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THE ACOUSTIC WAVE GENERATION BY A THIN LAYER WITH VARIABLE TEMPERATURE

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This study considers the problem of the thermal sound generation in the air by an unmoving solid body with a time-variable temperature. Applying the linear relations between the thermal energy stream supplied to a thin layer of a solid body in touch with a gaseous medium and an energy stream of the acoustic wave, and then taking advantage of electrothermo-acoustic analogies, the phenomena of the energy transport were represented in the form of an equivalent linear electric circuit with lumped constants. On the basis of the equivalent circuit, both the properties of the considered thermal source of the acoustic wave were investigated and the problem of thermo-acoustic cooling of the solid body was discussed. It was shown that, in view of the low efficiency of the temperature–pressure conversion, it is necessary to generate large layer temperature changes to obtain the mean values of sound intensity, whereas the maximum of the modulus of the transmittance function of the source occurs even for very thin layers. It was also shown that in the course of cooling of thin layers the amounts of energy: that carried by the acoustic wave and that supplied to the environment as a result of external conduction are comparable, so that in the energy balance of a thin layer with varying temperature, it is necessary to take into account the two factors which bring about an energy loss.

Notation

c_0 sound velocity, $c_0^2 = \gamma P_0/\varrho_0$
c_p specific heat of gas at constant pressure
c_v specific heat of gas at constant volume
c_w specific heat of a solid body layer
C_1 thermal capacity of a solid body layer $C_1 = dgc_w$
d thickness of a solid body layer
g density of a solid body layer
h enthalpy of unit volume of the gaseous medium, the coefficient of the external conduction of a solid body
H transmittance of the thermal source of the acoustic wave $H = \hat{p}/\hat{I}_z$
I efficiency of energy flux in acoustic field
I_z efficiency of the controlled heat flux source
$j = \sqrt{-1}$
k heat conduction coefficient of the solid body

N_p pressure level in $2\cdot 10^{-5}$ Pa[dB]
p acoustic pressure
P_0 pressure in unperturbed gas
P_r Prandtl number, $P_r = c_p \nu \varrho \varkappa^{-1}$
R gas constant, $R/\mu = c_p - c_v$
R_1 heat resistance of a solid layer, $R_1 = d/k$
R_2 heat resistance corresponding to external conduction $R_2 = 1/h$
t time
T variable component of the gas temperature
T_0 temperature in unperturbed gas
T_p initial temperature of the cooling solid body
v acoustic velocity
x spatial coordinate distance in the gas from the solid body surface
Z_{ta} thermal-acoustic impedance in the harmonic excitation method, $Z_{ta} = \hat{\theta}/\hat{I}$
$\mathscr{L}\{\cdot\}$, $\mathscr{L}^{-1}\{\cdot\}$ operators of the simple and inverse Laplace transformations
γ adiabate exponent, $\gamma = c_p/c_v$
θ amplitude of the sinusoidally variable temperature of the boundary layer of the gas medium
$\varkappa$ heat conduction coefficient of the gas
Λ $-(d/dt)\ln(T(t)/T_p)$
μ molar mass
ν kinematic viscosity coefficient
ϱ variable component of the total density of the gas medium
ϱ_0 density of the gas medium in an unperturbed state
φ velocity potential $v = \varphi_x$
$\phi(z)$ probability integral, $\phi(z) = \frac{2}{\sqrt{\pi}}\int\limits_0^z e^{-x^2}dx$
ω angular frequency
$(\cdot)' = d/dx(\cdot)$
$(\hat{\cdot})$ complex amplitude in the harmonic excitation method
$(\bar{\cdot})$ Laplace transform of a given time function

1. Introduction

A stationary solid body with the time variable external surface temperature generates an acoustic wave in the surrounding gas medium. The present study carries out a quantitative analysis of the relation between a constant heat flux fed to the body (causing changes in its temperature) and a pressure level, of the generated acoustic wave. The results of this analysis made it possible to determine the fundamental properties of the thermal source of the acoustic wave.

Another problem discussed in the study is related to the above effect — namely that a solid body at a temperature higher than that of the ambient gaseous medium cools faster than results from the classical external conduction, since it gives away part of the energy to the generated acoustic wave (the thermo-acoustic cooling effect). It is justified to neglect the component energy carried out by the acoustic wave for thick layers whose temperature varies slowly in the course of cooling. On the other hand, the situation changes essentially for thin layers, and the time course of the temperature of such a layer deviates from that expected by the classical analysis of the problem.

To determine the fundamental relations characterizing the thermal source of an acoustic wave, an idealized case was considered, in which a flat layer of a solid body is in contact with a gaseous medium. A time variable heat flux with constant density is supplied to the whole Surface of the layer (Fig. 1). Moreover, the notion of

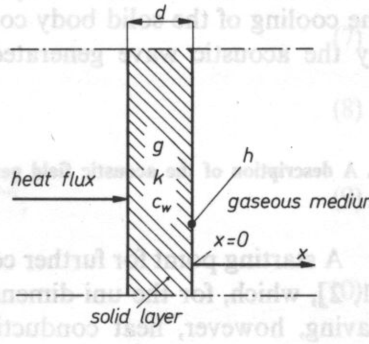

Fig. 1. The cross-section of the thermal source of a plane acoustic wave: d — solid layer thickness, g — solid body density, k — conduction coefficient, c_w — specific heat, h — external conduction coefficient

supplied heat flux denotes a flux penetrating into the solid body, neglecting the problem of the reflection of part of the incident flux (if such a way of energy supply were applied). As a result of temperature changes, the solid layer becomes a source of a plane acoustic wave propagating in the gaseous medium.

