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METHOD OF CALCULATING THE ACOUSTICAL WAVE REFLECTION COEFFICIENT FROM A NOT-SHARP BOUNDARY OF TWO MEDIA

MAREK HAGEL

Institute of Automatics of the Silesion Technical University
44-100 Gliwice, ul. W. Pstrowskiego 16

The paper presents a numerical calculation method of the reflection coefficient of a plane, longitudinal acoustical wave from a plane-parallel, non-homogeneous transient layer (not-sharp boundary) positioned between two homogeneous, half-spatial media. Changes in the physical properties of the transient layer, determined by the changes in its material parameters, occur along its thickness and can by described by arbitrary one-variable functions.

Results of theoretical calculations of the reflection coefficient were given for chosen cases of material parameter changes in the transient layer and they were compared with results of measurements conducted on a physical model.

The presented method is accurate, universal and simple. It can be useful for the choice of ultrasonic wave frequency and for the measurement accuracy evaluation in certain applications of ultrasonic level meters, as well as for the determination of the shape of an ultrasonic pulse reflected from a not-sharp transient layer.

1. Introduction

The echo method is one of the methods of level measurement (determination of the position of the boundary of two media in space) applied in the ultrasonic technique. It is based on the measurement of the ultrasonic pulse transition time on the path: sending head – measured level – receving head. This method can be applied only when the pulse is reflected from the boundary of two media. Hence, the reflection coefficient of an acoustical wave from the studied media boundary is an important factor, which influences the choice of the construction parameters of ultrasonic level meters. Calculations of the reflection coefficient are not difficult in the case of a sharp boundary between the media, i. e. there is a discontinuous change of the physical properties on the boundary [11]. However, in certain cases one medium passes into the second through a non-homogeneous transient layer, called further on a not-sharp boundary [4]. Such cases can be encountered e. g. during the measurements of sediment levels, investigations of the bottom of water reservoirs and in medicine. The problem of calculating the wave reflection coefficient from a not-sharp boundary of two media has been undertaken in several papers [2], [3], [7], [8], [10]. However, all these publications do not contain the confirmed experimentally general analytical expressions allowing the calculation of the wave reflection coefficient for arbitrary parameters of the transient layer. Therefore, this paper presents a numerical method of calculating the of coefficient reflection from a not-sharp boundary of two media, for a case of a perpendicular incidence of a plane, longitudinal wave on a plane-parallel transient layer. Changes of the physical properties of this layer are determined by the changes of its material parameters (density and elasticity coefficients), take place along its thickness and can be described by arbitrary one variable functions. Measurements of the reflection coefficient on a physical model were done in order to check the obtained calculation results.

2. A mathematical model of the reflection of an acoustical wave from a not-sharp boundary of media

In order to calculate the acoustical wave coefficient of reflection from a not-sharp boundary of media it was accepted, that between two half plane, continuous, homogenous, non-dispersive and lossless media A and C a plane-parallel transient layer B exists. It differs from media A and C, because its material parameters (density and elasticity coefficients) can change along its thickness in an arbitrary manner.

The wave equation in layer B, for a linear, one-dimension problem, with neglect of the body force, is [6]:

$$\varrho_B(x) \left| \frac{\partial^2 u_B(x, t)}{\partial t^2} = \frac{\partial}{\partial x}\left\{[\lambda_B(x)+2\mu_B(x)]\frac{\partial u_B(x, t)}{\partial x}\right\}, \right. \tag{1}$$

where $u_B(x, t)$ – displacement of medium particles, $\varrho_B(x)$ – density, $\lambda_B(x)$, $\mu_B(x)$ – Lamé coefficients, t – time, x – linear coordinate.

Let us assume that in medium A a continuous, plane, sinusoidal and longitudinal acoustical wave A_A propagates in a direction opposite to axis x, and at the same time perpendicularily to the boundary of media A and B (Fig. 1).

