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Research Papers

Sound Wave Radiation from Partially Lined Duct

Burhan TIRYAKIOGLU[(1)*], Ahmet DEMIR[(2)]

[(1)] *Department of Applied Mathematics*
Marmara University
Istanbul, Turkey
*Corresponding Author e-mail: burhan.tiryakioglu@marmara.edu.tr

[(2)] *Department of Mechatronics Engineering*
Karabuk University
Karabuk, Turkey

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The radiation of sound waves from partially lined duct is treated by using the mode-matching method in conjunction with the Wiener-Hopf technique. The solution is obtained by modification of the Wiener-Hopf technique and involves an infinite series of unknowns which are determined from an infinite system of linear algebraic equations. Numerical solution of this system is obtained for various values of the problem parameters, whereby the effects of these parameters on the sound diffraction are studied. A perfect agreement is observed when the results of radiated field are compared numerically with a similar work existing in the literature.

Keywords: Wiener-Hopf; Fourier transform; duct; saddle point.

1. Introduction

Radiation of sound waves is a problem which has been extensively studied in the literature. Zorumski (1973) investigated the radiation of sound from the circular duct with an infinite flange for higher modes. The diffraction effect at the opening of a soft cylindrical duct was obtained analytically by Snakowska (2008). The acoustic impedance of an unflanged cylindrical duct for multimode wave was analysed (Snakowska *et al.*, 2017). In their study, a hybrid method was applied successfully for the solution and some numerical results were also given graphically.

In particular, the problem of radiation of sound waves by semi-infnite ducts has been used as a model for many engineering applications, such as noise reduction in architectural and experimental aerodynamics, in road transportation, in radar target scattering, in modern aircraft jet and turbofan engines, etc. (Büyükaksoy, Polat, 1997; Büyükaksoy, Demir, 2006; Demir, Rienstra, 2010).

The first analytical solution of the radiation from a semi-infinite unflanged rigid pipe was obtained by Levine and Schwinger (1948). An analytical solution was obtained based on the Wiener-Hopf technique (Noble, 1958). The Wiener-Hopf technique was applied later in papers by Weinstein (1969), Rienstra and Peake (2005), Snakowska *et al.* (2017), etc.

To overcome the unwanted noise pollution, various methods can be applied. One of the effective methods which, proved experimentally, is coating the duct with acoustically absorbing material. Rawlins (1978), who showed the effectiveness of this method, considered the radiation of plane wave from an unflanged rigid cylindrical duct with an acoustically absorbing internal surface.

The contributions from the impedance discontinuities are accounted for through the solution of a Modified Wiener-Hopf Equation (MWHE) (Büyükaksoy *et al.*, 2008; Tiryakioglu, Demir, 2016). Due to the difficulties in obtaining and solving the MWHE, the infinite lining is usually preferred rather than the finite one. In practice, however, these linings should be of finite length. In this way, it is both less costly and more realistic. Demir and Buyukaksoy (2003) have studied the cylindrical pipe with a rigid outer surface and finite lined inner surface. In this study, the inner surface of the pipe was examined with the mode matching technique. As a result of this study, an infinite system of infinite set of unknown coefficients was obtained. The system is cut out at appropriate number and solved both analytically and numerically.

This work aims at introducing more realistic conditions to the study of (Demir, Büyükaksoy, 2003). Here, the acoustic liner materials are examined both internally and externally in different lengths and admittances. As a result of the finite outer lining, the related boundary-value problem is formulated as a MWHE of the third kind and then reduced to a pair of simultaneous Fredholm integral equations of the second kind which are susceptible to a treatment by iterations (Büyükaksoy, Polat, 1998). The solution of the MWHE involves branch-cut integrals with unknown integrands and infinitely many unknown coefficients satisfying three infinite systems of linear algebraic equations. The branch-cut integrals are evaluated numerically. The effect of these finite linings on the radiation phenomenon for fundemental mode is presented graphically. The results are found to be in good agreement with the results of the study of (Demir, Buyukaksoy, 2003) for rigid outer surface.

2. Analysis

2.1. Formulation of the problem

We consider the radiation of sound waves by a semi-infinite circular cylindrical duct. Duct walls are assumed to be infinitely thin and they occupy the region $\{\rho = a, z \in (-\infty, l_2)\}$ (see Fig. 1). The outer and inner surfaces of cylinder are assumed to be lined partially with an acoustically absorbing material. The liner admittances are characterized by β_1 and β_2, respectively. From the symmetry of the geometry of the problem and of the incident field, the total field will be independent of azimuth θ everywhere in the circular cylindrical coordinate system (ρ, θ, z). We shall therefore introduce a scalar potential $u(\rho, z)$ which defines the acoustic pressure and velocity by $p = -(\partial/\partial t)\rho_0 u$ and $\mathbf{v} = \operatorname{grad} u$, respectively, where ρ_0 is the density of the undisturbed medium. Time dependence is assumed to be $e^{-i\omega t}$ and suppressed throughout this paper.

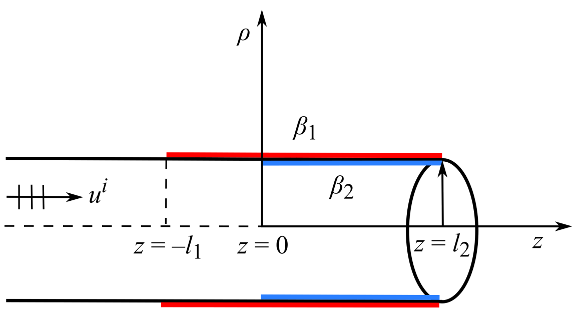

Fig. 1. Geometry of the problem.

For the sake of analytical convenience, the total field will be expressed as follows:

$$u^{\mathrm{T}}(\rho, z) = \begin{cases} u_1(\rho, z); & \rho > a, \quad z \in (-\infty, \infty), \\ u_2(\rho, z); & \rho < a, \quad z > l_2, \\ u_3(\rho, z); & \rho < a, \quad 0 < z < l_2, \\ u_4(\rho, z) + u^i(\rho, z); & \rho < a, \quad z < 0, \end{cases} \tag{1_1}$$

where $u^i(\rho, z)$ is the incident field which propagates the positive z direction

$$u^i(\rho, z) = A_{mn} J_m\left(\frac{j_{mn}}{a}\rho\right) e^{i\sigma_{mn} z}, \tag{1_2}$$

where j_{mn} is the n-th root of the equation

$$J'_m(j_{mn}) = 0 \tag{1_3}$$

and σ_{mn} stands for

$$\sigma_{mn} = \sqrt{k^2 - (j_{mn}/a)^2}. \tag{1_4}$$

Here $k = \omega/c$ denotes the wave number of the medium and c is the speed of the sound. A_{mn} stands for the amplitude of the incident wave which will be taken equal to 1 in the analysis.