The first stage of solving the above problem is to determine the dependence between the time variable temperature on the border of the gaseous medium and the pressure in the generated acoustic wave. This dependence was obtained by solving the hydrodynamics equations with the following boundary condition

$$T(x = 0, t) = f(t), \tag{1}$$

$$v(x = 0, t) = 0, \tag{2}$$

where all these quantities describing the state of the medium are functions of the spatial variable x and the time t, v denotes the particle velocity, T is the temperature and $f(t)$ a function describing temperature changes in the boundary layer of the medium. In analysis of wave phenomena in a gas, a frequently assumed model of the medium, is an ideal gas undergoing no dissipation processes. In the considered problem, such a model is not appropriate, since the assumed boundary conditions (1) and (2) would then be mutually contradictory: they can be satisfied only if one takes into account the heat conduction of the gas and the resultant fact that all the quantities characterizing the acoustic field are expressed by the sums of two components. The two components are the solutions of auxiliary second-order equations, corresponding to the so-called pressure and temperature modes of vibration in the medium (see solutions (12) and (13)).

The value of temperature changes in the surface in contact with the gas depends on the quantity of the supplied heat flux, on the heat capacity of the solid layer, on the value of the external conduction and on the quantity of the load generated by the gaseous medium as a result of energy convection in the generated acoustic wave. Applying electro-thermo-acoustic analogies, the considered processes of energy transport wave represented in the form of a linear component electric circuit with lumped constants. On the basis of the above circuit, both the properties of the considered thermal source of the acoustic wave were investigated and the problem of the cooling of the solid body considered, taking into account the energy convection by the acoustic wave generated in the medium.

2. A description of the acoustic field generated by temperature changes in the stationary boundary layer of the medium

A starting point for further considerations are linearized Navier-Stokes equations [1, 2], which, for the uni-dimensional problem of the thermodynamically ideal gas, having, however, heat conductions and viscosity, are in the form

$$\varrho_t + \varrho_p \cdot v_x = 0, \tag{3}$$

$$v_t + \frac{1}{\varrho_0} \cdot p_x = \frac{4}{3} v_{xx}, \tag{4}$$

$$\varrho_0 c_v T_t - T_0 \frac{R}{\mu} \varrho_t = \varkappa T_{xx}, \tag{5}$$

$$\frac{\mu}{R} p = \varrho_0 T + T_0 \varrho, \tag{6}$$

where ϱ, p and T denote variable components of the total quantities of the density, of pressure and temperature; ϱ_0, P_0 and T_0 are the constant components of these quantities, R is the gas constant ($R/\mu = c_p - c_v$ where c_p and c_v denote respectively the specific heats for constant pressure and constant volume, μ is the molar mass, v is the velocity, $\varkappa$ is the heat conduction coefficient, ν is the kinematic viscosity coefficient. Along with the boundary conditions (1) and (2), the system of equations (3)–(6) makes it possible to determine the relation between temperature changes in the boundary layer of the medium and the quantities describing the generated acoustic field.

A similarly formulated problem was considered by TRILLING [3]. Since the final results of the Trilling study were obtained by the method of approximate determination of the inverse Laplace transformation, there occurs a difference between these results and those obtained by the authors of the present study. Because of the above fact and since the present authors were aware that the problems of the tempeature boundary conditions have quite seldom occurred in the acoustic literature, it was decided that the solution of the hydrodynamic equations (3)–(6) along with the temperature boundary conditions (1)–(2), should be cited in brief form.

Therefore, let all the quantities characterizing the state of the gas have the form: $a(x, t) = \hat{a}(x)\exp(j\omega t)$, where $\hat{a}(x)$ denotes the complex amplitude of a given quantity, while ω is the angular frequency. After this substitution differential equations with partial derivatives transform into the corresponding simple differential equations connecting complex amplitudes, namely:

$$j\omega\hat{\varrho}+\varrho_0\hat{v}' = 0, \tag{7}$$

$$j\omega\hat{v}+\frac{1}{\varrho_0}\hat{p}' = \frac{4}{3}\nu\hat{v}'', \tag{8}$$

$$j\omega\varrho_0\, c_v\hat{T}-j\omega T_0\frac{R}{\mu}\hat{\varrho} = \varkappa\hat{T}'', \tag{9}$$

$$\frac{\mu}{R}\hat{p} = T_0\hat{\varrho}+\varrho_0\hat{T}, \tag{10}$$

where these equations introduced the denotation $(\cdot)' = d/dx(\cdot)$ The above system of equations can be reduced to an equivalent fourth-order differential equation. As $\hat{\varphi}$ denotes the complex amplitude of the velocity potential ($\hat{v} = \hat{\varphi}$), this equation is in the form

$$\frac{\omega^2}{c_0^2}\hat{\varphi}+\left[1+j\omega\left(\frac{\varkappa}{\varrho_0 c_v c_0^2}+\frac{4}{3}\frac{\nu}{c_0^2}\right)\right]\hat{\varphi}''-\left(\frac{4}{3}\frac{\varkappa\nu}{\varrho_0 c_v c_0^2}+\frac{\varkappa}{j\omega\varrho_0 c_p}\right)\hat{\varphi}'''' = 0, \tag{11}$$

where $c_0 = \sqrt{\gamma P_0/\varrho_0}$ denotes the sound velocity and $\gamma = c_p/c_v$ is the adiabate exponent.

