$ to some device of impedance $Z(l) = Z_l$ (the waveguide outlet can be regarded as such a load) then, analyzing the plane wave reflection phenomena and having in mind the assumed time dependence $e^{-i\omega t}$, the impedance at $z = 0$ is equal to [6]

$$Z(0) = Z_c \frac{Z_l - iZ_c \tan kl}{Z_c - iZ_l \tan kl}\,, \tag{61}$$

where Z_c means the adequate characteristic impedance.

The last formula is basic and frequently applied in the theory of acoustic and electromagnetic systems – transmission lines and electrical circuits and attends for calculating the wave velocity or material constants. It is derived by considering reflection coefficient of a plane wave at a plane interface between two media, which takes the same form for sound and electromagnetic waves [6]. It does not applies to empty conducting ducts, as they do not carry plane waves (TEM).

3.3. *Power transmission in ducts*

Wave, propagating through medium, carries energy. In electromagnetic field the complex energy flux is described by the Poynting vector $\mathbf{S}$ [1]

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}^* . \tag{62}$$

It points in the direction of energy flow and its magnitude is the power per unit area crossing a surface that is normal to it. It is also called the surface power density or wave intensity. The real part of the time-averaged power density is given by

$$\mathrm{Re}\,\langle \mathbf{S} \rangle_T = \frac{1}{2}\mathrm{Re}\,(\mathbf{E} \times \mathbf{H}^*). \tag{63}$$

The averaged complex power of electromagnetic wave transmitted through the surface turns to be

$$\mathcal{P}^{\mathrm{EM}} = \frac{1}{2} \int\limits_{\mathcal{S}} (\mathbf{E} \times \mathbf{H}^*) \cdot \mathrm{d}\mathbf{s}, \tag{64}$$

and can be expressed by means of wave impedance

$$\mathcal{P}^{\mathrm{EM}} = \frac{1}{2} \int\limits_{S} E_\perp H_\perp^* \,\mathrm{d}s = \frac{1}{2} \int\limits_{S} Z^{\mathrm{EM}} |H_\perp|^2 \,\mathrm{d}s. \tag{65}$$

In acoustics, the complex power flux density is a product of the acoustic pressure and conjugate particle velocity [14]

$$\mathbf{I}(t) = p\mathbf{v}^*, \tag{66}$$

thus the real part of its time-average value, called sound intensity vector, is equal to

$$\mathrm{Re}\,\langle \mathbf{I} \rangle_T = \frac{1}{2}\mathrm{Re}\,(p\mathbf{v}^*). \tag{67}$$

The averaged power transmitted through the surface

$$\mathcal{P}^{\mathrm{ak}} = \frac{1}{2} \int\limits_{\mathcal{S}} \mathbf{I} \cdot \mathrm{d}\mathbf{s} = \frac{1}{2} \int\limits_{\mathcal{S}} p\mathbf{v}^* \cdot \mathrm{d}\mathbf{s}, \tag{68}$$

can be, according to (53), expressed by means of the acoustic impedance

$$\mathcal{P}^{\mathrm{ak}} = \frac{1}{2} \int\limits_{\mathcal{S}} p\mathrm{v}_n \,\mathrm{d}s = \frac{1}{2} \int\limits_{\mathcal{S}} Z^{\mathrm{ak}} |\mathrm{v}_n|^2 \,\mathrm{d}s. \tag{69}$$

Thus, the consecutive acousto-electromagnetic analogies have been derived

$$\begin{aligned} p &\longleftrightarrow E_\perp \ , \\ \mathrm{v}_n &\longleftrightarrow H_\perp \ , \end{aligned} \tag{70}$$

where $E_\perp$, $H_\perp$ mean components normal to the direction of propagation (in assumed geometry to the z-axis), while v_n means particle velocity normal to the cross-section area, thus $\mathrm{v}_n = \mathrm{v}_z$.

In case of multimodal excitation, when more than one mode propagates along the duct, the wave potential is a summ of subsequent modes potential, as is expressed in (22). The power transmitted is a sum of powers of consecutive modes, even though power is not, in general, the additive quantity.

Calculating power along to (69) for sound waves and (65) for electromagnetic waves, one obtains terms of the form [17]

$$\begin{aligned} \int_0^{2\pi} e^{im\varphi} e^{im'\varphi} \, \mathrm{d}\varphi \;\sim\; \delta_{mm'} \,, \\ \int_0^a J_m(\beta_{mn}\varrho) J_{m'}(\beta_{m'n'}\varrho)\varrho \, \mathrm{d}\varrho \;\sim\; \delta_{mm'}\delta_{nn'}, \end{aligned} \tag{71}$$

if $\beta_{mn} = \mu_{mn}/a$ (hard duct, H-waves) or $\beta_{mn} = \nu_{mn}/a$ (soft duct, E-waves). As a result one obtains

$$\mathcal{P}\left(\sum_{m,n} \Phi_{mn}\right) = \sum_{m,n} \mathcal{P}(\Phi_{mn}), \tag{72}$$

where in the last formula Φ_{mn} stands for potential of (m, n) mode of sound or electromagnetic wave. In conclusion, consecutive modes may be treated as excited by independent (incoherent) sources.

The *explicite* formula for sound waves in hard duct is given by [17]

$$\langle \mathcal{P} \rangle_T = \frac{1}{2}\pi a^2 \rho_0 c k \left[\sum_{m=0} \sum_{n=1} \gamma_{mn} |A_{mn}|^2 \left(1 - \frac{m^2}{\mu_{mn}^2}\right) + k|A_{00}|^2 \right], \tag{73}$$

the last term representing the plane wave and summation running for all real values of axial wave numbers γ_{mn}.

4. Conclusions

Despite all differencies, there are some basic aspects of wave propagation in ducts common for sound and electromagnetic waves, for which the unified waveguide theory may be developed and the method of analogies applied.

The most important of these aspects are:

- analogies between velocity potential in hard and soft acoustic duct and the Hertz potentials of electromagnetic waves in conductive duct,
- decomposition of wave mode potential into plane waves,
- additivity of power transmitted along the duct in case of multimodal excitation,
- analogies between acoustic and electromagnetic wave impedance allowing for coherent description of in-duct phenomena.

The derived analogies may be helpful in applying solution obtained for one kind of waves to the other, still having in mind all essential differencies.

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