2.2. Derivation of the modified Wiener-Hopf equation

The unknown fields $u_1(\rho, z)$, $u_2(\rho, z)$, $u_3(\rho, z)$ and $u_4(\rho, z)$ satisfy the Helmholtz equation for $z \in (-\infty, \infty)$

$$\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{\partial^2}{\partial z^2} + k^2\right] u_j(\rho, z) = 0, \quad j = 1, 2, 3, 4. \tag{2}$$

For determination of unknown fields, we need boundary conditions and continuity relations. One can write these equations from the geometry of the problem

$$\begin{aligned}
\frac{\partial}{\partial \rho} u_1(a_+, z) &= 0, & z &< -l_1, \\
\frac{\partial}{\partial \rho} u_4(a_-, z) &= 0, & z &< 0, \\
\left(ik\beta_1 + \frac{\partial}{\partial \rho}\right) u_1(a_+, z) &= 0, & -l_1 &< z < l_2, \\
\left(ik\beta_2 - \frac{\partial}{\partial \rho}\right) u_3(a_-, z) &= 0, & 0 &< z < l_2, \\
\frac{\partial}{\partial \rho} u_1(a_+, z) - \frac{\partial}{\partial \rho} u_2(a_-, z) &= 0, & z &> l_2, \\
u_1(a_+, z) - u_2(a_-, z) &= 0, & z &> l_2, \\
\frac{\partial}{\partial z} u_2(\rho, l_2) - \frac{\partial}{\partial z} u_3(\rho, l_2) &= 0, & \rho &< a, \\
u_2(\rho, l_2) - u_3(\rho, l_2) &= 0, & \rho &< a, \\
\frac{\partial}{\partial z} u_3(\rho, 0) - \frac{\partial}{\partial z} u_4(\rho, 0) & \\
= i\sigma_{mn} A_{mn} J_m((j_{mn}/a)\rho), & & \rho &< a, \\
u_3(\rho, 0) - u_4(\rho, 0) & \\
= A_{mn} J_m((j_{mn}/a)\rho), & & \rho &< a.
\end{aligned} \tag{3}$$

In addition to these boundary and continuity relations one has to take into account the following radiation condition:

$$u \sim \frac{e^{ikr}}{r}, \qquad r = \sqrt{\rho^2 + z^2} \to \infty.$$

Consider the Fourier transform of the Helmholtz equation satisfied by the scattered field $u_1(\rho, z)$ in the region $\rho > a$ for $z \in (-\infty, \infty)$; namely,

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + K^2(\alpha)\right] F(\rho,\alpha) = 0, \tag{4}$$

where $K(\alpha)$ is the square-root function

$$K(\alpha) = \sqrt{k^2 - \alpha^2}, \qquad K(0) = k,$$

which is defined in the complex α-plane cut as shown in Fig. 2 and $F(\rho,\alpha)$ is the Fourier transform of the field $u_1(\rho, z)$ defined to be

$$F(\rho,\alpha) = \int_{-\infty}^{\infty} u_1(\rho,z)\, e^{i\alpha z}\, \mathrm{d}z = e^{-i\alpha l_1} F^-(\rho,\alpha) + F_1(\rho,\alpha) + e^{i\alpha l_2} F^+(\rho,\alpha), \qquad (5)_1$$

with

$$F^-(\rho,\alpha) = \int_{-\infty}^{-l_1} u_1(\rho,z)\, e^{i\alpha(z+l_1)}\, \mathrm{d}z, \qquad (5)_2$$

$$F_1(\rho,\alpha) = \int_{-l_1}^{l_2} u_1(\rho,z)\, e^{i\alpha z}\, \mathrm{d}z, \qquad (5)_3$$

$$F^+(\rho,\alpha) = \int_{l_2}^{\infty} u_1(\rho,z)\, e^{i\alpha(z-l_2)}\, \mathrm{d}z. \qquad (5)_4$$

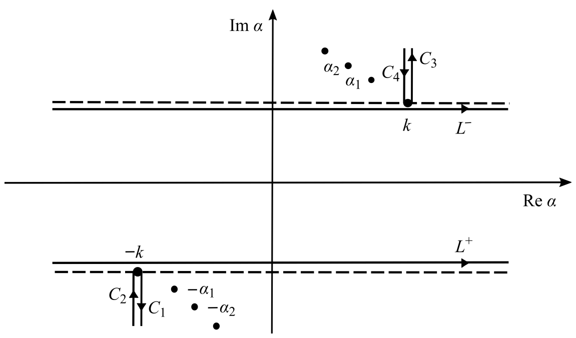

Fig. 2. Complex α-plane with branch cuts.

Owing to the analytical properties of Fourier integrals, $F^+(\rho,\alpha)$ and $F^-(\rho,\alpha)$ are regular functions in the upper half plane $\operatorname{Im}\alpha > \operatorname{Im}(-k)$ and in the lower half-plane $\operatorname{Im}\alpha < \operatorname{Im} k$, respectively. The general solution of Eq. (4) satisfying the radiation condition for $\rho \to \infty$ can be easily obtained as

$$e^{-i\alpha l_1} F^-(\rho,\alpha) + F_1(\rho,\alpha) + e^{i\alpha l_2} F^+(\rho,\alpha) = A(\alpha) H_m^{(1)}(K\rho) + B(\alpha) H_m^{(2)}(K\rho), \qquad (6)_1$$

where $A(\alpha)$ and $B(\alpha)$ are spectral coefficients to be determined and $H_m^{(1)}$, $H_m^{(2)}$ are the Hankel functions of the first and second type of order m, respectively (Abramowitz, Stegun, 1964). The second term $B(\alpha)H_m^{(2)}(K\rho)$, which does not fulfill the boundary condition in infinity, is then omitted. We get

$$e^{-i\alpha l_1} F^-(\rho,\alpha) + F_1(\rho,\alpha) + e^{i\alpha l_2} F^+(\rho,\alpha) = A(\alpha) H_m^{(1)}(K\rho). \qquad (6)_2$$

Consider now the Fourier transform of Eqs $(3)_1$ and $(3)_3$; namely,

$$e^{-i\alpha l_1} F'^-(a,\alpha) = 0, \quad ik\beta_1 F_1(a,\alpha) + F_1'(a,\alpha) = 0, \tag{7}$$

where the $(')$ denotes the derivative with respect to ρ. By taking the derivative of Eq. $(6)_2$ with respect to ρ and using Eqs (7), we obtain, after putting $\rho = a$

$$e^{-i\alpha l_1} W^-(\alpha) + e^{i\alpha l_2} W^+(\alpha) = A(\alpha) H(\alpha), \qquad (8)_1$$

where

$$W^\pm(\alpha) = ik\beta_1 F^\pm(a,\alpha) + F'^\pm(a,\alpha), \qquad (8)_2$$

$$H(\alpha) = ik\beta_1 H_m^{(1)}(Ka) + K H_m^{(1)\prime}(Ka). \qquad (8)_3$$

Substituting Eq. $(8)_1$ into Eq. $(6)_2$ yields

$$e^{-i\alpha l_1} F^-(\rho,\alpha) + F_1(\rho,\alpha) + e^{i\alpha l_2} F^+(\rho,\alpha) = \left[e^{-i\alpha l_1} W^-(\alpha) + e^{i\alpha l_2} W^+(\alpha)\right] \frac{H_m^{(1)}(K\rho)}{H(\alpha)}. \tag{9}$$

In the region $\rho < a$, $z > l_2$ the field $u_2(\rho, z)$ satisfies the Helmholtz equation for $z \in (l_2, \infty)$ as denoted in Eq. (2). The Fourier transform of this equation for the region in question is