The characteristic equation corresponding to the differential equation (11) has the form of an algebraic biquadratic equation. The solution of equation (11) can be represented as the sum of the solutions of two second-order equations, each of which corresponds to the pair of roots of the characteristic equation. In determining these roots, simplifications were carried out, taking into account the real values of c_0, $\varkappa$, ν, ϱ_0, P_0 and T_0 for the air in normal conditions and neglecting very low-value components. Despite these simplifications, the obtained final dependencies ensure sufficient accuracy over a very broad frequency range ($\omega < 10^6$). The seconed-order equations obtained by this way and the relations between the complex amplitude of the velocity potential and the complex amplitude of the other quantities characterizing the acoustic field are as follows:

$$\hat{\varphi}''+\frac{\omega^2}{c_0^2}\hat{\varphi} = -j\frac{4}{3}\omega\nu\frac{\gamma}{c_0^2}\hat{\varphi}'', \tag{12a}$$

$$\hat{v} = \hat{\varrho}', \tag{12b}$$

$$\hat{p} = -j\omega\varrho_0\hat{\varphi}, \tag{12c}$$

$$\hat{\varrho} = -j\omega\varrho_0\frac{1}{c_0^2}\hat{\varphi}, \tag{12d}$$

$$T = -j\omega\frac{1}{c_p}\hat{\varphi}, \tag{12e}$$

and

$$\hat{\varphi}'' \frac{j\omega\varrho_0 c_p}{\varkappa}\hat{\varphi} = 0, \tag{13a}$$

$$\hat{v} = \hat{\varphi}' \tag{13b}$$

$$\hat{p} = j\omega\varrho_0\left(\frac{4}{3}\varrho_0\nu\frac{c_p}{\varkappa} - 1\right)\hat{\varphi}, \tag{13c}$$

$$\hat{\varrho} = -\varrho_0^2\frac{c_p}{\varkappa}\hat{\varphi}, \tag{13d}$$

$$\hat{T} = T_0\varrho_0\frac{c_p}{\varkappa}\hat{\varphi}. \tag{13e}$$

The values of the parameters ν and $\varkappa$ are not independent for the medium, and even for numerical studies, it is necessary to take into account their interdependencies. In particular, this interdependence is strong for gases, so that the dimensionless Prandtl number $P_r = c_p\nu\varrho\varkappa^{-1}$ takes for the most real gases values close to unity. For the air, in normal conditions, $P_r \cong 3/4$, hence, it follows that $\hat{p}$ in equation (13c) is 0 irrespective of the boundary condition and the value of the velocity potential amplitude.

Therefore, the two systems of equations (12) and (13) correspond to two different modes of the medium vibrations – namely, equation (12) corresponds to the pressure mode, equation 13 to the temperature mode.

Limiting below the considerations to the analysis of waves travelling from the source and assuming that $f(t)$ in equation (1) has the form: $f(t) = \theta\exp(j\omega t)$, it is obtained that (see [4])

$$\varphi_{\text{pressure}}(x) = A\exp\left(-j\frac{\omega}{c_0}x\right)\exp\left(-\frac{2}{3}\nu\gamma\omega^2 c_0^{-3}x\right), \tag{14}$$

$$\varphi_{\text{temp}}(x) = B\exp\left[-\sqrt{\frac{\omega\varrho_0 c_p}{2\varkappa}}\cdot(1+j)\cdot x\right], \tag{15}$$

where the constants A and B determined from the boundary conditions (1)–(2), have the form

$$A = \frac{\theta}{-j\dfrac{\omega}{c_p}\dfrac{T_0}{c_0}\sqrt{\dfrac{j\omega\varrho_0 c_p}{\varkappa}}} = -\theta\frac{c_0}{T_0}\sqrt{\frac{\varkappa}{j\omega\varrho_0 c_p}}, \qquad (16)$$

$$B = \theta\frac{\varkappa}{T_0\varrho_0 c_p}. \qquad (17)$$

From (16), (14), (12b), the pressure wave generated in the medium as a result of forced temperature, is described by the following relation:

$$\frac{\hat{p}}{p_0} = \frac{\theta}{T_0}\cdot\frac{2\gamma}{c_0}\sqrt{\frac{\omega\nu}{3}}\cdot\exp\left[-j\left(\frac{\omega}{c_0}x - \frac{\pi}{4}\right)\right]\cdot\exp\left(-\frac{2}{3}\nu\gamma\omega^2 c_0^{-3} x\right). \qquad (18)$$

In dependence (18), the occurring factor exp $[(-2/3)\nu\gamma\omega^2 c_0^3 x]$ and the related dissipation coefficient $\alpha = (2/3)\nu\omega^2 c_0^{-3}$ correspond to wave attenuation caused by the presence of dissipation effects in the medium, namely heat conduction and viscosity. In the literature, these two causes of wave energy dissipation were discussed separately (see [14, 15]). Two solutions of the problem of propagation in a viscous medium lead to the Stokes absorption coefficient $\alpha_1 = (2/3)\nu\omega^2 c_0^{-3}$, and in a heat-conducting medium, to the Kirchhoff absorption coefficient $\alpha_2 = \omega^2\varkappa(\gamma-1)/2c_0^3 c_v\varrho_0\gamma$. The total dissipation coefficient is the sum of the above factors: $\alpha_c = \alpha_1 + \alpha_2$. If one considers the relation between ν and $\varkappa$ resulting from $P_r = c_p\nu\varrho\varkappa^{-1} \approx 3/4$ it can readily be verified that α_c is the same as α gained here by another way.

It follows from dependencies (14) and (15) that velocity potentials, both in the pressure and temperature modes, contain exponential factors corresponding to the decay of both these quantities with increasing distance from the source. In both these modes the dissipation coefficients are given by the dependencies: $\alpha_{\text{pressure}} = \alpha = \frac{2}{3}\nu\gamma\omega^2 c_0^{-3}$, $\alpha_{\text{temperature}} = \sqrt{\dfrac{\omega\varrho_0 c_p}{2\varkappa}}$. Considering the values of the parameters occurring in these two expressions (see the Appendix) it appears that $\alpha_{\text{pressure}} = 3.5\cdot 10^{-13}\omega^2[\text{m}^{-1}]$, while $\alpha_{\text{temperature}} = 155\sqrt{\omega}\,[\text{m}^{-1}]$, therefore, for a very wide temperature frequency range $\alpha_{\text{temp}} \gg \alpha_{\text{pres}}$. Hence, the components of particular quantities characterizing the acoustic field related to the temperature mode rapidly decay and are essential only in a region in direct contact with the source, namely in an area where the two modes interact; where the acoustic wave is generated. The above remarks emphasize in addition the difference between these two vibration modes of the medium.

