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CALCULATION OF THE ACOUSTICAL FIELD OF A SEMI-INFINITE CYLINDRICAL WAVE-GUIDE BY MEANS OF THE GREEN FUNCTION EXPRESSED IN CYLINDRICAL COORDINATES

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The exact solution to the problem of the acoustic wave propagation is presented for a half-infinite cylindrical wave-guide with rigid walls, i.e., with taking into account the diffraction phenomena on the open end of wave-guide. The problem was solved by means of the theory of acoustic field without sources and the use is made of the Green's function method in the cylindrical space coordinates, leading to two integral equations which are solvable with the help of the Wiener-Hopf method.

The wave number considered was taken to be a complex quantity, and the reduced forms of the final formulae are presented for the limiting case of real wave number.

Notations

$f(\bar{u})$ – directivity factor,
$f_m(z)$ – function of apparent sources,
$F_m(w)$ – Fourier transform of functions of apparent sources,
$g_m(z)$ – source function,
$\tilde{G}(\varrho, \varrho', w)$ – Fourier transform of Green function,
k – constant,
$l_m(z)$ – nucleus of the integral equation,
$L(w)$ – Fourier transform of the nucleus of the integral equation,
$L_+(w), L_-(w)$ – factors,
l, m – integers, indexes of wave mode,
N – order of the highest acceptable mode,
w – partial wave number,

v — radial wave number,
$S(w)$ — function determining factors L_+, L_-,
$X(w)$, $Y(w)$ — real and imaginary part of function $S(w)$,
γ_m — radial wave number of mode numbered m,
ε — imaginary part of wave numbered k,
η — imaginary part of variable w,
μ_m — m root of the Bessel function $J_1(z)$,
$\Psi(z)$ — jump of the potential on the wall of the wave-guide,
$\Omega(v)$ — function equaling $\mathrm{tg}^{-1} \dfrac{N_1(v)}{J_1(v)} + \dfrac{\pi}{2}$.

Other notations used in this paper are typical and have not been included in the above list.

1. Introduction

The determination of a wave-guide acoustic field consists from the mathematical point of view in the solution of a wave equation with given boundary conditions, generally applied to the normal component of the vibrational velocity on the walls. Such solutions are known only for a few cases, where the boundary conditions are accepted on highly symmetrical planes (e.g. infinite wave-guides). In other cases the symmetry of vibrating systems is corrected by supplementing them with infinite acoustic baffles. However, only few problems have an exact solution.

From among papers concerned with theoretical and experimental acoustics in the field of phenomena taking place in cylindrically symmetrical wave-guides, the fundamental work of Rayleigh should be mentioned [1]. Rayleigh calculated the quantity called the "correction for the open end", which is the measure of the phase shift of a plane wave due to the reflection at the wave-guide orfice supplied additionally with an infinite rigid acoustic baffle. The method of separation of variables, applied to the wave equation expressed in cylindrical coordinates [2], gives a solution, which points out that not only a plane wave can propagate in the wave-guide, but also higher wave modes can occur. They appear above certain limit frequencies, depending on the pipe radius. It is by intuition evident that the same modes can also occur in a half-infinite wave-guide and that their generation can be related to diffraction effects taking place at the orfice. This proves that the Rayleigh method applied in certain cases especially with waves shorter than the doubled pipe radius can give completely erroneous results.

Levin's and Schwinger's, and Wajnsztejn's papers published in the 40-fies have contributed in particular to significant progress in the mathematic theory of vibrations in a pipe. The first two scientists [3] have achieved an exact solution of the wave equation with boundary conditions characteristic for a semi-infinite cylindrical pipe with perfectly rigid walls, under an assumption that only a plane wave propagates in the direction of the orfice and in the reverse direction. Of course, this limits the practical application of the results to waves not shorter than the pipe radius, when the generation of higher Bessel modes is impossible. WAJNSZTEJN, on the other hand, in his works [4, 5] gave an exact solution to the problem of electromagnetic wave propagation in a flat and cylindrical wave-guide, and then on the analogy of electric waves (in a flat wave-guide), or magnetic waves (in a cylindrical wave-guide) and acoustic waves, he established an expression for the acoustic potential, postulating, also by analogy, such a form of the jump of the potential on the wall of the wave-guide, which would result in integral equations identical with equations obtained for electromagnetic waves.

