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Acoustic Heating Produced in the Thermoviscous Flow of a Shear-Thinning Fluid

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This study is devoted to the instantaneous acoustic heating of a shear-thinning fluid. Apparent viscosity of a shear-thinning fluid depends on the shear rate. That feature distinguishes it from a viscous Newtonian fluid. The special linear combination of conservation equations in the differential form makes it possible to derive dynamic equations governing both the sound and non-wave entropy mode induced in the field of sound. These equations are valid in a weakly nonlinear flow of a shear-thinning fluid over an unbounded volume. They both are instantaneous, and do not require a periodic sound. An example of a sound waveform with a piecewise constant shear rate is considered as a source of acoustic heating.

Keywords: acoustic heating, non-Newtonian liquids, shear-thinning fluid.

1. Introduction

Time-independent fluids constitute the class of fluids characterized by the fact that their shear rate depends only on the shear stress and is a single valued function of it (Collyer, 1973; Cheremisihoff, 1988; Wilkinson, 1960). Solutions of melts of high molecular weight, particular high polymers and suspensions of solids in liquids fall into this group. In contrast with Newtonian fluids, the non-Newtonian properties of many fluids from this group are caused by the viscous dissipation of energy due to collisions between large particles or colloidal structures. The non-Newtonian group consists of the following types of fluids: shear-thinning, shear-thickening (for them, the flow curve is nonlinear, i.e., the curve describing dependence of the shear stress on the shear rate departs from a straight line, see Fig. 1), plastic (Benito, Bruneau *et al.*, 2008), Bingham plastic (Bingham, 1916), and fluids with a yield stress and nonlinear flow curve (Barnes, 1999).

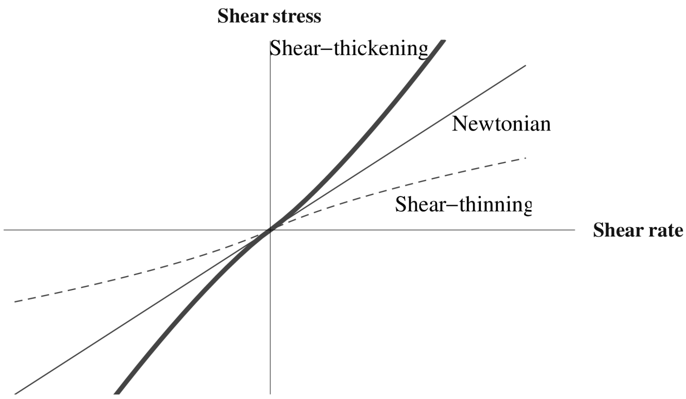

Fig. 1. Flow curves for some time independent fluids.

A shear-thinning material is one in which viscosity decreases with an increase in the rate of shear. Materials that exhibit shear-thinning are called pseudoplastic. This property is found in certain complex solutions, such as lava, ketchup, whipped cream, blood, paint, and nail polish. It is also a common property of polymer solutions and molten polymers. The logarithmic plot between shear stress and shear rate is generally linear for these fluids, of the slope n less than unity. For Newtonian fluids, $n = 1$. The shear-thickening fluids are far less common than the shear-thinning varieties, and they behave in the opposite manner, namely, their apparent viscosity increases with the increase in the shear rate. Shear-thickening materials were first recognized by Reynolds in 1885 (Reynolds, 1885), and he called them dilatant fluids because the model he used to describe their properties requires a dilation upon shearing. For shear-thickening fluids, $n > 1$.

It is well known that sound attenuates linearly in the standard thermoviscous flow of a Newtonian fluid. Acoustic heating is an increase of the ambient fluid temperature caused by *nonlinear* losses in acoustic energy. It is not an acoustic quantity but a value referred to the entropy, or thermal mode. Acoustic heating in the standard thermoviscous fluid flows is well studied theoretically and experimentally for the cases where periodic sound is the origin of heating (Rudenko, Soluyan, 1977; Makarov, Ochmann, 1996). Acoustic heating originates from attenuation and nonlinearity. Interest in acoustic heating of non-Newtonian fluids has grown in the recent years in connection with many biomedical and technical applications. They require accurate estimations of heating during medical therapy which applies sound of different kinds including non-periodic ones, in particular, impulses (Hartman *et al.*, 1992; Rudenko, 2007). Not only biological, but many technical liquids belong to the shear-thinning group, such as the already mentioned dilute solutions of high polymers, printing ink, soap solution and glycerol, cellulose, acetate and napalm, paper pulp and detergent slurries, as well as many food products like mayonnaise (Collyer, 1973). Some of them are very shear-thinning fluids, for example, concentrated cement slurries, mineral suspensions, oil-well cements (Barnes, 1999; Roberts *et al.*, 2001).