Example 1: The dependence of the the pressure level in the wave in the air on the temperature change amplitude on the boundary

Substitution in formula (18) of the values of the constants characteristic of the air in normal conditions (see the juxtaposition of the constants in the Appendix) gives

$$\hat{p} = 6.38 \cdot 10^{-3}\, \theta\sqrt{\omega}\, \exp\left[-j\left(\frac{\omega}{c_0}x - \frac{\pi}{4}\right)\right] \exp(-3.5\ 10^{-13}\omega^2 x)\ [\text{Pa}]. \tag{19}$$

It can be seen that the pressure wave is weakly attenuated, and even for $x = 0$ it is shifted in phase by $\pi/4$ with respect to the temperature on the boundary. Neglecting the attenuation and expressing the pressure in decibels with espect to $2 \cdot 10^{-5}$ Pa, it follows directly from dependence (19) that

$$N_p = 58 + 20\log(\sqrt{f}\theta)\text{dB}. \tag{20}$$

To illustrate the value of the pressure level for a given amplitude of temperature changes it was assumed that for $f = 1000$ Hz and $\theta = 1$K. Then, $N_p = 88$dB.

3. The thermo-acoustic impedance

In the case of negligible dissipation effects, the energy flux density in the acoustic field, is given by the dependence [1]:

$$I = \varrho v\left(\frac{v^2}{2} + h\right), \tag{21}$$

where h is the enthalpy of unit volume of the medium. The dissipation effects in a gas with properties close to those of the air are small (see (19)). Because of this, use will be made of dependence (21) and the relation between the pressure and enthalpy of a gas unit volume subject to adiabatic transformation, and also the relation between the pressure and the acoustic velocity of an ordinary plane wave [1, 2], namely

$$h = \frac{c_0}{\gamma - 1}\left(1 + \frac{p}{p_0}\right)^{(\gamma-1)/\gamma}, \tag{22}$$

$$v = \frac{2c_0}{\gamma - 1}\left[\left(1 + \frac{p}{p_0}\right)^{(\gamma-1)/2\gamma} - 1\right]. \tag{23}$$

From relations (21), (22) and (23), with an accuracy up to components of the second order of smallness, it is obtained that

$$I = \frac{c_0}{\gamma - 1}p + \frac{(3\gamma - 1)}{4(\gamma - 1)\varrho_0 c_0}p^2. \tag{24}$$

Considering the $p/P_0 \ll 1$ and applying (18) it is possible to determine the ratio between the complex amplitude of the boundary layer temperature of the gaseous medium and the complex amplitude of the flux energy in the acoustic wave, namely the thermo-acoustic impedance, i.e.,

$$Z_{ta} \overset{df}{=} \frac{\theta}{\hat{I}} = \frac{\gamma-1}{2\gamma}\,\frac{T_0}{P_0}\sqrt{\frac{3}{\nu}}\,\frac{1}{\sqrt{\omega}}\,e^{-j\frac{\pi}{4}} = \frac{Z_0}{\sqrt{\omega}}\,e^{-j\frac{\pi}{4}}, \tag{25}$$

where

$$Z_0 = [(\gamma-1)/(2\gamma)]\,(T_0/P_0)\sqrt{3/\nu}.$$

The determination of the impedance as a ratio between the temperature amplitude temperature surplus and the energy flux is typical of an electric modelling of heat transport processes. The impedance determined in this way has all the properties of the complex electric impedance; in particular, it is subject to the same rules of series and parallel connections.

It can be seen from the obtained dependence that as the frequency increases, the impedance Z_{ta} decreases, therefore, for a given amplitude of temperature changes at the solid body surface, for increasing frequency, there is an increase in the energy flux carried by the acoustic wave.

4. The electric equivalent circuit of the thermal source of the acoustic wave. The frequency properties of the thermal wave source

The following assignments were made:

— a difference in the potentials in the electric equivalent circuit corresponds to the increase in temperature (i.e., the variable component T of the total temperature), and — the heat flux density corresponds to the electric voltage intensity.

Assuming that the thickness of the solid layer is much thinner than the temperature wavelength in this layer, the electric equivalent circuit of the thermal wave source has the form shown in Fig. 2.

The lumped elements of the circuit are connected with the physical parameters of a solid body and the gaseous medium in the following way:

$R_1 = d/k$ is the heat resistance of a solid layer, where k is the heat conduction coefficient of the solid body, and d is the layer thickness; $C_1 = dgc_w$ is the thermal capacity of the solid body, g is the density of the solid body, c_w is the specific heat of

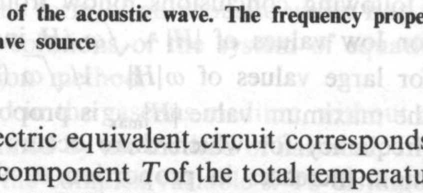

Fig. 2. An electric equivalent circuit of a thermal source working in a steady state at harmonic forcing

the solid body; $R_2 = 1/h$ is a resistance corresponding to the external conduction, and h is the external conduction coefficient; Z_{ta} is the thermo-acoustic impedance, given by dependence (25); $I_=$ is the efficiency of the controlled heat flux source.

The above lumped elements were determined for a solid layer with unit surface.

It can be noted that a consequence of the assumption of the small thickness of the solid layer is the lack of the influence of the heat resistance R_1 on the properties of the considered wave source.