This paper presents a method of obtaining an exact solution to this problem with the sole application of the acoustic field theory, for a region without sources, including constraints of the mathematical solution resulting from its physical interpretation as the potential. The theory of Green functions expressed in cylindrical coordinates has been applied.

2. Formulation of the problem in the form of an integral equation

Our analysis will concentrate on the acoustic field inside a cylindrical wave-guide stretching from $z = 0$ to ∞, with its axis of symmetry coinciding with the z-axis. The wall of the wave-guide is described by the equation of the side surface of a semi-infinite cylinder with a radius a (Fig. 1):

$$\sum = \{(\varrho, \varphi, z) : \varrho = a, z \geqslant 0\}. \tag{1}$$

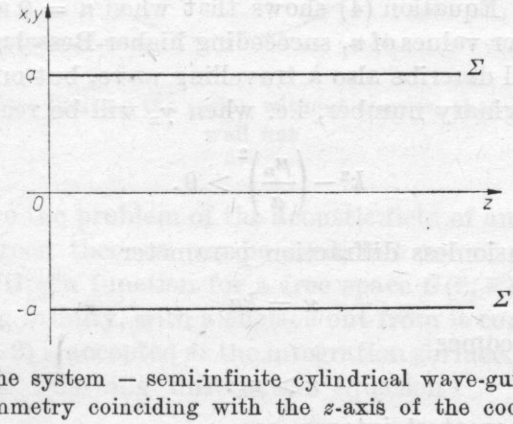

Fig. 1. Geometry of the system – semi-infinite cylindrical wave-guide with radius a and the axis of symmetry coinciding with the z-axis of the coordinate system

Let us consider a case when the wave-guide is axially excited (the velocity distribution is independent from the angular variable) to vibrate with a determinate circular frequency w. The expression for the acoustic potential inside an infinite wave-guide, obtained from the wave equation, is

$$\Delta\Phi(\bar{r}, t) = \frac{1}{c^2}\frac{\partial^2\Phi(\bar{r}, t)}{\partial t^2} \tag{2}$$

with the condition of the decay of the radial component of the vibration velocity on the surface of an infinite wave-guide

$$\left.\frac{\partial\Phi(\bar{r}, t)}{\partial\varrho}\right|_{\varrho=a} = 0, \quad -\infty < z < +\infty. \tag{3}$$

Considering only harmonic vibrations and assuming that the time dependence of the potential is expressed by factor $\exp(-i\omega t)$, we obtain the following solution [2], which satisfies physical conditions of the potential:

$$\Phi_n(\varrho, z) = A_n \frac{J_0\left(\frac{\mu_n}{a}\varrho\right)}{J_0(\mu_n)} \cdot e^{-i\gamma_n z} \tag{4}$$

where γ_n is the radial wave number related to the vave numbered k, by the following relationship

$$\gamma_n = \sqrt{k^2 - \left(\frac{\mu_n}{a}\right)^2} \tag{5}$$

and μ_n is the n root of the Bessel function $J_1(x)$. The $J_0(\mu_n)$ factor, which appeared in the denominator in (4) is a standarizer, so constant A denotes the amplitude. Index n numbers successive allowed wave modes. Of course, in a general case, the potential of an incident wave can be a superposition of the potentials of individual modes. Equation (4) shows that when $n = 0$ a plane wave is obtained, while for other values of n, succeeding higher Bessel modes occur. Moreover, formula (4) will describe also a travelling wave, but only when the exponent will be an imaginary number, i.e. when γ_n will be real, that is when

$$k^2 - \left(\frac{\mu_n}{a}\right)^2 > 0. \tag{6}$$

Introducing a dimensionless diffraction parameter

$$\varkappa = ka, \tag{7}$$

the condition (6) becomes

$$\varkappa > \mu_n. \tag{8}$$

Denoting by N the greatest integer, so

$$\mu_N < \varkappa \leqslant \mu_{N+1} \tag{9}$$

then N determines the order of the highest Bessel mode which can propagate in the wave-guide without loss, with an assigned diffraction parameter $\varkappa$.