Part of the incident wave A_A is reflected from layer B (wave A_{A1}) and a part of it passes to medium C (wave A_C). Displacements of medium particles in waves A_A, A_{A1}, A_C can be presented in a complex form:

$$
\begin{aligned}
u_A(x, t) &= U_A e^{j\omega(x-d)/c_A} e^{j\omega t}, \\
u_{A1}(x, t) &= U_{A1} e^{-j\omega(x-d)/c_A} e^{j\omega t}, \\
u_C(x, t) &= U_C e^{j\omega x/c_C} e^{j\omega t},
\end{aligned}
\tag{2}
$$

where U_A, U_{A1}, U_C – displacement amplitudes in the incident, reflected and transmitted waves, ω – pulsation, c_A, c_C – wave phase velocities in media A and C, d – thickness of the transient layer.

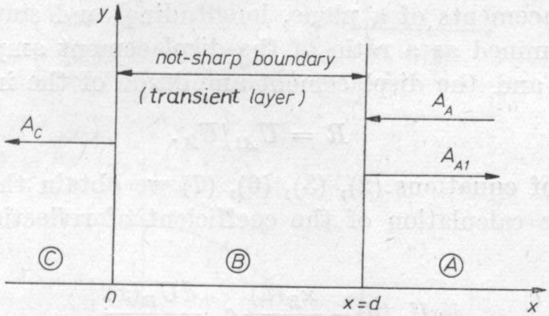


Fig. 1. Reflection of an acoustical wave from a not-sharp transient layer

Because initial phase displacements can occur in the incident, reflected and transmitted waves, the displacement amplitudes U_A, U_{A1} and U_C are complex numbers in a general case. The particle displacements in layer B can be expressed by [8]:

$$u_B(x, t) = U_B(x)\, e^{j\omega t}. \tag{3}$$

Placing equation (3) in (1) we obtain:

$$\frac{d^2 U_B(x)}{dx^2} = -\frac{1}{\varkappa_B(x)} \frac{d\varkappa_B(x)}{dx} \frac{dU_B(x)}{dx} - \omega^2 \frac{\varrho_B(x)}{\varkappa_B(x)} U_B(x), \tag{4}$$

where

$$\varkappa_B(x) = \lambda_B(x) + 2\mu_B(x).$$

In order to reach a full mathematical description of the not-sharp boundary, equation (4) has to be supplemented by the continuity conditions for displacements and stresses, for $x = 0$ and $x = d$:

for $x = 0$

$$
\begin{aligned}
u_B(0, t) &= u_C(0, t), \\
\varkappa_B(0) \frac{\partial u_B(x, t)}{\partial x}\bigg|_{x=0} &= \varkappa_C \frac{\partial u_C(x, t)}{\partial x}\bigg|_{x=0},
\end{aligned}
\tag{5}
$$

for $x = d$

$$u_B(d, t) = u_A(d, t) + u_{A1}(d, t),$$

$$\varkappa_B(d) \frac{\partial u_B(x, t)}{\partial x}\bigg|_{x=d} = \varkappa_A \frac{\partial}{\partial x}[u_A(x, t) + u_{A1}(x, t)]\bigg|_{x=d}, \tag{6}$$

where

$$\varkappa_A = \varkappa_A + 2\mu_A, \quad \varkappa_C = \lambda_C + 2\mu_C,$$

λ_A, λ_C, μ_A, μ_C – Lamé coefficients in media A and C.