On the basis of the equivalent circuit and dependences (18) and (25), it is easy to determine the transmittance $H = p/I_=$, namely

$$H = \frac{\hat{p}}{I_=} = \frac{\gamma/c_0}{1+(h+j\omega dgc_w)Z_{ta}}. \tag{26}$$

Example 2: The frequency properties of the thermal source working in the air

After including in dependencies (25) and (26) the values of the constants (see the Apppendix), the dependence of $|H|$ on the frequency and thickness of the layer is given by the expression

$$|H| = \frac{1.51\cdot 10^{-3}\omega^{0.5}}{\sqrt{10^{10}d^2\omega^2+5.2\cdot 10^4 d\omega^{1.5}+0.135\omega+0.52\omega^{0.5}+1}}. \tag{27}$$

The following conclusions follow from dependence (27):
– for low values of $|H| \sim \sqrt{\omega}$ (H increases proportionally to $\sqrt{\omega}$),
– for large values of $\omega |H| \sim 1/\sqrt{\omega}$ (H decreases proportionally to $1/\sqrt{\omega}$);
– the maximum value $|H|_{\max}$ is proportional to $1/\sqrt{d}$, and – $\omega_{\max} \sim 1/d$, namely the frequency for which there occurs a maximum modulus of the transmittance function, increases in proportion to the inverse of the layer thickness.

Fig. 3 represents the plot of $|H|$ against frequency for a few thickness of the solid layer d.

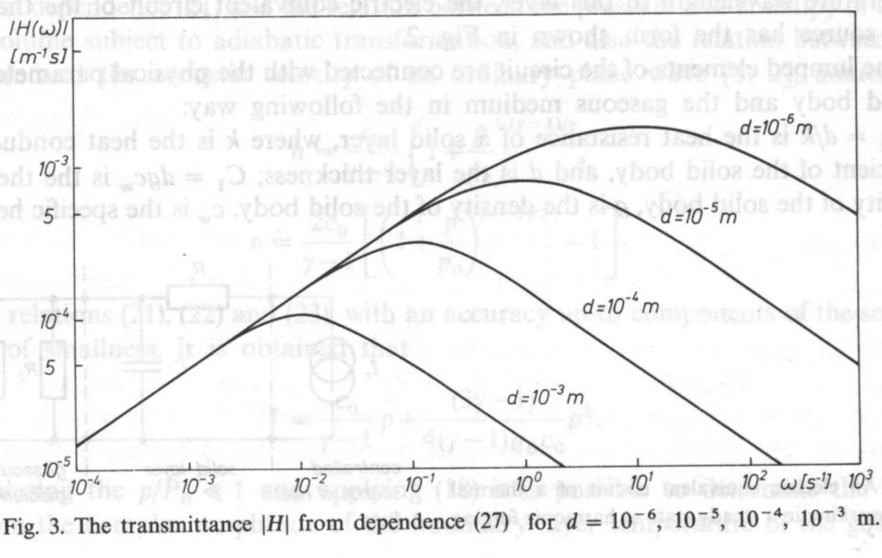

Fig. 3. The transmittance $|H|$ from dependence (27) for $d = 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}$ m.

5. The thermoacoustic cooling effect

Let the heated solid body be in contact with an unbounded gaseous medium, and let the loss of thermal energy be compensated by the external source of energy, so that the temperature of the solid body may be constant at the time of the observation. From the energy balance, it is possible to determine directly the power carried off to the ambient medium by unit surface area for the temperature difference between the solid body and the gaseous medium, namely 1K, i.e. the thermal coefficient of external conduction. If at a certain moment there is a break in the energy supply to the solid body, its temperature begins to decrease. A classical analysis of the phenomenon leads to the conclusion that the surplus of the temperature over the ambient temperature T depends on time in the following way:

$$T(t) = T_1 \exp\left(-\frac{S}{Vgc_w} ht\right),$$

where Vgc_w is the thermal capacity of the solid body, while S is the surface area of the contact with the gaseous medium; moreover, this analysis neglects the thermo-acoustic effect.

Below, the time course of the temperature of the cooling body will be determined, including the effect of energy convection in the form of an acoustic wave. For this purpose, it is more convenient to apply the solutions of the system of equations (3)-(6) obtained by the Laplace transformation method.

If zero-value initial conditions are assumed in the gaseous medium, without any changes it is possible to apply the solutions gained by the method of harmonic input signal; moreover, $j\omega$ will be now replaced by the complex variable s. To distinguish between the operator quantities and the complex amplitudes of quantities which are functions of time, two types of overhead indexes were introduced: $(\hat{\ })$ denotes the complex amplitude occurring in the harmonic excitation method, and $(\bar{\ })$ represents the Laplace transform of a given quantity.

Connecting equations (24), (28) and (32) it is possible to determine the operator thermo-acoustic impedance an equivalent to the impedance given by formula (25), namely

$$Z_{ta}(s) = \frac{\bar{\theta}(s)}{\bar{I}(s)} = Z_0 \frac{1}{\sqrt{s}}, \tag{34}$$

where

$$Z_0 = \frac{\gamma-1}{2\gamma} \frac{T_0}{P_0} \sqrt{\frac{3}{v}}.$$

Considering the dependence between the heat flux received by the solid layer and changes in its temperature

$$I_{sc} = gdc_w \, dT/dt, \tag{35}$$

where I_{sc} denotes a heat flux assimilated by unit surface area of the solid body, while the operator equivalent circuit for energy transport processes in the thermal source has the form shown in Fig. 4. The additional current source in this diagram with the efficiency $gdc_w T_p$ corresponds to the initial condition related to the initial temperature T_p of the solid body at the time $t = 0$.

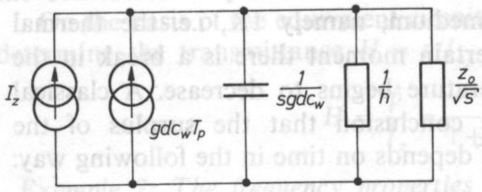

Fig. 4. The operator equivalent circuit of the thermal source of an acoustic wave

An analysis of the electric diagram shown in Fig. 4 makes it possible to determine the operator dependence corresponding to the time course of the temperature of the solid body during cooling after switching off the external source of the heat flux ($I_z = 0$), namely

$$T(s) = \frac{T_p}{s + \sqrt{s}/(Z_0 gdc_w) + h/(gdc_w)}. \tag{36}$$

The irrational form of the right side of dependence (36) causes the fact that the methods applied to determine the inverse transform which are usually used in the analysis of electric circuits are here useless.