This study is devoted to the nonlinear dissipation of sound energy in a shear-thinning fluid. The mathematical technique has been worked out and applied previously by the author in some problems of thermoviscous nonlinear flow (Perelomova, 2003; 2006; 2008). It allowed to separate individual equations governing sound, vorticity and entropy modes in Newtonian and non-Newtonian (relaxing) fluids. The method and results concerning the flow over a shear-thinning fluid are described in Secs. 3, 4. The formal correspondence of the governing equations of both the sound and entropy mode induced by it to that in a Newtonian fluid at $n = 1$ is traced (Sec. 5). A simple example considers sound waveforms with a piecewise constant shear rate.

2. Dynamic equations governing shear-thinning fluid flow

The continuity, momentum and energy conservation equations describing a thermoviscous fluid flow without external forces are:

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \mathbf{v}) = 0,\\
&\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \frac{1}{\rho}\left(-\boldsymbol{\nabla} p + \mathrm{Div}\,\mathbf{P}\right),\\
&\frac{\partial e}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})e = \frac{1}{\rho}\left(-p(\boldsymbol{\nabla}\cdot\mathbf{v}) + \chi\Delta T + \mathbf{P} : \mathrm{Grad}\,\mathbf{v}\right),
\end{aligned}
\tag{1}
$$

where $\mathbf{v}$ denotes velocity of fluid, ρ, p are density and pressure, e, T mark internal energy per unit mass and temperature, correspondingly, χ is the thermal conductivity, x_i, t are spatial coordinates and time. Operator Div denotes tensor divergency and Grad is a dyad gradient. $\mathbf{P}$ is the tensor of viscous stress. In the model of a shear-thinning fluid, the viscous stress tensor relates to the shear rate in the following manner (Collyer, 1973; Wilkinson, 1960):

$$
\mathbf{P}_{i,k} = \begin{cases} \dfrac{\eta}{2}\left(\dfrac{\partial v_i}{\partial x_k} + \dfrac{\partial v_k}{\partial x_i}\right)^n, & \left(\dfrac{\partial v_i}{\partial x_k} + \dfrac{\partial v_k}{\partial x_i}\right) \geq 0,\\[2ex] -\dfrac{\eta}{2}\left|\dfrac{\partial v_i}{\partial x_k} + \dfrac{\partial v_k}{\partial x_i}\right|^n, & \left(\dfrac{\partial v_i}{\partial x_k} + \dfrac{\partial v_k}{\partial x_i}\right) < 0, \end{cases} \qquad 0 < n < 1, \quad \eta = \mathrm{const.} \tag{2}
$$

Two thermodynamic functions $e(p,\rho)$, $T(p,\rho)$ complement the system (1). They may be written as a series of excess internal energy $e' = e - e_0$ and temperature

$T' = T - T_0$ in powers of excess pressure and density $p' = p - p_0$, $\rho' = \rho - \rho_0$ (ambient quantities are marked by index 0):

$$\begin{aligned} e' &= \frac{E_1}{\rho_0}p' + \frac{E_2 p_0}{\rho_0^2}\rho' + \frac{E_3}{p_0\rho_0}p'^2 + \frac{E_4 p_0}{\rho_0^3}\rho'^2 + \frac{E_5}{\rho_0^2}\rho' p', \\ T' &= \frac{\Theta_1}{\rho_0 C_V}p' + \frac{\Theta_2 p_0}{\rho_0^2 C_V}\rho' + \frac{\Theta_3}{p_0\rho_0 C_V}p'^2 + \frac{\Theta_4 p_0}{\rho_0^3 C_V}\rho'^2 + \frac{\Theta_5}{\rho_0^2 C_V}\rho' p', \end{aligned} \tag{3}$$