The coefficient of reflection R, from not-sharp boundary between media A and C for displacements of a plane, longitudinal and sinusoidal acoustical wave can be determined as a ratio of the displacement amplitude of the reflected wave U_{A1}, and the displacement amplitude of the incident wave U_A:

$$R = U_{A1}/U_A. \tag{7}$$

On the basis of equations (2), (3), (6), (7) we obtain the general expression leading to the calculation of the coefficient of reflection R from a not sharp boundary:

$$R = \frac{j\omega U_B(d) - \dfrac{\varkappa_B(d)}{\varkappa_A} c_A \dfrac{dU_B(x)}{dx}\bigg|_{x=d}}{j\omega U_B(d) + \dfrac{\varkappa_B(d)}{\varkappa_A} c_A \dfrac{dU_B(x)}{dx}\bigg|_{x=d}}. \tag{8}$$

3. Calculation of the reflection coefficient

The calculation of the reflection coefficient R was conducted numerically. A linear second order differential equation in the form (4) was solved with the function coefficients and boundary conditions expressed by equations (5) and (6). Equation (4) was integrated numerically along coorinate x, begining from the boundary condition for $x = 0$. The integration Runge--Kutt procedure was applied. The results, in the form of values U_B(d) and $dU_B(x)/dx|_{x=d}$, were put in equation (8). The following functions describing changes of material parameters, $\varrho_B(x)/\varrho_C$ and $\varkappa_B(x)/\varkappa_C$ in medium B (Fig. 2), were accepted in the course of calculations:

a. $$p(x) = 1 + (a-1)\frac{x}{d},$$

b. $$r(x) = \frac{1}{2}\left[1 + a - (a-1)\cos\frac{\pi}{d}x\right],$$

c. $$s(x) = a - (a-1)\cos\frac{\pi x}{2d},$$

d. $$t(x) = \sqrt{a},$$

where a – ratio of the values of material parameters in medium A and C:

$$a_\varrho = \varrho_A / \varrho_C, \quad a_\varkappa = \varkappa_A / \varkappa_C,$$

ϱ_A, ϱ_C – densities of media A and C.

The calculation programme was written in language FORTRAN 1900.

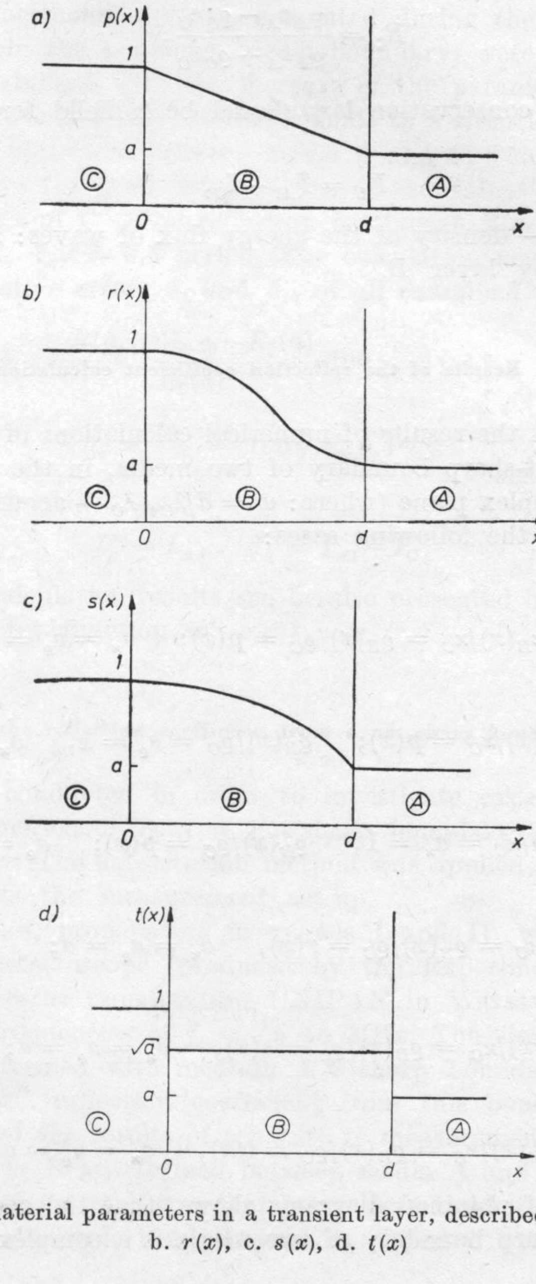


Fig. 2. Changes of material parameters in a transient layer, described by function: a. $p(x)$, b. $r(x)$, c. $s(x)$, d. $t(x)$