In order to determine the acoustic field of a semi-infinite wave-guide, the wave equation (2) has to be solved with a boundary condition of the decay of the normal component of the vibration velocity on the wave-guide surface, i.e. only for $z \geqslant 0$:

$$\left.\frac{\partial \Phi(\bar{r}, t)}{\partial \varrho}\right|_{\varrho=a} = 0, \quad z \geqslant 0. \tag{10}$$

An assumption is made that the sound wave, which propagates towards the open end, has a potential expressed by formula (4), i.e. it is a single wave mode. It undergoes diffraction at the orfice — part of the energy is radiated outside, and part returns to the wave-guide as a reflected wave. We postulate that it can consist of all Bessel modes, which can propagate in the given wave-guide. Therefore, the solution of the diffraction problem lies in the determination of complex amplitudes of modes in a wave returning from the open end.

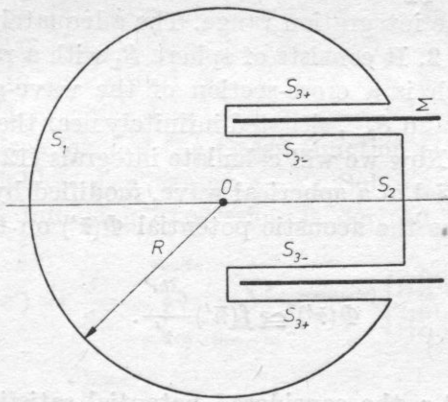

Fig. 2. Integration surface limiting the region without sources; that is with the wave-guide wall out

In order to solve the problem of the acoustic field of an investigated wave-guide the second Green theorem can be used, but one of the scalar functions is substituted by a Green function for a free space $G(\bar{r}, \bar{r}')$ and a sphere with a radius approaching infinity, with a surface out from it comprising the wall of the wave-guide (Fig. 2) is accepted as the integration surface. The Green function $G(\bar{r}, \bar{r}')$ satisfies the following differential equation

$$(\Delta + k^2)G(\bar{r}, \bar{r}') = -\delta(\bar{r} - \bar{r}') \tag{11}$$

where the function on the right side is the δ Dirac distribution. Using the mentioned above theorem and including the fact that the spatial part of the expression for the potential fulfills the Helmholtz differential equation, it can be written

$$\Phi(\bar{r}) = \oint_S [G(\bar{r}, \bar{r}')\bar{n}'\nabla'\Phi(\bar{r}') - \Phi(\bar{r}')\bar{n}'\nabla' G(\bar{r}, \bar{r}')]d\sigma', \tag{12}$$

The "prim" mark at the deloperator means, that the differentiation is done in respect to variables marked "prim", so $\bar{n}'$ is the unit vector normal to the $d\sigma$ surface element. If we want to use the form of the Green function for a free space

$$G(\bar{r}, \bar{r}') = \frac{1}{4\pi} \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} \tag{13}$$

then there should be no sources in the range limited by surface S. The surface of the wave-guide with apparent sources related to the potential discontinuity has been cut out from the integration range. The adequately chosen integration surface is shown in Fig. 2. It consists of sphere S_1 with a radius R approaching infinity, circle S_2, which is a cross-section of the wave-guide at $z = R$ and cylindrical surfaces, S_{3+} and S_{3-}, situated infinitely near the inner and outer side of the wave-guide wall. Now we will calculate integrals (12) on individual parts of surface S. The potential of a spherical wave, modified by a directivity factor $f(n')$, can be accepted as the acoustic potential $\Phi(\bar{r}')$ on the surface S_1