where $E_1, \ldots, E_5$, $\Theta_1, \ldots, \Theta_5$ are dimensionless coefficients, C_V marks the heat capacity per unit mass under constant volume. Series (3) allows to consider a wide variety of Newtonian or non-Newtonian fluids in the general form: the discrepancy is manifested by the coefficients different for different fluids. The common practice in nonlinear acoustics is to focus on the equations of the second order of acoustic Mach number $M = v_0/c_0$, where v_0 is the magnitude of fluid velocity, $c_0 = \sqrt{\dfrac{(1-E_2)p_0}{E_1\rho_0}}$ is the infinitively small signal velocity (without account for thermoviscous phenomena), respectively. The present study is also confined by considering of nonlinearity not higher than the second order, so that in the series (3) only terms up to the second order are held. As it will be concluded below, the acoustic force of heating includes terms caused by a thermal conductivity proportional to χM^2, and terms caused by a viscosity proportional to ηM^n. They are large as compared with those caused by the thermal conduction in a shear-thinning fluid. The expressions for coefficients E_1 and E_2 in terms of compressibility, κ, and thermal expansion β, are as follows:

$$\begin{aligned} E_1 &= \frac{\rho_0 C_V \kappa}{\beta}, \\ E_2 &= -\frac{C_p \rho_0}{\beta p_0} + 1, \end{aligned} \tag{4}$$

C_p denotes the heat capacity per unit mass under constant pressure,

$$\begin{aligned} \kappa &= -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T, \\ \beta &= \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p. \end{aligned} \tag{5}$$

A small variation in entropy is a total differential that provides a relationship between the first coefficients in the series (3):

$$\Theta_2 = \frac{C_V \rho_0 T_0}{E_1 p_0} - \frac{(1-E_2)\Theta_1}{E_1}. \tag{6}$$

We shall consider small dimensionless parameters responsible for viscosity, $\mu = \eta\Omega/(\rho_0 c_0^2)$, and thermal conductivity, $\delta = \dfrac{\chi T_0 \Omega}{c_0^4 E_1^2 \rho_0}$ (Ω is the characteristic frequency of perturbation). They are supposed to be of the same order as M. That means that the considered domain of characteristic frequencies provides a weak absorption of signals in a medium. We shall consider weakly nonlinear flows discarding cubic and higher order nonlinear terms in all expansions. The resulting model equations will account for the combined effects of nonlinearity, viscosity and thermal conductivity.

3. Definition of modes in the planar flow of infinitely-small amplitude

Let us consider the one-dimensional flow along the axis OX. The multidimensional flow in regard to the problem of acoustic heating will be briefly discussed in Concluding Remarks. All the formulae further below in the text, including links of modes and governing equations, are written in the leading order with respect to powers of the small parameters M, μ and δ. It is convenient to rearrange the formulae in the dimensionless quantities in the following way:

$$p^* = \frac{p'}{c_0^2 \cdot \rho_0}, \qquad \rho^* = \frac{\rho'}{\rho_0}, \qquad v^* = \frac{v}{c_0}, \qquad x^* = \frac{\Omega x}{c_0}, \qquad t^* = \Omega t. \tag{7}$$

Starting from Eq. (8), the upper indexes (asterisks) denoting the dimensionless quantities will be omitted everywhere in the text. In the dimensionless quantities, Eqs. (1) with a account for Eqs. (2), (3) read:

$$\begin{gathered}
\underbrace{\frac{\partial v}{\partial t} + \frac{\partial p}{\partial x}}_{\text{linear}} = \underbrace{-v\frac{\partial v}{\partial x} + \rho\frac{\partial p}{\partial x}}_{O(M^2)} + \underbrace{2^{n-1} n\mu \left|\frac{\partial v}{\partial x}\right|^{n-1} \left(\frac{\partial^2 v}{\partial x^2}\right)}_{O(M^{n+1})}, \\
\underbrace{\frac{\partial p}{\partial t} + \frac{\partial v}{\partial x} - \delta_1 \frac{\partial^2 p}{\partial x^2} - \delta_2 \frac{\partial^2 \rho}{\partial x^2}}_{\text{linear}} = \underbrace{-v\frac{\partial p}{\partial x} + (D_1 p + D_2 \rho)\frac{\partial v}{\partial x}}_{O(M^2)} + \underbrace{2^{n-1}\frac{\mu}{E_1}\left|\frac{\partial v}{\partial x}\right|^{n+1}}_{O(M^{n+2})} \\
+ \underbrace{\delta_3 \frac{\partial^2 p^2}{\partial x^2} + \delta_4 \frac{\partial^2 \rho^2}{\partial x^2} + \delta_5 \frac{\partial^2 (\rho p)}{\partial x^2}}_{O(M^3)}, \\
\underbrace{\frac{\partial \rho}{\partial t} + \frac{\partial v}{\partial x}}_{\text{linear}} = \underbrace{-v\frac{\partial \rho}{\partial x} - \rho\frac{\partial v}{\partial x}}_{O(M^2)}.
\end{gathered} \tag{8}$$