The physical significance and the accuracy of numerical calculations was controlled with the application of two methods:

a) when the width of the transient layer d decreaces to zero, then the value of the reflection coefficient R should tend to the values of the reflection coefficient R_ζ for a sharp boundary between two media:

$$R_\zeta = \frac{\varrho_A c_A - \varrho_C c_C}{\varrho_A c_A + \varrho_C c_C}, \tag{9}$$

b) the energy conservation law should be fulfielld for waves A_A, A_{A1}, A_C, i. e.,

$$I_A = I_{A1} + I_C, \tag{10}$$

where I_A, I_{A1}, I_C – density of the energy flux of waves: incident, reflected and transmitted by layer B.

4. Results of the reflection coefficient calculation

Fig. 3a-f shows the results of numerical calculations of the reflection coefficient from a not-sharp boundary of two media, in the form of diagrams $R(a, w)$ in the complex plane (where: $w = d/\lambda_C$, λ_C – acoustical wave length in medium C), for the following cases:

a. case $P1$

$$\varkappa_B(x)/\varkappa_C = \varrho_B(x)/\varrho_C = p(x); \quad a_\varkappa = a_\varrho = a;$$

b. case $P2$

$$\varkappa_B(x)/\varkappa_C = p(x); \quad \varrho_B(x)/\varrho_C = a_\varrho = 1, \quad a_\varkappa = a;$$

c. case $P3$

$$\varkappa_B(x)/\varkappa_C = a_\varkappa = 1, \quad \varrho_B(x)/\varrho_C = p(x); \quad a_\varrho = a;$$

d. case $P4$

$$\varkappa_B(x)/\varkappa_C = \varrho_B(x)/\varrho_C = r(x), \quad a_\varkappa = a_\varrho = a;$$

e. case $P5$

$$\varkappa_B(x)/\varkappa_C = \varrho_B(x)/\varrho_C = s(x), \quad a_\varkappa = a_\varrho = a;$$

f. case $P6$

$$\varkappa_B(x)/\varkappa_C = \varrho_B(x)/\varrho_C = t(x), \quad a_\varkappa = a_\varrho = a;$$

The analysis of obtained diagrams shows, that the coefficient of reflection from a not-sharp boundary of two media is a complex quantity and its values depend on the ratio of the width of the transient layer to the length of the acoustical wave in medium C, the ratio of the values of material parameters on media A and C, and on the functions describing the changes of material parameters in layer B. In cases $P1$–$P5$ the values of the modules of the reflection coefficient rapidly decrease to zero with the increase of the value of parameter w (for $w \approx 1$, so $\lambda_C \approx d$, $|R(a, w)| = 0.1-0.3\ R_\zeta(a)$). The values of phase displacements φ(a, w), generated during the reflection of the acoustical wave from the not-sharp media boundary, were in an $(0-\pi)$ interval and were stabilized with the increase of the parameter w value.

Case P6 is a particular case. It corresponds to a situation in which a homogeneous layer B is present between media A and C. The values of its material parameters are the geometrical means of the values of material parameters of media A and C – the reflection coefficient $R(a, w)$ is then a periodic function with a $w = 0.5$ period. The calculation accuracy determined on the basis of relative errors δ_1 and δ_2, of all examined cases, was:

$$\delta_1 = \frac{R(a,\ w)|_{w\to 0} - R_\zeta(a)}{R_\zeta(a)}\, 100\,\% \leqslant 0.1\,\%, \tag{11}$$