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + K^2(\alpha)\right] G^+(\rho,\alpha) = f(\rho) - i\alpha g(\rho), \tag{10}$$

where

$$f(\rho) = \frac{\partial}{\partial z} u_2(\rho, l_2),$$

$$g(\rho) = u_2(\rho, l_2).$$

In Eq. $(10)_1$, $G^+(\rho,\alpha)$ is a regular function in the upper half of the complex α-plane which is defined as

$$G^+(\rho,\alpha) = \int_{l_2}^{\infty} u_2(\rho,z) e^{i\alpha(z-l_2)}\, \mathrm{d}z. \tag{11}$$

Particular solutions to Eq. $(10)_1$ can be found easily by using Green's function which satisfies the Helmholtz equation

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+K^2(\alpha)\right]\widetilde{G}(\rho,\rho',\alpha)=0, \qquad \rho\neq\rho', \qquad \rho,\rho'\in(0,a) \tag{12}$$

with the following conditions:

$$\widetilde{G}(0,\rho',\alpha)\sim\text{bounded},$$

$$\widetilde{G}(\rho'+0,\rho',\alpha)-\widetilde{G}(\rho'-0,\rho',\alpha)=0,$$

$$\frac{\partial}{\partial\rho}\widetilde{G}(\rho'+0,\rho',\alpha)-\frac{\partial}{\partial\rho}\widetilde{G}(\rho'-0,\rho',\alpha)=\frac{1}{\rho'},$$

$$\left(ik\beta_1+\frac{\partial}{\partial\rho}\right)\widetilde{G}(a,\rho',\alpha)=0.$$

The solution is

$$\widetilde{G}(\rho,\rho',\alpha)=\frac{1}{J(\alpha)}Q(\rho,\rho',\alpha), \tag{13_1}$$

with

$$Q(\rho,\rho',\alpha)=\frac{\pi}{2}\begin{cases}J_m(K\rho)\left[J(\alpha)Y_m(K\rho')-Y(\alpha)J_m(K\rho')\right], & 0\le\rho\le\rho',\\ J_m(K\rho')\left[J(\alpha)Y_m(K\rho)-Y(\alpha)J_m(K\rho)\right], & \rho'\le\rho\le a,\end{cases} \tag{13_2}$$

where J_m and Y_m are the Bessel and Neumann functions of order m and $J(\alpha)$, $Y(\alpha)$ are given below:

$$J(\alpha)=ik\beta_1J_m(Ka)+KJ'_m(Ka), \tag{13_3}$$

$$Y(\alpha)=ik\beta_1Y_m(Ka)+KY'_m(Ka). \tag{13_4}$$

Note that we have

$$ik\beta_1Q(a,t,\alpha)+Q'(a,t,\alpha)=0. \tag{14_1}$$

The solution of Eq. $(10)_1$ can now be written as

$$G^+(\rho,\alpha)=\frac{1}{J(\alpha)}\left[B(\alpha)J_m(K\rho)+\int_0^a\left(f(t)-i\alpha g(t)\right)Q(t,\rho,\alpha)t\,\mathrm{d}t\right]. \tag{14_2}$$

In Eq. $(14)_2$, $B(\alpha)$ is a spectral coefficient to be determined. Combining Eqs $(3)_5$ and $(3)_6$, we may write

$$ik\beta_1G^+(a,\alpha)+G'^+(a,\alpha)=ik\beta_1F^+(a,\alpha)+F'^+(a,\alpha). \tag{14_3}$$

$B(\alpha)$ can be solved uniquely from Eq. $(14)_3$ as

$$B(\alpha)=W^+(\alpha). \tag{14_4}$$

Inserting now Eq. $(14)_4$ into Eq. $(14)_2$ we get

$$G^+(\rho,\alpha)=\frac{1}{J(\alpha)}\left[W^+(\alpha)J_m(K\rho)+\int_0^a\left(f(t)-i\alpha g(t)\right)Q(t,\rho,\alpha)t\,\mathrm{d}t\right]. \tag{15}$$

Although the left-hand side of Eq. (15) is regular in the half plane $\operatorname{Im}\alpha>\operatorname{Im}(-k)$, the regularity of the right hand side is violated by the presence of simple poles (see Fig. 3) lying at the upper half-plane, namely, at $\alpha=\alpha_{mp}$ satisfying

$$ika\beta_1J_m(\gamma_{mp})+\gamma_{mp}J'_m(\gamma_{mp})=0, \qquad \alpha_{mp}=\sqrt{k^2-\left(\frac{\gamma_{mp}}{a}\right)^2}, \qquad \operatorname{Im}\alpha_{mp}\ge\operatorname{Im}k. \tag{16}$$

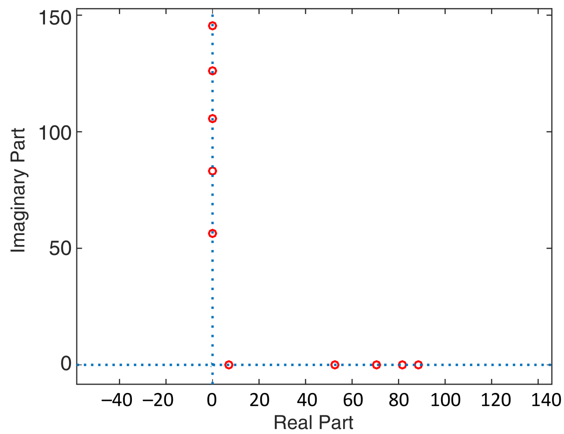

Fig. 3. Zeros of the function $J(\alpha)$ for $f=5000$ Hz, $a=0.2$ m, $\beta_1=2i$, $m=2$.

In order to provide regularity of the right hand side of Eq. (15) in the upper half of the α-plane, these poles have to be eliminated by imposing that their residues are equal to zero. This gives

$$W^+(\alpha_{mp})=\frac{a}{2}J_m(\gamma_{mp})\left[1-\frac{(\beta_1ka)^2+m^2}{\gamma_{mp}^2}\right]\cdot\left[f_{mp}-i\alpha_{mp}g_{mp}\right], \tag{17_1}$$

with

$$\begin{bmatrix}f_{mp}\\g_{mp}\end{bmatrix}=\frac{2}{a^2J_m^2(\gamma_{mp})\left[1-\frac{(\beta_1ka)^2-m^2}{\gamma_{mp}^2}\right]}\cdot\int_0^a\begin{bmatrix}f(t)\\g(t)\end{bmatrix}J_m\left(\frac{\gamma_{mp}}{a}t\right)t\,\mathrm{d}t. \tag{$17_{2,3}$}$$

Using the continuity relation $(3)_6$ together with Eq. (9) and taking into account Eq. (15) give

$$-\frac{e^{i\alpha l_2}}{2J(\alpha)}\int_0^a (f(t)-i\alpha g(t))\,J_m(Kt)t\,\mathrm{d}t = \frac{a}{2}e^{-i\alpha l_1}\cdot F^-(a,\alpha)N(\alpha)+\frac{e^{i\alpha l_2}W^+(\alpha)}{M(\alpha)}-\frac{a}{2}F_1(a,\alpha), \tag{18}$$

where

$$M(\alpha)=\pi i J(\alpha)H(\alpha),$$

$$N(\alpha)=-\frac{KH_m^{(1)\prime}(Ka)}{H(\alpha)}.$$

Owing to Eqs $(17)_{2,3}$, $f(\rho)$ and $g(\rho)$ can be expanded into Dini series as follows (Büyükaksoy, Polat, 1998):

$$\begin{aligned} f(\rho) &= \sum_{p=1}^{\infty} f_{mp}J_m\left(\frac{\gamma_{mp}}{a}\rho\right),\\ g(\rho) &= \sum_{p=1}^{\infty} g_{mp}J_m\left(\frac{\gamma_{mp}}{a}\rho\right). \end{aligned} \tag{19}$$