$$\Phi(\bar{r}') \cong f(\bar{n}') \frac{e^{ikr'}}{r'}. \tag{14}$$

For great values of r, the considered potential satisfies the Sommerfeld's radiation and finity conditions; the Green function fulfills the so-called sharpened Sommerfeld's radiation and finity conditions; the (12) integral on the surface of sphere S_1 vanishes, what has been proved among others by RUBINOWICZ [6]. The integral on surface S_2 tends to zero for $R \to \infty$ due to the finite value of the potential, finite integration surface and the decrease of the Green function with inverse proportion to distance. Thus, the value of potential $\Phi(\bar{r})$ will be determined only by an integral on surfaces S_{3+} and S_{3-}. Considering that the side surface of the cylinder satisfies the boundary condition (10) of the decay of the normal component of the vibration velocity, and that $\bar{n}'\nabla' = \partial/\partial\varrho$ for elements of the surface S_{3+}, and $\bar{n}'\nabla' = -\partial/\partial\varrho$ for elements of the surface S_{3-}, the expression (12), which determines the acoustic potential, can be written in the form

$$\Phi(\bar{r}) = 2\pi a \int_0^\infty \Psi(z') \frac{\partial}{\partial\varrho'} G(\bar{r}, \bar{r}')\bigg|_{\varrho'=a} dz' \tag{15}$$

where the function $\Psi(z')$ determines the jump of the potential on the surface of the wave-guide

$$\Psi(z') = \Phi(\varrho', z')|_{\varrho'\to a_+} - \Phi(\varrho', z')|_{\varrho'\to a_-}. \tag{16}$$

The analysis of expression (15) shows that the acoustic potential in an arbitrary point of the field is univocally defined by the jump (discontinuity) of the potential on the surface of the wave-guide. The application of this form allows the notation of the boundary condition (10) in the form of a homogeneous integral equation

$$\int_0^\infty \Psi(z') \frac{\partial}{\partial\varrho}\frac{\partial}{\partial\varrho'} G(\bar r, \bar r')\Big|_{\substack{\varrho'=a\\ \varrho=a}} dz' = 0; \quad z \geqslant 0. \tag{17}$$

Thus, the calculation of the acoustic potential of a semi-infinite cylindrical wave-guide has been reduced to the determination of the value of the jump of the potential $\Psi(z')$ on the wave-guide surface, i.e. to the solution of integral equation (17). Because the problem is cylindrically symmetric, then the Green function expressed in cylindrical coordinates can be applied, i.e. the Green function for a cylinder, which has been discussed in detail in papers [2], [7]. It results from the free form of the Green function (13) that this is a function of the following variables $(\varrho, \varrho', \varphi-\varphi', z-z')$. Solving equation (11) in cylindrical coordinates, the following expression is derived for the Green function:

$$G(\varrho, \varrho', \varphi-\varphi', z-z') = \frac{i}{8\pi} \int_{-\infty+i\eta}^{\infty+i\eta} \sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')} \begin{Bmatrix} H_m^{(1)}(v\varrho) J_m(v\varrho') \\ H_m^{(1)}(v\varrho') J_m(v\varrho) \end{Bmatrix} e^{iw(z-z')} dw, \quad \begin{matrix} \varrho > \varrho' \\ \varrho < \varrho'. \end{matrix} \tag{18}$$

This is the Green function for a cylinder. It has the form of an inverse Fourier transform, while the integration path is a line parallel to $\mathrm{Re}\, w$, and coefficient η satisfies the inequality

$$-\mathrm{Im}\, k < \eta < \mathrm{Im}\, k. \tag{19}$$

When the excitation is axial (the case under investigation), then the infinite series under the integral is reduced to one term for $m = 0$. Then we obtain

$$G(\varrho, \varrho', z-z') = \frac{i}{8\pi} \int_{-\infty+i\eta}^{\infty+i\eta} \begin{Bmatrix} H_0^{(1)}(v\varrho) J_0(v\varrho') \\ H_0^{(1)}(v\varrho') J_0(v\varrho) \end{Bmatrix} e^{iw(z-z')} dw, \quad \begin{matrix} \varrho > \varrho' \\ \varrho < \varrho' \end{matrix} \tag{20}$$