$$\delta_2 = \frac{I_p - I_0}{I_0}\, 100\,\% \leqslant 0.4\,\%, \tag{12}$$

where

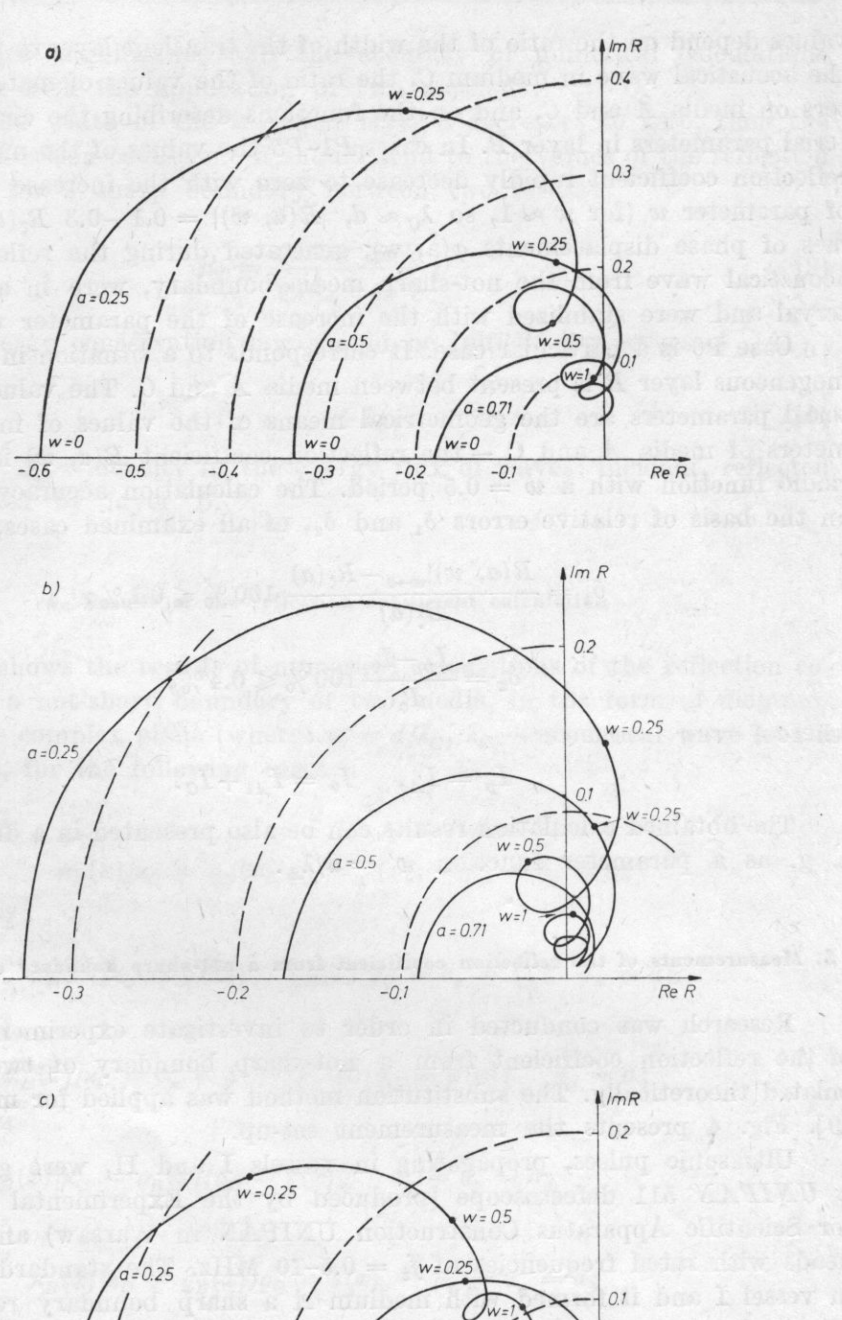
$$I_p = I_A, \quad I_0 = I_{A1} + I_C.$$

The obtained calculation results can be also presented in a different form e. g. as a parameter function $w' = d/\lambda_A$.

5. Measurements of the reflection coefficient from a not-sharp boundary of two media

Research was conducted in order to investigate experimentally values of the reflection coefficient from a not-sharp boundary of two media calculated theoretically. The substitution method was applied for measurements [9]. Fig. 4 presents the measurement set-up.