Certainly, neglecting the energy transport in the form of an acoustic wave (corresponding to the impedance Z_0 taking an infinitely large value), the problem becomes simplified, leading to the well-known solution

$$T(t) = T_p \exp\left(-\frac{h}{gdc_w} t\right). \tag{37}$$

The calculation of the inverse Laplace transform from (36) is made easier by a certain conclusion from Efros theorem [5] which can be written in the following form:

$$\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}} A(\sqrt{s})\right\} = \frac{1}{\sqrt{\pi t}} \int_0^\infty a(\tau) \exp\left(-\frac{\tau^2}{4t}\right) d\tau, \tag{38}$$

where $A(s) = \mathscr{L}\{a(t)\}$ and $\mathscr{L}$ and $\mathscr{L}^{-1}$ denote respectively the operators of the ordinary and inverse Laplace transformations.

Using (38), it can readily be shown that

$$\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}+a}\right\} = \frac{1}{\sqrt{\pi t}} + a \exp(a^2 t)\, [\phi(a \cdot \sqrt{t}) - 1], \tag{39}$$

where

$$\phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx,$$

is the so-called probability integral.

Therefore, representing (36) in the form

$$\bar{T}(s)/T_p = 1/[(\sqrt{s}+a_1)(\sqrt{s}+a_2)] = \\ = 1/[(a_2-a_1)(\sqrt{s}+a_1)] - 1/[(a_2-a_1)(\sqrt{s}+a_2)] \tag{40}$$

and applying formula (39), the following formula of $T(t)/T_p$ is obtained:

$$\frac{T(t)}{T_p} = \frac{1}{a_2-a_1}\{a_1 \cdot \exp(a_1^2 t) \cdot [\phi(a_1\sqrt{t})-1] - a_2 \exp(a_2^2 t)[\phi(a_2\sqrt{t})-1]\} \tag{41}$$

in a special case if

$$\Delta = [1/(Z_0 g d c_w)]^2 - 4h/(g d c_w) = 0, \tag{42}$$

the equivalent of dependence (40) becomes

$$\bar{T}(s)/T_p = 1/(\sqrt{s}+a)^2, \tag{43}$$

and the time course is as follows:

$$T(t)/T_p = 2a\sqrt{t}/\sqrt{\pi} + (2a^2 t + 1)\exp(a^2 t)[1 - \Phi(a\sqrt{t})]. \tag{44}$$

The time course $T(t)$ essentially depends on the layer thickness d; for a certain characteristic thickness $d = d_0$ the discriminent Δ given by dependence (42) takes a zero value and $T(t)$ is given by formula (44). For greater thickness, a_1 and a_2 occurring in (40) and (41) are complex conjugate quantities, and for smaller thickness, a_1 and a_2, are real.

Example 3. The temperature of the solid layer during cooling in the air

Substitution in dependences (41) or (44) of parameters characterizing the air and the solid layer (see the Appendix) gives dependences describing the analyzed relation. For these parameters, the characteristic thickness is $d_0 = 3.35 \cdot 10^{-7}$m. Figs. 5 and 6 represent the temperature $T(t)$ for two layer thicknesses. Each of these figures shows three curves: $T(t)$ from dependence (41) or (44), $T_{cl}(t)$ from dependence (37) and the logarithmic decrement of attenuation $\Lambda = -(d/dt) \ln (T(t)/T_p)$. In the case of the exponential decrease of the temperature takes a constant value independent of time. Therefore, the respective curves obtained from dependences taking into account the thermo-acoustic effect illustrate the quantity of deviation of the temperature of a thin layer on the curve resulting from the classical analysis.

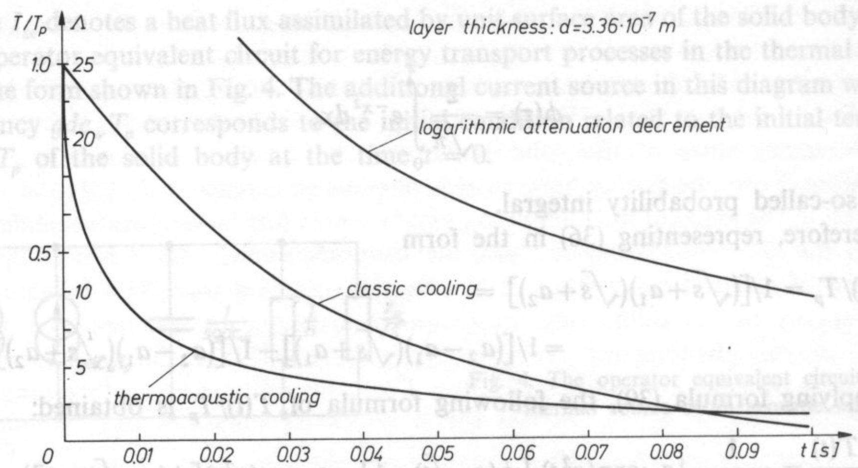

Fig. 5. The plots of $T(t)$ from (41), of T_{cl} from (37) and of Λ for $d = 3.36 \cdot 10^{-6}$ m

The following conclusions can be drawn from the results obtained in this case:

(i) $$\bigvee t_c > 0, \text{ that } \bigwedge t \in (0, t_c), T(t) < T_{cl}(t),$$

and $$\bigwedge t > t_c, T(t) > T_{cl}(t),$$

meaning that initially the thermo-acoustic cooling is faster than classical in view of the convection of part of the energy in the form of an acoustic wave, and then, as a result of heating the medium, and at the same time, of a considerable decrease in the velocity of temperature changes and, hence, a considerable decrease in the