The right-hand side of the set (8) consists of leading-order nonlinear terms. The dynamic equations in the rearranged form include dimensionless quantities

$$
\begin{aligned}
\delta_1 &= \frac{\chi \Theta_1 \Omega}{\rho_0 c_0^2 C_V E_1},\\
\delta_2 &= \frac{\chi \Theta_2 \Omega}{\rho_0 c_0^2 C_V (1-E_2)},\\
\delta_3 &= \frac{\Theta_3 \chi \Omega}{E_1 \rho_0 c_0^2 C_V}\frac{1-E_2}{E_1},\\
\delta_4 &= \frac{\Theta_4 \chi \Omega}{(1-E_2)\rho_0 c_0^2 C_V},\\
\delta_5 &= \frac{\Theta_5 \chi \Omega}{E_1 \rho_0 c_0^2 C_V},\\
D_1 &= \frac{1}{E_1}\left(-1 + 2\frac{1-E_2}{E_1}E_3 + E_5\right),\\
D_2 &= \frac{1}{1-E_2}\left(1 + E_2 + 2E_4 + \frac{1-E_2}{E_1}E_5\right).
\end{aligned}
\tag{9}
$$

The sum of the two first coefficients is the linear attenuation due to the thermal conductivity, $\delta = \delta_1 + \delta_2$. The linearized version of Eq. (8) describes a flow of an infinitely-small magnitude, when $M \to 0$:

$$
\begin{gathered}
\frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} = 0,\\
\frac{\partial p}{\partial t} + \frac{\partial v}{\partial x} - \delta_1 \frac{\partial^2 p}{\partial x^2} - \delta_2 \frac{\partial^2 \rho}{\partial x^2} = 0,\\
\frac{\partial \rho}{\partial t} + \frac{\partial v}{\partial x} = 0.
\end{gathered}
\tag{10}
$$

The dispersion equation follows from Eq. (10). Its roots determine three independent modes (or types of the linear flow) of infinitely small-signal disturbances in an unbounded fluid. In one dimension, there exist the acoustic (two branches) and the thermal (or entropy) modes. In general, any perturbation of the field variables contains contributions from any of the three modes. That allows decomposition of the equations which govern individual modes in the linear part using the specific properties of modes, namely, relationships between components of velocity and excess quantities of two thermodynamic functions, for example, pressure and density. The method developed by the author in the studies (Perelomova, 2003; 2006; 2008), makes it possible to split the initial system (1) into individual dynamic equations for every mode. Studies of motions of infinitely-small amplitudes begin usually with representing of all perturbations as planar waves, where $\widetilde{\varepsilon}(k)$ is the Fourier-transform of any perturbation ε':

$$\varepsilon'(x,t) = \widetilde{\varepsilon}\,\exp i(\omega t - kx). \tag{11}$$

Every type of motion is determined in fact by one of the roots of the dispersion equation, $\omega(k)$ (k is the wave number) (Rudenko, Soluyan, 1977; Makarov, Ochmann, 1996; Chu, Kovasznay, 1958). This then fixes the relationships between the hydrodynamic perturbations. They are independent on time and derived in the articles (Perelomova, 2003; 2006; 2008). The dispersion relations describing the sound progressive in the positive or negative direction of the axis OX will be marked by indices 1 and 2, and that for the entropy mode will be marked by index 3. They take the following form:

$$\omega_{a,1} = k + i\frac{\delta k^2}{2}, \qquad \omega_{a,2} = -k + i\frac{\delta k^2}{2}, \qquad \omega_e = -ik^2\delta_2. \tag{12}$$

They uniquely determine relations of velocity, excess density and pressure attributable to any mode valid in any time of a hydrodynamic field evolution,

$$\begin{gathered}
\psi_{a,1} = \begin{pmatrix} v_{a,1} \\ p_{a,1} \\ \rho_{a,1} \end{pmatrix} = \begin{pmatrix} 1 - \dfrac{\delta}{2}\dfrac{\partial}{\partial x} \\ 1 - \delta\dfrac{\partial}{\partial x} \\ 1 \end{pmatrix} \rho_{a,1}, \\
\psi_{a,2} = \begin{pmatrix} -1 - \dfrac{\delta}{2}\dfrac{\partial}{\partial x} \\ -1 + \delta\dfrac{\partial}{\partial x} \\ 1 \end{pmatrix} \rho_{a,2}, \\
\psi_e = \begin{pmatrix} \delta_2\dfrac{\partial}{\partial x} \\ 0 \\ 1 \end{pmatrix} \rho_e.
\end{gathered} \tag{13}$$