Ultrasonic pulses, propagating in vessels I and II, were generated by a *UNIPAN* 511 defectoscope (produced by the Experimental Department for Scientific Apparatus Construction UNIPAN in Warsaw) and ultrasonic heads with rated frequencies of $f_z = 0.5$–10 MHz. The standard was placed in vessel I and it formed with medium A a sharp boundary reflecting the ultrasonic wave. The reflection coefficient from this boundary was calculated on the basis of the results of separate te measurements. In vessel II a model transient layer B was formed between media A and C, where the top edge of the standard and the transient layer were theoretically at the same distance from the ultrasonic head. Layers A, B and C were made from ge-


a)
Im R
0.4
w=0.25
0.3
w=0.25
0.2
a=0.25
a=0.5
w=0.5
0.1
w=1
a=0.71
w=0
w=0
w=0
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
Re R
b)
Im R
0.2
w=0.25
a=0.25
0.1
w=0.25
w=0.5
a=0.5
w=1
a=0.71
-0.3
-0.2
-0.1
Re R
c)
Im R
0.2
w=0.25
w=0.5
a=0.25
w=0.25
0.1
w=1
a=0.5
w=0.5
w=0.25
a=0.71
w=0
w=0
w=0
-0.3
-0.2
-0.1
Re R

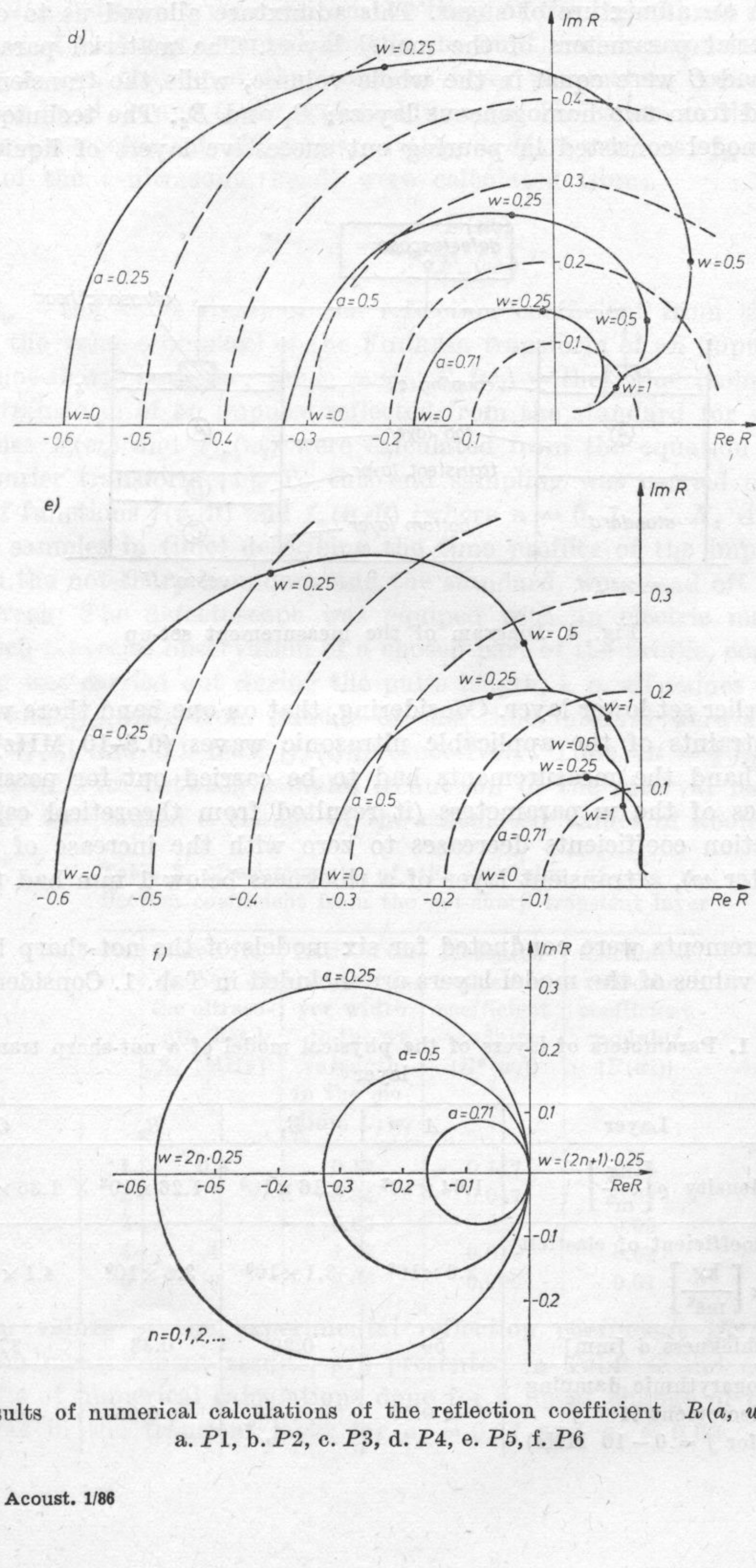


Fig. 3. Results of numerical calculations of the reflection coefficient $R(a, w)$ for cases: a. $P1$, b. $P2$, c. $P3$, d. $P4$, e. $P5$, f. $P6$