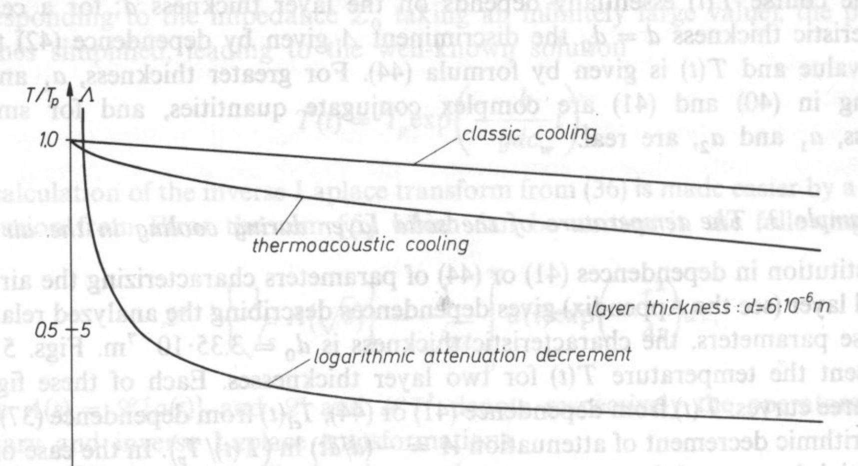

Fig. 6. The plots of $T(t)$, $T_{cl}(t)$ and Λ for $d = 6 \cdot 10^{-6}$ m

pressure component in the wave, the thermo-acoustic cooling, from a certain time moment t_c, proceeds more slowly than the classical cooling;

(ii) $$T(0) = T_{cl}(0) = T_p, \lim_{t\to\infty} T(t) = \lim_{t\to\infty} T_{cl}(t) = 0$$

(iii) $$\lim_{t\to\infty} \Lambda = \infty, \lim_{t\to\infty} \Lambda = 0$$

(iv) for greater layer thicknesses $T(t)$ and $T_{cl}(t)$ decrease more slowly and so does the difference between these curves

(v) for $d \gg 1$ $a_{1,2} \cong \sqrt{\dfrac{h}{gdc_w}}$, and dependence (41) becomes (37).

The theory of cooling presented in this chapter is an extention of the classical theory and, as follows from quantitative analysis, the introduced complementation (i.e., considering the energy convection by the acoustic wave) must be included in the analysis of cooling of thin layers. Certainly, the presented problem is not only concerned with the question of cooling of bodies. Additional energy load should be considered for all processes in which the temperature of a solid body immersed in a gaseous medium varies in time.

6. Final conclusions

An inspiration to undertake the analysis presented in this study was an interest in the problem of sound generation by means of nonmechanical methods, i.e., by sources which do not contain moving mechanical elements (see [1, 6, 7, 8]).

One of the possibilities of nonmechanical sound generation is thermal generation. The idea of thermal generation has repeatedly been discussed in the literature [2, 3, 6], and practical examples of the implementation of this method can be: e.g., the pistophone, and the parametric source of the acoustic wave using coherent light absorption ([9], see also [10, 11]).

One of the imaginable models of thermal source of the acoustic wave in a gaseous medium is a solid layer immersed in this medium in which the temperature is a function of time. A separate problem is the question of the implementation of temperature changes. As an example, control of temperature can be performed by using absorption in this layer of light with variable intensity, or the emission of Joule heat by electric current with variable intensity (analogously to the pistophone). In these two cases, energy is supplied by an external source, while cooling is effected by removing energy to the ambient medium.

Another possibility is ensured by the use of the Peltier effect in the metal-semiconductor junction (see, e.g. [12]). Since in this effect, depending on the direction of the flowing currents, the thermal energy is absorbed or emitted, the metal-semiconductor junction is a practical implementation of the source of a heat flux with a variable sign, and the metal layer which forms a connection of two semiconductor elements of types n and p is heated and cooled in an active way. Moreover, the quantity of the heat flux emitted or absorbed is linearly dependent on the density of the current flowing through the junction for a relatively wide range of this density.

The detailed analysis of the operation of the thermal source, presented in the first part of this study, led to the conclusion that the transmission band of such a transducer was limited by two factors: for low frequencies – by the low efficiency of the temperature-pressure conversion, and for high frequencies – by the heat capacity of the layer. As a consequence of the common effect of these two factors, even for thin layers with thickness of about 10^{-6}m, the transmittance function modulus of the transducer has a maximum for $\omega = 10\text{s}^{-1}$, decreasing for both increasing and decreasing frequencies.

It is difficult to implement practically the electro-thermo-acoustic using the thermo-acoustic effect described above, for the following reasons:

– the unfavourable course of the transmittance of the transducer:

– the necessity of causing large temperature changes of the layer, to gain the mean levels of sound intensity (the low efficiency of the temperature-pressure conversion, resulting from the effect of the stationary boundary condition) and

– the limitation of the current density range for which it is possible to observe the Peltier cooling effect, and also the necessity of using a thin metal cramp linking two semiconductor elements, for which there arises the problem of the surface nonuniformity of the current density distribution in the p-n junction.

The essential result of this study is the construction of a model of the phenomenon of the thermal energy transport in a metal layer, considering the energy carried by the acoustic wave generated in the gaseous medium. In the form of an electric equivalent circuit for harmonic cases, such a model is shown in Fig. 2, and that for pulsed courses and for initial problems, in Fig. 4. In these diagrams, the introduced thermo-acoustic impedance Z_{ta} connects the surplus layer temperature exceeding the ambient temperature with the energy carried by the acoustic wave.