Substituting Eqs (19) into Eq. (18) and evaluating the resulting integrals term by term we get the following modified Wiener-Hopf equation valid in the strip $\operatorname{Im}(-k)<\operatorname{Im}\alpha<\operatorname{Im}k$:

$$\frac{a}{2}e^{-i\alpha l_1}F^-(a,\alpha)N(\alpha)+\frac{e^{i\alpha l_2}W^+(\alpha)}{M(\alpha)}-\frac{a}{2}F_1(a,\alpha) = e^{i\alpha l_2}\frac{a}{2}\sum_{p=1}^{\infty}\frac{J_m(\gamma_{mp})}{\alpha_{mp}^2-\alpha^2}\left[f_{mp}-i\alpha g_{mp}\right]. \tag{20}$$

2.3. Approximate solution of the modified Wiener-Hopf equation

By using the Wiener-Hopf factorisation method, the kernel functions $M(\alpha)$ and $N(\alpha)$ can be written as

$$\begin{aligned} M(\alpha) &= \frac{M^+(\alpha)}{M^-(\alpha)},\\ N(\alpha) &= \frac{N^+(\alpha)}{N^-(\alpha)}, \end{aligned} \tag{21}$$

where $M^+(\alpha)$, $N^+(\alpha)$ and $M^-(\alpha)$, $N^-(\alpha)$ are the split functions regular and free of zeros in the upper ($\operatorname{Im}\alpha>\operatorname{Im}(-k)$) and lower ($\operatorname{Im}\alpha<\operatorname{Im}k$) halves of the complex α-plane, respectively (Demir, Rienstra, 2010). Note that, when we let $|\alpha|\to\infty$ in their respective regions of regularity, we have

$$\begin{aligned} M^\pm(\alpha) &= O(\pm\alpha^{1/2}),\\ N^\pm(\alpha) &= O(1). \end{aligned} \tag{22}$$

The multiplication of both sides of Eq. (20) by $e^{-i\alpha l_2}/M_-(\alpha)$ and decomposition of the resulting equation into the Wiener-Hopf form leads to

$$\frac{W^+(\alpha)}{M_+(\alpha)} = -\frac{1}{2\pi i}\frac{a}{2}\int_{L_+}\frac{N(\tau)e^{-i\tau(l_2+l_1)}F^-(a,\tau)}{(\tau-\alpha)M_-(\tau)}\,\mathrm{d}\tau + \frac{1}{2\pi i}\frac{a}{2}\int_{L_+}\frac{1}{(\tau-\alpha)M_-(\tau)}\cdot\sum_{p=1}^{\infty}\frac{J_m(\gamma_{mp})}{\alpha_{mp}^2-\tau^2}\left[f_{mp}-i\tau g_{mp}\right]\mathrm{d}\tau, \tag*{$(23)_1$}$$

then, multiplying Eq. (20) with $e^{i\alpha l_1}/N_+(\alpha)$, we write

$$\frac{a}{2}\frac{F^-(a,\alpha)}{N_-(\alpha)} = \frac{1}{2\pi i}\int_{L_-}\frac{e^{i\tau(l_2+l_1)}W^+(\tau)}{N_+(\tau)M(\tau)(\tau-\alpha)}\,\mathrm{d}\tau - \frac{a}{2}\frac{1}{2\pi i}\int_{L_-}\frac{e^{i\tau(l_2+l_1)}}{N_+(\tau)(\tau-\alpha)}\cdot\sum_{p=1}^{\infty}\frac{J_m(\gamma_{mp})}{\alpha_{mp}^2-\tau^2}\left[f_{mp}-i\tau g_{mp}\right]\mathrm{d}\tau. \tag*{$(23)_2$}$$

For $k(l_1+l_2)\gg 1$, the coupled system of Fredholm integral equations of the second kind in Eqs $(23)_1$ and $(23)_2$ is susceptible to a treatment by iterations

$$\begin{aligned} W^+(\alpha) &= W_1^+(\alpha)+W_2^+(\alpha)+\cdots,\\ F^-(a,\alpha) &= F_1^-(a,\alpha)+F_2^-(a,\alpha)+\cdots. \end{aligned} \tag{24}$$

The first iterations gives

$$\frac{W_1^+(\alpha)}{M_+(\alpha)} = \frac{a}{2}\sum_{p=1}^{\infty}\frac{J_m(\gamma_{mp})\left[f_{mp}+i\alpha_{mp}g_{mp}\right]}{2\alpha_{mp}(\alpha+\alpha_{mp})M_-(-\alpha_{mp})} \tag*{$(25)_1$}$$

and

$$\frac{a}{2}\frac{F_1^-(a,\alpha)}{N_-(\alpha)} = -\frac{a}{2}\cdot\sum_{p=1}^{\infty}\frac{J_{mp}(\gamma_{mp})\left[f_{mp}-i\alpha_{mp}g_{mp}\right]e^{i\alpha_{mp}(l_2+l_1)}}{2\alpha_{mp}N_+(\alpha_{mp})(\alpha-\alpha_{mp})}, \tag*{$(25)_2$}$$

while the second iteration reads

$$\frac{W_2^+(\alpha)}{M_+(\alpha)} = \frac{a}{2}\cdot\sum_{p=1}^{\infty}\frac{J_m(\gamma_{mp})\left[f_{mp}-i\alpha_{mp}g_{mp}\right]e^{i\alpha_{mp}(l_2+l_1)}}{2\alpha_{mp}N_+(\alpha_{mp})}I_1(\alpha), \tag*{$(25)_3$}$$

$$\frac{a}{2}\frac{F_2^-(a,\alpha)}{N_-(\alpha)} = \frac{a}{2}\cdot\sum_{p=1}^{\infty}\frac{J_m(\gamma_{mp})\left[f_{mp}+i\alpha_{mp}g_{mp}\right]}{2\alpha_{mp}M_-(-\alpha_{mp})}I_2(\alpha), \tag*{$(25)_4$}$$

with

$$I_1(\alpha)=\frac{1}{2\pi i}\int\limits_{L_+}\frac{N_-(\tau)N(\tau)e^{-i\tau(l_2+l_1)}}{(\tau-\alpha)(\tau-\alpha_{mp})M_-(\tau)}\,\mathrm{d}\tau, \qquad (25)_5$$

$$I_2(\alpha)=\frac{1}{2\pi i}\int\limits_{L_-}\frac{M_+(\tau)e^{i\tau(l_2+l_1)}}{N_+(\tau)M(\tau)(\tau-\alpha)(\tau+\alpha_{mp})}\,\mathrm{d}\tau. \qquad (25)_6$$