latine with an admixture of sugar. This admixture allowed us to obtain various material parameters of the model layers. The material parameters of media A nad C were equal in the whole volume, while the transient layer B was formed from two homogeneous layers: B_1 and B_2. The technique of making the model consisted in pouring out successive layers of liquid gelatine

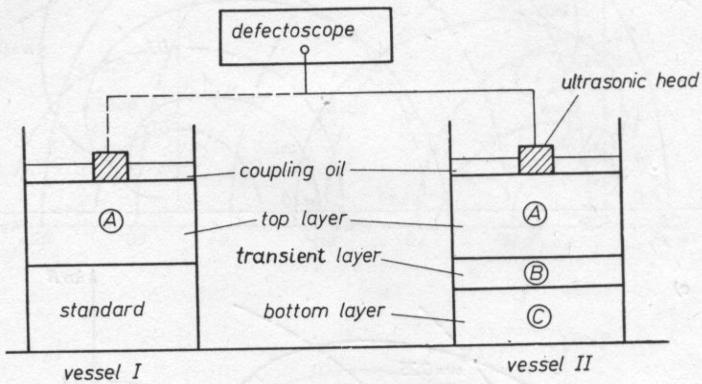


Fig. 4. Diagram of the measurement set-up

onto an earlier set lower layer. Considering, that on one hand there were apparatus constraints of the applicable ultrasonic waves (0.5–10 MHz), and on the other hand the measurements had to be carried out for possibly small (0–2) values of the w parameters (it resulted from theoretical calculations, that reflection coefficients decreases to zero with the increase of the value of parameter w), a transient layer of a thickness below 1 mm had to be modelled.

Measurements were conducted for six models of the not-sharp boundary. The mean values of the model layers are included in Tab. 1. Considering parti-

Table 1. Parameters of layers of the physical model of a not-sharp transient layer

No	Layer	A	B_1	B_2	C
1	density $\varrho \left[\frac{kg}{m^3}\right]$	1.04×10^3	1.16×10^3	1.26×10^3	1.35×10^3
2	coefficient of elasticity $\varkappa \left[\frac{kg}{ms^2}\right]$	2.6×10^9	3.1×10^9	3.6×10^9	4.1×10^9
3	thickness d [mm]	50	0.29	0.33	77
4	logarythmic damping decrement Λ (for $f = 0-10$ MHz)	< 0.06	< 0.06	< 0.06	<0.06