where v is the radial wave number

$$v = \sqrt{k^2 - w^2}. \tag{21}$$

The form of the obtained Green function differs in dependence on whether the point of the field lies inside ($\varrho < \varrho'$) or outside ($\varrho > \varrho'$) the wave-guide. This occurs because from among a family of solutions of equation (11) we have to chose those, which fulfill the physical conditions of the problem. Both solutions are symmetrical with respect to the change from ϱ to ϱ', what corresponds to the change of location of the source and the observation point. The expression in braces describes the propagation of cylindrical waves along the radial coordinate ϱ.

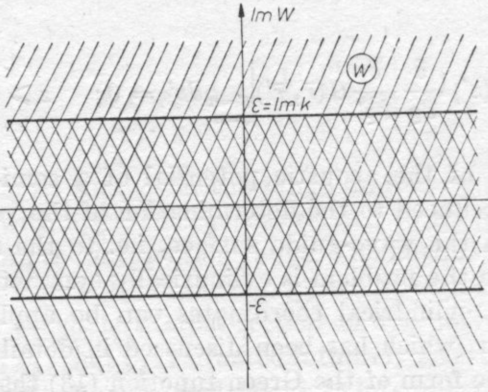

Fig. 3. Analyticity regions of Fourier transforms, $L(w)$ and $F_l(w)$. Common analyticity region $-\varepsilon < \mathrm{Im} < \varepsilon$.

Using expression (20) in the equation describing the potential (15) and the boundary condition (17), two integral equation are obtained

$$\Phi(\varrho, z) = \frac{ai}{4}\int_0^\infty \Psi(z')\,dz' \int_{-\infty+i\eta}^{\infty+i\eta} v \begin{Bmatrix} H_0^{(1)}(v\varrho)J_1(va) \\ H_1^{(1)}(va)J_0(v\varrho)\end{Bmatrix} e^{iw(z-z')}\,dw, \qquad \begin{matrix}\varrho > a \\ \varrho < a\end{matrix}, \tag{22}$$

$$\int_0^\infty \Psi(z')\,dz' \int_{-\infty+i\eta}^{\infty+i\eta} v^2 H_1^{(1)}(va)J_1(va)\,e^{iw(z-z')}\,dw = 0, \quad z \geqslant 0. \tag{23}$$

The acoustic potential can be found by solving the second equation, i.e. fining the function $\Psi(z')$. The problem of solving a wave equation with a boundary condition of the decay of the normal derivative on the side surface of a semi-infinite cylinder has been reduced to the problem of solving a pair of integral equations, (22) and (23).

3. The determination of source functions on the surface of a wave-guide

The jump of the potential on the surface of the wave-guide can be accepted as the occurrence of apparent sources on this surface. The function describing sources on the surface of the wave-guide is marked $g(z)$. When a single allowed wave mode (e.g. 1-st) propagates in the wave-guide, then this function can be expressed as a sum

$$g_l(z) = f_l(z) + \Phi_l(a, z) \tag{24}$$

where $f_l(z)$ determines apparent sources, which appear on the surface of the wave-guide due to diffraction, and $\Phi_l(a, z)$ is the value of the potential of the wave inciding on the wave-guide wall. For $z < 0$ sources do not occur, therefore

$$g_l(z) = 0, \quad z < 0. \tag{25}$$

It results from the above discussion that the function of sources equals the sought value of the potential jump on the surface of the wave-guide and zero on its extension

$$g_l(z) = \begin{cases} \Psi(z), & z \geqslant 0, \\ 0 \quad , & z < 0. \end{cases} \tag{26}$$