Consider first the asymptotic evaluation of $I_1(\alpha)$. According to Jordan's Lemma, the integration line L_+ can be deformed into the branch-cut C_1+C_2 through the branch point $\tau=-k$ (see Fig. 2). $I_1(\alpha)$ can be rearranged as follows:

$$I_1(u)=-\frac{1}{2\pi i}\left[\int\limits_{C_1}\frac{N_-(\tau)N(\tau)e^{-i\tau(l_2+l_1)}}{(\tau-\alpha_{mp})M_-(\tau)}\frac{\mathrm{d}\tau}{(\tau-\alpha)} + \int\limits_{C_2}\frac{N_-(\tau)N(\tau)e^{-i\tau(l_2+l_1)}}{(\tau-\alpha_{mp})M_-(\tau)}\frac{\mathrm{d}\tau}{(\tau-\alpha)}\right]. \qquad (26)_1$$

By using the property

$$H_n^{(1)}\left(e^{i\pi}z\right)=-e^{-in\pi}H_n^{(2)}(z), \qquad n \text{ integer} \qquad (26)_2$$

and making the following substitution:

$$k+\tau=te^{-i\pi/2}, \qquad t>0 \qquad (27)$$

the integral in Eq. $(26)_1$ can be reduced to the following one written over $\mathbb{R}^+$

$$I_1(u)=-\frac{2k\beta_1}{\pi^2 a}e^{ik(l_2+l_1)} \cdot\int\limits_0^\infty\frac{N_-(-k-it)}{(k+it+\alpha_{mp})M_-(-k-it)P(t)}\frac{e^{-t(l_2+l_1)}}{k+it+\alpha}\,\mathrm{d}t, \qquad (28)$$

where

$$P(t)=\left(ik\beta_1 H_m^{(1)}(Ka)+KH_m^{(1)\prime}(Ka)\right)\cdot\left(ik\beta_1 H_m^{(2)}(Ka)+KH_m^{(2)\prime}(Ka)\right).$$

When l_1+l_2 is large, the main contribution to the integral in Eq. (28) comes from the end point $t=0$. Hence, $I_1(u)$ can be approximated by

$$I_1(\alpha)=-\frac{2k\beta_1}{\pi^2 a(k+\alpha_{mp})}e^{ik(l_2+l_1)}\cdot\frac{N_-(-k)}{M_-(-k)}\xi_1\left(a,l_{1,2},\beta_1;\alpha\right), \qquad (29)_1$$

where the function

$$\xi_1\left(a,l_{1,2},\beta_1;\alpha\right)=\int\limits_0^\infty\frac{e^{-t(l_2+l_1)}}{P(t)(k+it+\alpha)}\,\mathrm{d}t \qquad (29)_2$$

is to be evaluated numerically. By proceeding similarly, we get the following approximate expression for $I_2(u)$. The result can be written as

$$I_2(\alpha)=\frac{1}{i\pi^2(k+\alpha_{mp})}e^{ik(l_2+l_1)}\frac{M_+(k)}{N_+(k)}\xi_2\left(a,l_{1,2},\beta_1;\alpha\right) - \frac{1}{2}\sum_{n=1}^\infty\frac{M_+(\alpha_{mn})e^{i\alpha_{mn}(l_2+l_1)}}{N_+(\alpha_{mn})(\alpha_{mn}-\alpha)(\alpha_{mn}+\alpha_{mp})\alpha_{mn}A^*}, \qquad (29)_3$$

with

$$\xi_2\left(a,l_{1,2},\beta_1;\alpha\right)=\int\limits_0^\infty\frac{e^{-t(l_2+l_1)}}{P(t)(k+it-\alpha)}\,\mathrm{d}t, \qquad (29)_4$$

and

$$A^*=\left(1-\frac{(\beta_1 ka)^2+m^2}{\gamma_{mn}^2}\right). \qquad (29)_5$$

The approximate solution to the system $(23)_1$ and $(23)_2$ can now be written as

$$\begin{aligned}\frac{W^+(a)}{M_+(\alpha)} &\simeq \frac{a}{2}\sum_{p=1}^\infty\frac{B^*}{2\alpha_{mp}(\alpha+\alpha_{mp})M_-(-\alpha_{mp})} + \frac{a}{2}\sum_{p=1}^\infty\frac{C^*e^{i\alpha_{mp}(l_2+l_1)}}{2\alpha_{mp}N_+(\alpha_{mp})}I_1(\alpha),\\ \frac{F^-(a,\alpha)}{N_-(\alpha)} &\simeq -\sum_{p=1}^\infty\frac{C^*e^{i\alpha_{mp}(l_2+l_1)}}{2\alpha_{mp}N_+(\alpha_{mp})(\alpha-\alpha_{mp})} + \sum_{p=1}^\infty\frac{B^*}{2\alpha_{mp}M_-(-\alpha_{mp})}I_2(\alpha),\end{aligned} \qquad (30)$$

where

$$B^*=J_m(\gamma_{mp})[f_{mp}+i\alpha_{mp}g_{mp}],$$
$$C^*=J_m(\gamma_{mp})[f_{mp}-i\alpha_{mp}g_{mp}].$$

2.4. Determination of the expansion coefficients

In region $\rho<a$, $0<z<l_2$, $u_3(\rho,z)$ can be expressed as

$$u_3(\rho,z)=\sum_{n=1}^\infty\left[a_{mn}e^{i\chi_{mn}z}+b_{mn}e^{-i\chi_{mn}z}\right]J_m\left(\frac{\xi_{mn}}{a}\rho\right), \qquad (31)_1$$

with

$$\chi_{mn}=\sqrt{k^2-\left(\frac{\xi_{mn}}{a}\right)^2}. \qquad (31)_2$$

Here ξ_{mn}'s are the roots of the characteristic equation

$$ika\beta_2 J_m(\xi_{mn})-\xi_{mn}J_m'(\xi_{mn})=0. \qquad (31)_3$$

Consider now the continuity relations in Eqs $(3)_7$ and $(3)_8$, namely

$$\begin{aligned} f(\rho)&=\frac{\partial}{\partial z}u_2(\rho,l_2)=\frac{\partial}{\partial z}u_3(\rho,l_2),\\ g(\rho)&=u_2(\rho,l_2)=u_3(\rho,l_2).\end{aligned} \qquad (32)$$

Then, substituting Eqs (19), $(31)_1$ and its derivative with respect to z, into Eqs (32) we obtain

$$\sum_{p=1}^{\infty} f_{mp} J_m\left(\frac{\gamma_{mp}}{a}\rho\right) = i\sum_{n=1}^{\infty}\chi_{mn} D^* J_m\left(\frac{\xi_{mn}}{a}\rho\right),$$
$$\sum_{p=1}^{\infty} g_{mp} J_m\left(\frac{\gamma_{mp}}{a}\rho\right) = \sum_{n=1}^{\infty} E^* J_m\left(\frac{\xi_{mn}}{a}\rho\right), \tag{33}$$

where

$$D^* = \left[a_{mn}e^{i\chi_{mn}l_2} - b_{mn}e^{-i\chi_{mn}l_2}\right],$$
$$E^* = \left[a_{mn}e^{i\chi_{mn}l_2} + b_{mn}e^{-i\chi_{mn}l_2}\right].$$