The layer thickness range which can be analyzed within the presented theory is limited. Over the range of greater thickness, this limitation results from the assumed model of the solid body for the heat transport phenomenon in the form of the lumped elements of capacity and resistance. Such a model is valid if the temperature wavelength in the body is much greater than the layer thickness. For pulsed excitation, this limits the maximum variability rate for the analyzed courses. There are no principal barriers against which one could not generalize the results by assuming a solid body model in the form of an adequate system with distributed constants. The low-thickness range is limited by the possibility of applying a macroscopic description of the heat transport in thin layers with atomic thickness.

The above model made it possible to carry out analysis both for the thermal source of the acoustic wave and the other problem discussed in this study, namely the thermo-acoustic cooling effect. The obtained results show that as the thin layer cools the amount of energy carried by the acoustic wave and given away to the ambient medium, as a result of external conduction, are comparable, so that in the energy balance of a thin layer with variable temperature, it is necessary to include the two factors causing the loss of energy.

The topical nature of the problems verging on thermal studies and acoustics was recently confirmed by a study of GERVAIS [13]. This author concentrated on the problems of the solution of the fundamental Navier-Stokes equations by the harmonic input signal method for plane and cylindrical waves.

The model of the thermal source of the acoustic wave assumed in this study does not always coincide with a real situation. E.g., a solid body can be contact with a gaseous medium on both sides, or energy can be supplied to the solid body in different ways etc. It is, therefore, necessary to modify correspondingly the substitute circuit for the energy processes in the source where, moreover, the basic elements of this basic remain uncharged. Therefore, if one were to consider the cooling of a thin layer by energy radiation to both sides, in the circuit and the respective dependencies, it is sufficient to replace the layer thickness by $d/2$ to obtain a correct description of the phenomenon.

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Appendix

In numerical examples illustrating the obtained results, the following values of the constants characteristic of the air in normal conditions were applied:

$$P_0 = 1.018\cdot 10^5\,\mathrm{Nm^{-2}},\ T_0 = 293\,\mathrm{K},\ \varrho_0 = 1.205\,\mathrm{kgm^{-3}}$$

$$c_0 = 344\,\mathrm{ms^{-1}},\ \gamma = 1.4,\ \frac{R}{\mu} = \frac{P_0}{\varrho_0 T_0} = 288.33 = c_p - c_v$$

$$c_p = 1009.16\,\mathrm{Jkg^{-1}K^{-1}},\ c_v = 720.83\,\mathrm{Jkg^{-1}K^{-1}},\ R = 8.36\,\mathrm{JK^{-1}}$$

$$\mu = 0.029\ \mathrm{kg\ mol^{-1}},\ \varkappa = 0.049\,\mathrm{mkgK^{-1}s^{-3}},\ \nu = 1.53\cdot 10^{-5}\,\mathrm{m^2s^{-1}}$$

In turn for the solid body, constants close to the parameters of a metal were assumed, namely

$$g = 5\cdot 10^3\,\mathrm{kgm^{-3}},\ k = 10^2\,\mathrm{Jm^{-1}s^{-1}K^{-1}},\ c_w = 3\cdot 10^2\,\mathrm{Jkg^{-1}K^{-1}},\ h = 15\ \mathrm{Wm^{-2}K^{-1}}.$$

References

[1] L. K. ZAREMBO, V. A. KRASILNIKOV, *Vvedenie v nielinejnuju akustiku*, Izd. Nauka, Moskva 1966.
[2] P. M. MORSE, K. U. INGARD, *Theoretical acoustics*, Mc Graw Hill Book Co. New York 1968.
[3] L. TRILLING, *On thermally induced sound fields*, Journ. of the Acoustic Soc. of America, 27, 3, 425–431 (1955).
[4] R. DYBA, B. ŻÓŁTOGÓRSKI, *Thermal boundary condition for linearized Navier-Stokes equations*, 11th ICA, Paris 1983, vol. 2, 27–30.
[5] J. OSIOWSKI, *An outline of operator calculus* (in Polish), WNT, Warsaw 1972, p. 165.

[6] H. M. MERKLINGER, *On nonlinear and thermal acoustic sources*, Abstract of the 7th International Symp. on Nonlinear Acoust., Backsburg 1976.

[7] K. RATHMAN, *Influence of velocity slip and temperature jump in rarefied gas acoustic oscillations in cylindrical tubes*, Journ. of Sound and Vibr., **103** 3, 448–452 (1985).

[8] J. S. TOMAR, A. K. GUPTA, *Effect of thermal gradient on frequencies of an orthotropic rectangular plate whose thickness varies in two directions*, Journ. of Sound and Vibr., **98**, 2, 257–262 (1985).

[9] T. G. MUIR, C. R. GULBERTSON, J. R. CLYNCH, *Experiments on thermoacoustic array with laser excitation*, Journ. of the Acoustic Soc. of America, **59**, 4, 735–742 (1976).

[10] A. M. AINDROW, R. J. DEWHURST, D. A. HUTCHINS, S. B. PALMER, *Laser-generated pluses at free metal surfaces*, J. Acoust. Soc. Am., **69**, 449–455 (1981).

[11] R. J. DEWHURST, D. A. HUTCHINS, S. B. PALMER, C. B. SCRUBY, *Quantitative measurements of laser-generated acoustic wave forms*, J. Appl. Phys., **53**, 4064–4071 (1982).

[12] A. F. IOFFE, *Semiconductor thermoelements and thermoelectric cooling*. Infosearch Ltd., London 1975.

[13] Y. GERVAIS, *Le generation de bruit par conduction thermique*, Acustica, **59**, 3, 199–205 (1986).

[14] I. MALECKI, *Physical foundations of technical Acoustics*, Pergamon Press, PWN, Warszawa 1969.

[15] R. WYRZYKOWSKI, *Linear theory of acoustic field of gaseous media* (in Polish), Rzeszów Society of the Friends of Sciences, Rzeszów 1972.

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