cle diffusion between layers and on the basis of the logarithmic damping decrement $\Lambda \ll 1$, it was accepted, that in the built models a continuous, approximately linear, change of the transient layer parameters occurs and damping in media A, B and C are neglectably small. The values of the experimental reflection coefficient $R^*(w_i)$ ($w_i = d/\lambda_{Ci} = df_{zi}/c_C$, where: f_{zi} – rated frequency of the i ultrasonic head) were calculated from:

$$R^*(w_i) = R_w \frac{F(w_i)}{F_w(w_i)}, \tag{13}$$

where R_w – the value (real) of the reflection coefficient from the standard, $F(w_i)$ – the value (complex) of the FOURIER transform of an impulse reflected from a not-sharp boundary for $w = w_i$, $F_w(w_i)$ – the value (complex) of the Fourier transform of an impulse reflected from the standard for $w = w_i$.

Values $F(w_i)$ and $F_w(w_i)$ were calculated from the equation of the discrete Fourier transform [1]. To this end sampling was carried out, i. e. the values of functions $f(n\Delta t)$ and $f_w(n\Delta t)$ (where $n = 0, 1, ..., N$, Δt – interval between samples in time) describing the time profiles of the impulses reflected from the not-sharp boundary and the standard, were read off the defectoscope screen. The defectoscope was equiped with an electric magnifier, through wich a precise observation of a chosen part of the profile, could be done. Sampling was carried out during the pulse length, i. e. all values of functions $|f(n\Delta t)|$ and $|f_w(n\Delta t)|$ from outside of this time interval were smaller than 0.1 max $|f(t)|$ and 0.1 max $|f_w(t)|$, respectively. Time $\Delta t = \frac{1}{4} f_{zi}$ was taken to be the interval between samples (reduction of the interval between samples Δt by two caused a change of the calculated values of about 0.5%).

Table 2. Measurements of the modulus of the reflection coefficient from the not-sharp transient layer

No	Rated frequency of the ultrasonic head f_{zi} [MHz]	Ratio of the transient layer width to the wave length in the medium C; w_i	Measured reflection coefficient modulus $\|R^*(w_i)\|$	Calculated reflection coefficient modulus $\|R(w_i)\|$
1	0.5	0.17	0.117	0.14
2	1	0.34	0.047	0.07
3	2	0.68	0.026	0.03
4	4	1.36	0.010	0.02
5	6	2.06	0.006	0.01

Mean values of the experimental reflection coefficient $|R^*(w_i)|$, calculated from measurement results, are presented in Table 2 and compared to the results of numerical calculations done for a linear change of the material parameters in the transient layer for $a_\varrho = 0.77$ and $a_\varkappa = 0.63$.

Taking into account the difficulty of building a physical model (small thickness of the transient layer), its departure from the theoretical model (among others: approximately linear change of the material parameters of the transient layer, nonplanar wave front), as well as the low accuracy of the measurements (several percent), it can be said, that the experimental results are qualitatively consistent with the results of numerical calculations, despite fairly considerable differences in the numerical values.

The values $\varphi^*(w_i)$ (calculated from equation (13)), of the phase shift $\varphi(w)$, generated during the reflection of the acoustical wave from a not-sharp boundary, were contained in the range $(\pi, -2.5\pi)$ rad. On the other hand it has to be taken into account, that the inaccuracy of the ultrasonic head setting in respect to the reflecting boundary (approximated at ± 0.5 mm), could cause phase shifts of the same order of magnitude. For this reason the measured values of the phase shifts were not presented.

6. Conclusions

The presented here numerical method of calculating the coefficient of reflection from a not-sharp boundary of two media is accurate, universal and simple. The diagrams of the reflection coefficient, calculated with its application, can be used for the selection of optimal rated frequencies of ultrasonic heads for level meters, measuring the level determined by a not-sharp boundary of media; for the analysis of their indication accuracy as well as for the determination of the shape of an ultrasonic pulse reflected from a not-sharp boundary between media [5].

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