The multiplication of both sides of Eqs (33) with $\rho J_m\left(\frac{\xi_{ml}}{a}\rho\right)$ and integration of the resulting relations with respect to ρ from $\rho = 0$ to $\rho = a$ yields for $n = l$

$$a_{mn} = \frac{a^2 e^{-i\chi_{mn}l_2}}{2i\chi_{mn}P_{mn}}\sum_{p=1}^{\infty}\frac{f_{mp}+i\chi_{mn}g_{mp}}{\gamma_{mp}^2-\xi_{mn}^2}\cdot \xi_{mn}J_m'(\xi_{mn})J_m(\gamma_{mp})\left(1+\frac{\beta_1}{\beta_2}\right),$$
$$b_{mn} = -\frac{a^2 e^{i\chi_{mn}l_2}}{2i\chi_{mn}P_{mn}}\sum_{p=1}^{\infty}\frac{f_{mp}-i\chi_{mn}g_{mp}}{\gamma_{mp}^2-\xi_{mn}^2}\cdot \xi_{mn}J_m'(\xi_{mn})J_m(\gamma_{mp})\left(1+\frac{\beta_1}{\beta_2}\right), \tag{34}$$

where

$$P_{mn} = \frac{a^2}{2}\left[\left(1-\frac{m^2}{\xi_{mn}^2}\right)J_m^2(\xi_{mn}) + J_m^{2\prime}(\xi_{mn})\right].$$

In region $\rho < a$, $z < 0$, $u_4(\rho, z)$ can be expressed as

$$u_4(\rho,z) = \sum_{n=0}^{\infty} c_{mn}e^{-i\sigma_{mn}z}J_m\left(\frac{j_{mn}}{a}\rho\right), \tag{35}$$

from the continuity relations which is given as $(3)_9$ and $(3)_{10}$ we write

$$\sum_{n=1}^{\infty}\left[a_{mn}+b_{mn}\right]J_m\left(\frac{\xi_{mn}}{a}\rho\right) = \sum_{p=0}^{\infty}F^* + A_{mr}J_m((j_{mr}/a)\rho),$$
$$i\sum_{n=1}^{\infty}\chi_{mn}\left[a_{mn}-b_{mn}\right]J_m\left(\frac{\xi_{mn}}{a}\rho\right) = -i\sum_{p=0}^{\infty}\sigma_{mp}F^* + i\sigma_{mr}A_{mr}J_m((j_{mr}/a)\rho), \tag{36}$$

where

$$F^* = c_{mp}J_m\left(\frac{j_{mp}}{a}\rho\right).$$

Similarly, one can obtain

$$a_{mn} = \frac{a^2}{2\chi_{mn}P_{mn}}\left\{\sum_{p=0}^{\infty}c_{mp}G^*(\chi_{mn}-\sigma_{mp}) + A_{mr}H^*(\chi_{mn}+\sigma_{mr})\right\},$$
$$b_{mn} = \frac{a^2}{2\chi_{mn}P_{mn}}\left\{\sum_{p=0}^{\infty}c_{mp}G^*(\chi_{mn}+\sigma_{mp}) + A_{mr}H^*(\chi_{mn}-\sigma_{mr})\right\}, \tag{37}$$

where

$$G^* = \frac{\xi_{mn}J_m'(\xi_{mn})J_m(j_{mp})}{j_{mp}^2-\xi_{mn}^2},$$
$$H^* = \frac{\xi_{mn}J_m'(\xi_{mn})J_m(j_{mr})}{j_{mr}^2-\xi_{mn}^2}.$$

If we consider Eqs (37) together with Eqs (34)

$$\sum_{p=1}^{\infty}\frac{f_{mp}+i\chi_{mn}g_{mp}}{\gamma_{mp}^2-\xi_{mn}^2}J_m(\gamma_{mp}) = \frac{\beta_2}{\beta_1+\beta_2}ie^{i\chi_{mn}l_2}\cdot\sum_{p=0}^{\infty}c_{mp}\frac{J_m(j_{mp})}{j_{mp}^2-\xi_{mn}^2}(\chi_{mn}-\sigma_{mp}) + A_{mr}\frac{\beta_2}{\beta_1+\beta_2}ie^{i\chi_{mn}l_2}\frac{J_m(j_{mr})}{j_{mr}^2-\xi_{mn}^2}(\chi_{mn}+\sigma_{mr}), \tag*{$(38)_1$}$$

$$\sum_{p=1}^{\infty}\frac{f_{mp}-i\chi_{mn}g_{mp}}{\gamma_{mp}^2-\xi_{mn}^2}J_m(\gamma_{mp}) = -\frac{\beta_2}{\beta_1+\beta_2}ie^{-i\chi_{mn}l_2}\cdot\sum_{p=0}^{\infty}c_{mp}\frac{J_m(j_{mp})}{j_{mp}^2-\xi_{mn}^2}(\chi_{mn}+\sigma_{mp}) - A_{mr}\frac{\beta_2}{\beta_1+\beta_2}ie^{-i\chi_{mn}l_2}\frac{J_m(j_{mr})}{j_{mr}^2-\xi_{mn}^2}(\chi_{mn}-\sigma_{mr}). \tag*{$(38)_2$}$$

By substituting $\alpha = \alpha_1, \alpha_2, \alpha_3, \ldots$ in Eq. $(17)_1$ and using Eq. $(30)_1$ we obtain

$$\frac{J_m(\gamma_{mr})\left[1-\frac{(\beta_1 ka)^2+m^2}{\gamma_{mr}^2}\right]\left[f_{mr}-i\alpha_{mr}g_{mr}\right]}{M_+(\alpha_{mr})} = \sum_{p=1}^{\infty}\frac{J_m(\gamma_{mp})\left[f_{mp}+i\alpha_{mp}g_{mp}\right]}{2\alpha_{mp}(\alpha_{mr}+\alpha_{mp})M_-(-\alpha_{mp})} + \sum_{p=1}^{\infty}\frac{J_m(\gamma_{mp})\left[f_{mp}-i\alpha_{mp}g_{mp}\right]e^{i\alpha_{mp}(l_2+l_1)}}{2\alpha_{mp}N_+(\alpha_{mp})}I_1(\alpha_{mr}). \tag*{$(38)_3$}$$

Expressions given by Eqs $(38)_1$, $(38)_2$ and $(38)_3$ are the required linear systems of algebraic equations which permit us to determine f_{mp}, g_{mp}, and c_{mp}.

2.5. Far field

To calculate the field outside the duct one has to consider $\rho > a$, $z \in (-\infty, \infty)$, and $\rho < a$, $z > l_2$. In the present paper we limit ourselves to the $\rho > a$. For the field for $\rho < a$ (Snakowska, 1992).