The linear equation describing a fluid's excess density in propagating in the positive direction of the axis OX acoustic wave agrees with $\omega_{a,1}$ from Eq. (12) and takes the form:

$$\frac{\partial \rho_{a,1}}{\partial t} + \frac{\partial \rho_{a,1}}{\partial x} - \frac{\delta}{2}\frac{\partial^2 \rho_{a,1}}{\partial x^2} = 0. \tag{14}$$

It may be readily rearranged into the diffusion equation by special choice of new variables, the retarded time $\tau = t - x$ and the slow varying distance from a transducer, Mx (Rudenko, Soluyan, 1977). In the leading order, it takes the form

$$\frac{\partial \rho_{a,1}}{\partial \tau} - \frac{\delta}{2}\frac{\partial^2 \rho_{a,1}}{\partial x^2} = 0. \tag{15}$$

The density perturbation in the entropy motion also satisfies the diffusion equation (δ_2 is negative for any fluid):

$$\frac{\partial \rho_e}{\partial t} + \delta_2 \frac{\partial^2 \rho_e}{\partial x^2} = 0. \tag{16}$$

Equations for every type of motion may be also extracted from the system (8) in accordance to relationships specific for each mode. That may be formally proceeded by means of projecting of the equations into specific sub-spaces. The linear dynamic equations are obviously independent.

4. Dynamic equations in a weakly nonlinear flow

4.1. Weakly nonlinear dynamic equation of sound

The nonlinear terms in every equation from the right-hand side of system (8) include parts attributable to every mode. We fix relations determining any mode in a linear flow and will consider every excess quantity as a sum of specific excess quantities of all modes. For example, total perturbation of density is a sum of specific parts,

$$\rho = \rho_{a,1} + \rho_{a,2} + \rho_e. \tag{17}$$

The consequent decomposing of the governing equations for both branches of sound and the thermal mode may be still done by means of linear projecting, see for details (Perelomova, 2008). Projecting points out the way of linear combination of equations. This way allows to keep terms belonging to the appropriate mode in the linear part and to reduce all other terms there. In respect to the first (rightwards progressive) acoustic mode, the scheme for deriving the dynamic equation in terms of excess density is as follows: apply $\frac{1}{2} + \frac{\delta}{2}\frac{\partial}{\partial x}$ to the first equation from the set from Eqs. (8), $\frac{1}{2} + \frac{\delta_2}{2}\frac{\partial}{\partial x}$ to the second equation, and $-\frac{\delta_2}{2}\frac{\partial}{\partial x}$ to the third equation and take their sum. The linear terms of the second acoustic and the entropy modes become completely reduced. Expressing all acoustic quantities in terms of excess density by use of links ($\psi_{a,1}$ from Eqs. (13)) one readily obtains the leading-order equation analogous to the well-known Burgers one:

$$\begin{aligned} \frac{\partial \rho_{a,1}}{\partial t} + \frac{\partial \rho_{a,1}}{\partial x} - \frac{\delta}{2}\frac{\partial^2 \rho_{a,1}}{\partial x^2} = -\frac{1 - D_1 - D_2}{2}\rho_{a,1}\frac{\partial \rho_{a,1}}{\partial x} \\ + 2^{n-2} n\mu \left|\frac{\partial \rho_{a,1}}{\partial x}\right|^{n-1} \left(\frac{\partial^2 \rho_{a,1}}{\partial x^2}\right). \end{aligned} \tag{18}$$

The nonlinear term in the right-hand side of Eq. (18) may be considered as a result of self-action of sound which corrects the linear equation governing sound (14) by nonlinear terms. For the flow over a shear-thinning fluid, the last term in the right-hand side of Eq. (18) is large as compared with the first one. It is of importance that Eq. (18) does tend to the Burgers equation with a non-zero standard viscosity when n tends to 1 (limit of a Newtonian fluid). In this case, the last term becomes linear and may be united with this one standing by δ,

$$\frac{\partial \rho_{a,1}}{\partial t} + \frac{\partial \rho_{a,1}}{\partial x} - \frac{\mu + \delta}{2} \frac{\partial^2 \rho_{a,1}}{\partial x^2} = -\frac{1 - D_1 - D_2}{2} \rho_{a,1} \frac{\partial \rho_{a,1}}{\partial x}. \tag{19}$$

The dynamics of sound in a newtonian fluid depends therefore on the total attenuation, $\mu + \delta$ (Rudenko, Soluyan, 1977; Makarov, Ochmann, 1996).