By taking the Fourier transfrom of $F(\rho, \alpha)$, the far field in the region $\rho > a$ can be obtained from Eq. $(8)_1$

$$u_1(\rho, z) = \frac{1}{2\pi}\int_\Gamma ik\beta_1 F^-(a,\alpha)\frac{H_m^{(1)}(K\rho)}{H(\alpha)}e^{-i\alpha(z+l_1)}\,\mathrm{d}\alpha + \frac{1}{2\pi}\int_\Gamma W^+(\alpha)\frac{H_m^{(1)}(K\rho)}{H(\alpha)}e^{-i\alpha(z-l_2)}\,\mathrm{d}\alpha, \tag{39}$$

where Γ is a straight line parallel to the real α-axis, lying in the strip $\mathrm{Im}\,(-k) < \mathrm{Im}\,\alpha < \mathrm{Im}\,k$. Utilising the asymptotic expansion of $H_m^{(1)}(K\rho)$ as $k\rho \to \infty$ and using the saddle point technique (Snakowska, Idczak, 2006), we obtain

$$u_1(\rho, z) \approx \frac{k}{i\pi}e^{-(im\pi)/2}\left[\frac{ik\beta_1 F^-(a, -k\cos\theta_1)}{H(-k\cos\theta_1)}\frac{e^{ikr_1}}{kr_1} + \frac{W^+(-k\cos\theta_2)}{H(-k\cos\theta_2)}\frac{e^{ikr_2}}{kr_2}\right], \tag{40}$$

where $F^-(a,\alpha)$ and $W^+(\alpha)$ are given by Eqs (30), respectively. r_1, θ_1, and r_2, θ_2 are the spherical coordinates defined by

$$\rho = r_1 \sin\theta_1, \qquad z + l_1 = r_1\cos\theta_1 \tag{41$_1$}$$

and

$$\rho = r_2 \sin\theta_2, \qquad z - l_2 = r_2\cos\theta_2. \tag{41$_2$}$$

In the far-field region we have (Turetken *et al.*, 2003)

$$\theta_1 \approx \theta_2,$$
$$r_1 = \begin{cases} r_2 + (l_1 + l_2)\cos\theta_1, & \text{for the phase term,} \\ r_2, & \text{for the amplitute term,}\end{cases} \tag{42}$$

and (40) reduces to

$$u_1(\rho, z) \approx D(\theta)\frac{e^{ikr_1}}{r_1}, \tag{43$_1$}$$

where $D(\theta)$ is directivity given by

$$D(\theta) = \frac{e^{-(im\pi)/2}}{i\pi}\left[\frac{I^*}{H(-k\cos\theta_1)}\right], \tag{43$_2$}$$

where

$$I^* = ik\beta_1 F^-(a, -k\cos\theta_1) + e^{-ik(l_1+l_2)\cos\theta_1}W^+(-k\cos\theta_1).$$

3. Computational Results

In this section some graphics displaying the effects of the surface admittances $\beta_{1,2}$ at some frequencies f on the radiated field are presented. The far field values are plotted at a distance 46 m away from the duct edge. The numerical results are produced for Sound Pressure Level (SPL), defined by

$$\mathrm{SPL} = 20\log\left|\frac{p}{2\sqrt{2}\cdot 10^{-5}}\right|,$$

where p is the amplitude of the acoustic pressure of the sound wave, with the observation angle θ_1 changing from 0 to π. Infinite series are truncated at some number N. Since the series converge rapidly its effect on the total field is nearly absent. Some parameter values remain unchanged in all examples. They are given below

Truncation Number	$N = 10$,
Speed of Sound	$c = 340.17$ m/s,
Density of Un. Med.	$\rho_0 = 1.255$ kg/m^3,
Far Radius	$r_1 = 46$ m,
Initial Point of Outer Imp.	$l_1 = 0.1$ m,
Duct Extension	$l_2 = 0.2$ m.

Figure 4 shows the variation of the sound pressure level against the observation angle θ_1 for values of $f = 2000$ Hz, $a = 0.1$ m, $\beta_2 = 0.1i$, and $m = 2$. It is seen that the sound pressure level decreases with lining compared to the rigid surface, especially after 90 degrees.

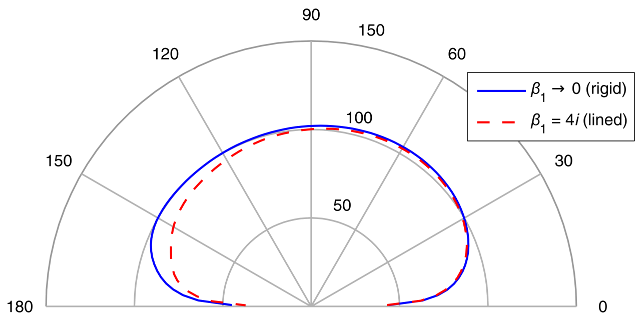

Fig. 4. Sound pressure level for rigid-lined duct with $f = 2000$ Hz, $a = 0.1$ m, $\beta_2 = 0.1i$, $m = 2$ (order of Bessel, Neumann and Hankel functions).

In Fig. 5, it is observed that the sound pressure level decreases at the beginning and end observation angles when the value of m is increased. For the high frequency value, the sound pressure level also decreases with lining like in Fig. 4.

From Fig. 6, one can see that a second mode is revealed when the duct radius is increased. Similarly, as the value of m increases, the sound pressure level at the begining and end angles also decreases.

Figure 7 depicts the variation of the sound pressure level against the observation angle θ_1 for values of $f = 3500$ Hz, $a = 0.2$ m, $\beta_2 = 0.5i$, and $m = 10$. For

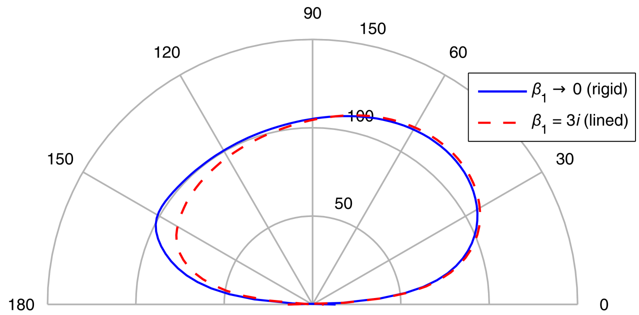

Fig. 5. Sound pressure level for rigid-lined duct with $f = 5000$ Hz, $a = 0.1$ m, $\beta_2 = 0.25i$, $m = 5$.

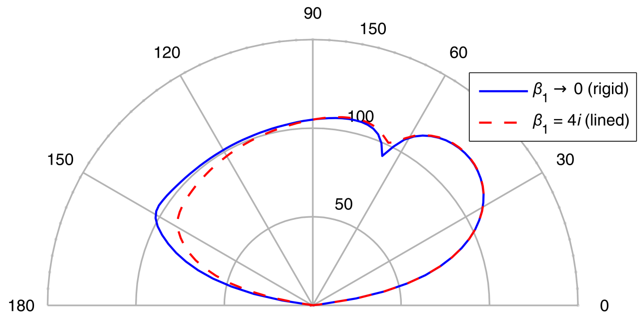

Fig. 6. Sound pressure level for rigid-lined duct with $f = 2000$ Hz, $a = 0.5$ m, $\beta_2 = 0.1i$, $m = 10$.

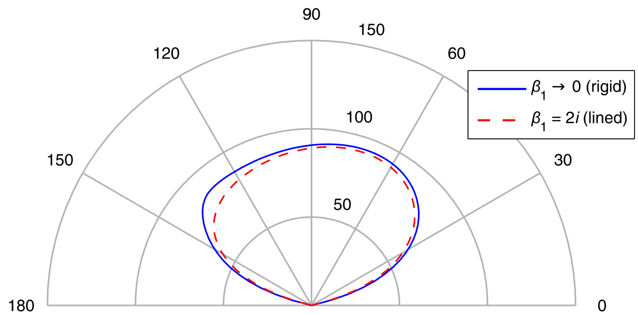

Fig. 7. Sound pressure level for rigid-lined duct with $f = 3500$ Hz, $a = 0.2$ m, $\beta_2 = 0.5i$, $m = 10$.

these parameters, the effect of the sound pressure level is observed at the angles approximately from 30 to 150.