4.2. *The thermal mode induced by sound*

In the context of acoustic heating, the magnitude of an excess density specific for the entropy mode is small as compared with that for sound. We consider the ratio of amplitudes of excess densities specifying the entropy motion and sound of order M. Modes (13) satisfy in the leading order the equality as follows:

$$\begin{pmatrix} -\delta \dfrac{\partial}{\partial x} & -1 & 1 \end{pmatrix} \begin{pmatrix} v_{a,1} + v_{a,2} + v_e \\ p_{a,1} + p_{a,2} + p_e \\ \rho_{a,1} + \rho_{a,2} + \rho_e \end{pmatrix} = \rho_e, \tag{20}$$

which points out a way of combination of Eqs. (8) in order to reduce all acoustic quantities in the linear part of equations. The important property of projection is not only to decompose specific perturbations in the linear part of equations, but to distribute nonlinear terms correctly between different dynamic equations. The links inside the sound should be supplemented by nonlinear quadratic terms making sound isentropic in the leading order. These corrections in a shear-thinning fluid are similar to the ones specific for the Riemann wave in an ideal gas. They were derived by Riemann (Riemann, 1953):

$$\begin{aligned} v_{a,1} &= \rho_{a,1} - \frac{\delta}{2} \frac{\partial \rho_{a,1}}{\partial x} - \frac{1}{4}(3 + D_1 + D_2)\rho_{a,1}^2, \\ p_{a,1} &= \rho_{a,1} - \delta \frac{\partial \rho_{a,1}}{\partial x} - \frac{1}{2}(1 + D_1 + D_2)\rho_{a,1}^2, \end{aligned} \tag{21}$$

but involve additional terms proportional to δ. The nonlinear corrections of the second (and higher than second) order depend on the equation of state. For simplicity, let the sound be progressive in the positive direction of OX: $p_a = p_{a,1}$, $\rho_a = \rho_{a,1}$, $v_a = v_{a,1}$. Linear combination of the left-hand sides of the equations in (8) in accordance to (20), (21) results in the leading-order equality

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\rho - p - \delta \frac{\partial v}{\partial x}\right) - \delta \frac{\partial^2 p}{\partial x^2} + \delta_1 \frac{\partial^2 p}{\partial x^2} + \delta_2 \frac{\partial^2 \rho}{\partial x^2} = \frac{\partial \rho_e}{\partial t} + \delta_2 \frac{\partial^2 \rho_e}{\partial x^2} \\ &\quad - \frac{\delta}{4}(3 + D_1 + D_2) \frac{\partial^2 \rho_a^2}{\partial x^2} + (1 + D_1 + D_2)\left(-\rho_a \frac{\partial \rho_a}{\partial x} + \frac{\delta_2}{2} \frac{\partial^2 \rho_a^2}{\partial x^2}\right). \end{aligned} \tag{22}$$

In the simple evaluations above, the corrected links (21) are used, as well as the dynamic equation (18) to exclude the partial derivative with respect to time in the nonlinear terms. In the context of acoustic heating, the sound is dominative, so that only acoustic quadratic terms should remain. Combining in a similar way the right-hand sides of the equations from the set (8), and comparing the result with Eq. (22), one obtains the dynamic equation governing an excess density attributable to the entropy mode,

$$\begin{aligned}\frac{\partial\rho_e}{\partial t}+\delta_2\frac{\partial^2\rho_e}{\partial x^2}-\frac{\delta}{4}(3+D_1+D_2)\frac{\partial^2\rho_a^2}{\partial x^2}&\\ +(1+D_1+D_2)\left(-\rho_a\frac{\partial\rho_a}{\partial x}+\frac{\delta_2}{2}\frac{\partial^2\rho_a^2}{\partial x^2}\right)&=-(1+D_1+D_2)\left(\rho_a\frac{\partial\rho_a}{\partial x}\right)\\ &-2^{n-1}\frac{\mu}{E_1}\left|\frac{\partial\rho_a}{\partial x}\right|^{n+1}+\delta\left(D_1\left(\frac{\partial\rho_a}{\partial x}\right)^2-\rho_a\frac{\partial^2\rho_a}{\partial x^2}\right)\\ &-(\delta_3+\delta_4+\delta_5)\left(\frac{\partial^2\rho_a^2}{\partial x^2}\right),\end{aligned}\tag{23}$$

which becomes simpler after ordering:

$$\begin{aligned}\frac{\partial\rho_e}{\partial t}+\delta_2\frac{\partial^2\rho_e}{\partial x^2}&=-2^{n-1}\frac{\mu}{E_1}\left|\frac{\partial\rho_a}{\partial x}\right|^{n+1}\\ &+\left(\left(\frac{\delta}{2}-\delta_2\right)(1+D_1+D_2)-2(\delta_3+\delta_4+\delta_5)\right)\rho_a\frac{\partial^2\rho_a}{\partial x^2}\\ &+\left(\frac{\delta}{2}(3D_1+D_2+3)-\delta_2(1+D_1+D_2)-2(\delta_3+\delta_4+\delta_5)\right)\left(\frac{\partial\rho_a}{\partial x}\right)^2.\end{aligned}\tag{24}$$

It is remarkable, that the dynamic equation for acoustic heating is a result of combining of energy and continuity equations in the absence of the thermal conduction. Otherwise, it is a result of combining of energy, continuity, and momentum equations accordingly to Eq. (20). The acoustic terms of the leftwards propagating sound become completely reduced in the linear part of the final equation. Consideration is restricted to an acoustic field represented by the rightwards progressive sound (i.e., in the positive direction of OX). It may be easily expanded on the leftwards one or any mixture of two acoustic branches.

5. Acoustic heating

5.1. An ideal gas

Solution of Eq. (24) governing the decrease in the ambient density ρ_e, is a fairly complex problem, because an excess acoustic density itself must satisfy Eq. (18). Both dynamic equations are nonlinear and account for attenuation due to the thermal conduction and viscosity which depends on the shear rate. The equation governing dynamics of ρ_e includes nonlinear acoustic terms standing by dissipative coefficients. They form the nonlinear source of acoustic heating and reflect the fact that the reason for the phenomenon are both nonlinearity and absorption. The diffusion equation (24) is instantaneous, it describes dynamics of the thermal mode in any time and does not require periodicity of sound. Although the shear-thinning fluid is certainly not a perfect gas, it would be useful to trace a governing formula for the sound and entropy mode in this limit. The coefficients participating in Eqs. (9) take the following form (Perelomova, 2006):

$$\begin{aligned} &\delta_1 = \frac{\gamma\delta}{\gamma - 1}, \qquad \delta_2 = -\frac{\delta}{\gamma - 1}, \qquad \delta_3 = 0, \qquad \delta_4 = \frac{\delta}{\gamma - 1}, \\ &\delta_5 = -\frac{\gamma\delta}{\gamma - 1}, \qquad E_1 = \frac{1}{\gamma - 1}, \qquad D_1 = -\gamma, \qquad D_2 = 0, \end{aligned} \tag{25}$$

where $\gamma = C_p/C_V$ is the ratio of specific heats of an ideal gas. Equations (18), (24) may be readily rearranged into the equations

$$\frac{\partial \rho_a}{\partial t} + \frac{\partial \rho_a}{\partial x} - \frac{\delta}{2}\frac{\partial^2 \rho_a}{\partial x^2} + \frac{\gamma + 1}{2}\rho_a \frac{\partial \rho_a}{\partial x} - 2^{n-2} n\mu \left| \frac{\partial \rho_{a,1}}{\partial x} \right|^{n-1} \left(\frac{\partial^2 \rho_{a,1}}{\partial x^2} \right) = 0, \tag{26}$$

$$\begin{aligned} \frac{\partial \rho_e}{\partial t} - \frac{\delta}{\gamma - 1}\frac{\partial^2 \rho_e}{\partial x^2} = &-\gamma\delta \left(\frac{\partial \rho_a}{\partial x} \right)^2 - \delta\rho_a \frac{\partial^2 \rho_a}{\partial x^2} \\ &- \delta\frac{\gamma - 5}{4}\frac{\partial^2 \rho_a^2}{\partial x^2} - 2^{n-1}\mu(\gamma - 1) \left| \frac{\partial \rho_a}{\partial x} \right|^{n+1}. \end{aligned} \tag{27}$$