In Fig. 8, it is seen that a second mode is revealed for high frequency and small duct radius when it is compared with Fig. 6. We can see that the main effect

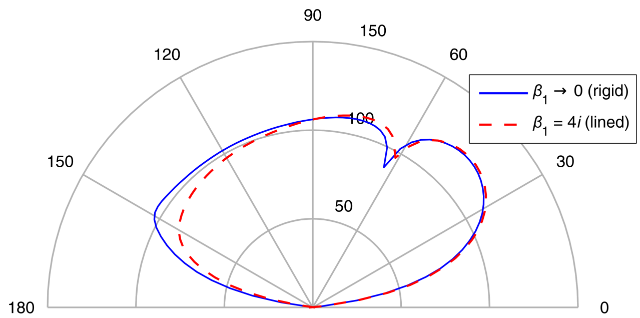

Fig. 8. Sound pressure level for rigid-lined duct with $f = 5000$ Hz, $a = 0.2$ m, $\beta_2 = 0.1i$, $m = 10$.

for the second mode depends on the frequency and the duct radius.

Figures 9 and 10 show the variation of the sound pressure level against the observation angle θ_1 for different parameter values. As it can be seen, the sound pressure level decreases with increasing value of m for angles from 0 to 55 and 130 to 180. Moreover, when the value of m increases, one can see that the first mode is dominant and the second mode is not revealed.

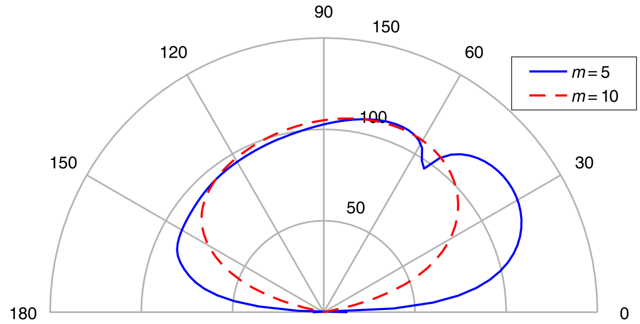

Fig. 9. Sound pressure level for $f = 3500$ Hz, $a = 0.2$ m, $\beta_1 = 4i$, $\beta_2 = 0.1i$.

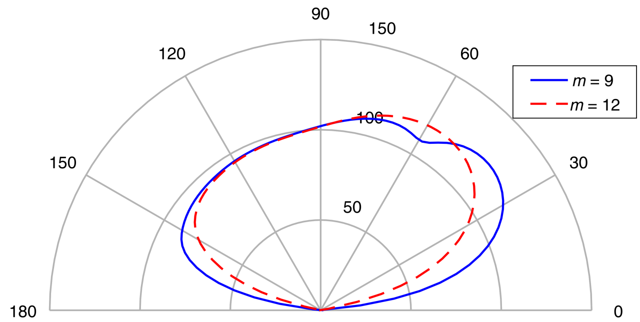

Fig. 10. Sound pressure level for $f = 5000$ Hz, $a = 0.2$ m, $\beta_1 = 2i$, $\beta_2 = 0.5i$.

Finally, Figs 11 and 12 depict an excellent agreement both in normalised radiated field and reflec-

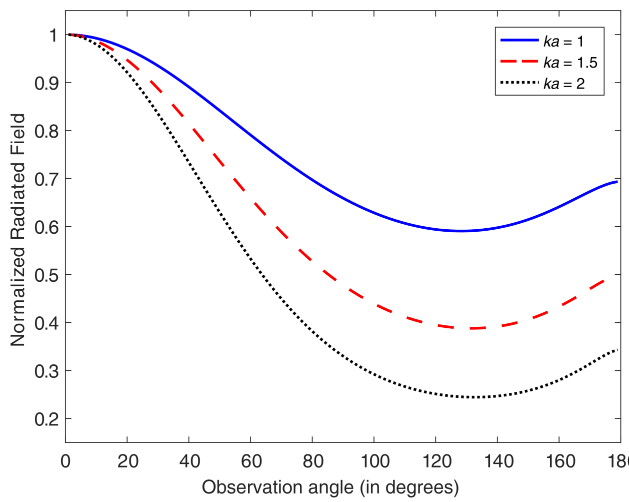

Fig. 11. Comparison of the normalised radiated field with the study of (Demir, Büyükaksoy, 2003) for $kl_1 = 0$, $kl_2 = 10$, $\beta_1 = 0$, $\beta_2 = 0.1i$, $m = 0$.

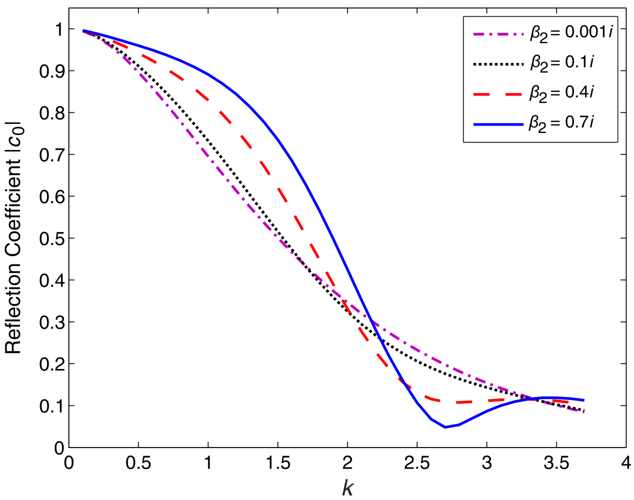

Fig. 12. Comparison of the refflection coefficient $|c_0|$ with the study of (Demir, Büyükaksoy, 2003) for $a = 1$, $l_1 = 0$, $l_2 = 1$, $\beta_1 = 0$, $m = 0$.

tion coefficient $|c_0|$, between the present paper ($\beta_1 = 0$) and the previous study (Demir, Büyükaksoy, 2003). With these graphs, we can see the accuracy of all mathematical operations (factorisations, Fredholm integral equations, iterations, etc.) which are much more complicated than the one presented by (Demir, Büyükaksoy, 2003).

4. Conclusions

A rigorous Modified Wiener-Hopf solution is presented for the problem of radiation of sound waves from a semi-infinite circular cylindrical duct whose outer and inner surface is treated by an acoustically absorbing material of a finite length. In this work, the lined region of the outer surface is assumed to be finite, which makes the problem more complicated. The problem is reduced to a modified Wiener-Hopf equation whose solution involves infinitely many expansion coefficients satisfying an infinite system of linear algebraic equations, solved iteratively by using the classical factorisation and decomposition procedures. A numerical solution to these systems is obtained for various values of the problem parameters such as rigid-lined cases, m (order of Bessel, Neumann and Hankel functions), etc.

As is well known, the inner absorbent lining provided a few decibel of sound wave reduction. In addition, the effect of partial outer lining on sound pressure level is clearly seen from Figs 4–8. Considering the cost and applicability, the importance of a finite coating is obvious.

When the outer lining is zero ($\beta_1 = 0$), which corresponds to the rigid case, the results obtained in this paper are compared with the results of (Demir, Büyükaksoy, 2003) and the agreement is perfect. In addition, these results show that the complex mathematical operations encountered are rigorously examined.

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