The periodic sound satisfying Eq. (26) with $n = 1$ (i.e., in a Newtonian fluid) starting from some distances from a transducer where the Reynolds number is small enough (Rudenko, Soluyan, 1977) is

$$\rho_{a,1} = M \exp(-(\mu + \delta)x/2) \sin(t - x). \tag{28}$$

Substituting $\rho_{a,1}$ into Eq. (27) and averaging the right-hand part over the dimensionless sound period 2π, yield

$$\left\langle \frac{\partial \rho_e}{\partial t} - \frac{\delta}{\gamma - 1}\frac{\partial^2 \rho_e}{\partial x^2} \right\rangle = -M^2 \frac{\mu + \delta}{2}(\gamma - 1) \exp((-\mu - \delta)x). \tag{29}$$

The angled brackets denote the averaged quantities. Equations (28), (29) referring to ideal gases, standard attenuation in Newtonian fluids and periodic sound, are well-known in the theory of heating (Rudenko, Soluyan, 1977; Makarov, Ochmann, 1996).

5.2. Acoustic heating relating exclusively to the non-Newtonian behavior of a shear-thinning fluid

Let us consider effects of only non-Newtonian viscous behavior in both governing equations, discarding those of thermal conductivity, which are well studied in regard to standard absorbing flows. Equations (18), (24) take the form

$$\frac{\partial \rho_a}{\partial t} + \frac{\partial \rho_a}{\partial x} - 2^{n-2} n\mu \left| \frac{\partial \rho_a}{\partial x} \right|^{n-1} \frac{\partial^2 \rho_a}{\partial x^2} = 0, \tag{30}$$

$$-\frac{\partial \rho_e}{\partial t} = -\frac{C_V \rho_0}{\Theta_2 p_0} \frac{\partial T_e}{\partial t} = \beta \frac{\partial T_e}{\partial t} = 2^{n-1} \frac{\mu}{E_1} \left| \frac{\partial \rho_a}{\partial x} \right|^{n+1}. \tag{31}$$

The hydrodynamic nonlinearity in equation for sound, Eq. (30), will be also ignored. The shear- thinning fluids manifest a greater heating than the standard absorbing fluid. The acoustic force of heating in this latter is proportional to μM^2, but in the shear-thinning fluid it is proportional to μM^{n+1}. It is positive and provides an increase in temperature associated with the thermal mode, T_e.

The simplest case for illustration is an acoustic excess density consisting of domains of linear functions such as $\partial^2 \rho_a / \partial x^2 = 0$. In this case, the acoustic waveforms propagates without a change of its form with the dimensionless velocity 1. That yields a piecewise constant acoustic force of heating. Even simplified equations in partial derivatives, (30), (31), which do not account for the thermal conductivity, are hardly expected to be solved analytically due to nonlinearity. The validity of this system is also limited by the initial point that the magnitude of an excess acoustic density is much greater than that of the entropy mode and the leftwards propagating sound.

6. Concluding remarks

The main result of this study is the equation governing acoustic heating, Eq. (24), along with the dynamic equation for the sound, Eq. (18). They are the result of a consequent decomposition of weakly nonlinear equations governing sound and the entropy mode. This study considers the one-dimensional flow of a shear-thinning fluid. One can readily generalize the results in the quasi-planar flow along the axis OX. The dynamic equation which governs a weakly diffracted sound beam propagating in the positive direction of axis OX takes the form:

$$\frac{\partial}{\partial \tau} \left(\frac{\partial \rho_a}{\partial x} - \frac{1 - D_1 - D_2}{2} \rho_a \frac{\partial \rho_a}{\partial \tau} - \frac{\delta}{2} \frac{\partial^2 \rho_{a,1}}{\partial \tau^2} - 2^{n-2} n\mu \left| \frac{\partial \rho_{a,1}}{\partial x} \right|^{n-1} \frac{\partial^2 \rho_a}{\partial \tau^2} \right) = \frac{1}{2} \Delta_\perp \rho_a, \tag{32}$$

where $\Delta_\perp$ is the Laplacian operating in the perpendicular to the axis OX plane. Equation (32) recalls the famous Kuznetsov–Zabolotskaya–Kuznetsov equation

(Hamilton, Morfey, 1998; Kuznetsov, 1971). The equation governing the entropy mode, Eq. (24), remains unchanged in the leading order. The acoustic heating grows with the increase of acoustic Mach number M and parameters responsible for attenuation, μ and δ.

As for the shear-thickening fluids, the thermal conductivity dominates over viscosity in the total attenuation. The interesting features of nonlinear sound propagation and nonlinear interactions of modes are expected in other kinds of non-Newtonian fluids such as Bingham plastic and fluids with a yield stress and non-linear flow curve (Collyer, 1973). These fluids are characterized by non-zero yield stress influencing even on the linear sound velocity.

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