Therefore, the integration range can be widened onto the interval $(-\infty, +\infty)$, and then expressions (22) and (23) will have a form convenient for further calculations. The boundary condition, in particular, will have the form of a convolution, so it will be simple to find its Fourier transform. Moreover, if we denote

$$l(z-z') = \frac{1}{2\pi} \int\limits_{-\infty+i\eta}^{\infty+i\eta} e^{iw(z-z')} v^2 H_1^{(1)}(va) J_1(va)\, dw \tag{27}$$

then the boundary condition for a l wave mode reaching the open end will be

$$\int\limits_{-\infty}^{\infty} g_l(z') l(z-z')\, dz' = 0, \quad z \geqslant 0, \tag{28}$$

while the expression for the acoustic potential will equal

$$\Phi_l(\varrho, z) = \frac{ai}{4} \int\limits_{-\infty}^{\infty} g_l(z') \int\limits_{-\infty+i\eta}^{\infty+i\eta} v \begin{Bmatrix} H_0^{(1)}(v\varrho) J_1(va) \\ H_1^{(1)}(va) J_0(v\varrho) \end{Bmatrix} e^{iw(z-z')} dw, \quad \begin{matrix} \varrho > a \\ \varrho < a \end{matrix}. \tag{29}$$

The function of sources is equal to

$$g_l(z) = f_l(z) + A_l e^{-i\gamma_l z} \tag{30}$$

because the form of the second term in expression (24) is accepted as explicit.

Equation (28) is an equation with a nucleus with a translated argument. It can be solved with the factorization method, which consists in the distribution of the Fourier transform (if it exists) of the investigated equation onto the product of analytical functions (factors), which do not have zeroes respectively in the upper and lower half-plane of the complex variable w. The upper half-plane will be noted by $\{w: \mathrm{Im}\, w > -\mathrm{Im}\, k\}$, and the lower by $\{w: \mathrm{Im}\, w < \mathrm{Im}\, k\}$. It should be noted that these half-planes have a common part, denoted by expression $\{w: -\mathrm{Im}\, k < w < \mathrm{Im}\, k\}$.

As it has been previously stated, expression (28) presents a convolution, so its Fourier transform can be easily found. Expression (27) has the form of an inverse transform, so the transform will be

$$L(w) = v^2 H_1(va) J_1(va). \tag{31}$$

Certain conditions have to be satisfied by both functions in order for the transform to exist. The analysis of the function of sources proves that $f_l(z)$ as a diffraction term, must tend to zero for $z \to \infty$, so it has a Fourier transform

$$F_l(w) = \int_{-\infty}^{\infty} f_l(z) e^{iwz} dz \tag{32}$$

while if $\mathrm{Im}\, k > 0$, then $\mathrm{Im}\, \gamma_l > 0$, and thus for $z \to \infty$ the second term in expression (30) approaches infinity. At the same time it results from transform $L(w)$ that

$$L(\gamma_n) = 0 \tag{33}$$

so set $\left\{\gamma_n = \sqrt{k^2 - \left(\frac{\mu_n}{a}\right)^2}\right\}$ is the set of roots of equation (31). Taking all that was up to now said into consideration we obtain the following form of the Fourier transform of the boundary condition

$$\int_{-\infty}^{\infty} e^{-iwz} dz \int_{-\infty}^{\infty} g_l(z') l(z-z') dz' = F_l(w) L(w). \tag{34}$$

The last equality is true if both transforms $F(w)$ and $L(w)$ have a common analyticity range. This range is the zone of the complex plane w, defined by equality $-\mathrm{Im}\, k < \mathrm{Im}\, w < \mathrm{Im}\, k$. Now, additional conditions, which the transform of the function of apparent sources, $F_l(w)$, must satisfy will be determined. The function of sources $g_l(z)$ equals zero when $z < 0$, therefore

$$f_l(z) = -A_l e^{-i\gamma_l z} \quad z < 0. \tag{35}$$

This equality will be satisfied if the transform $F_l(w)$ will have a first order pole with the residuum equal to $\frac{A_l}{i}$ in point $w = -\gamma_l$, and furthermore it will uniformly tend to zero on the lower half-plane, for $|w| \to \infty$. Then

$$\frac{1}{2\pi} \int_{-\infty+i\eta}^{\infty+i\eta} F_l(w) e^{iwz} dw + A_l e^{-i\gamma_l z} = 0, \quad z < 0. \tag{36}$$

The physical interpretation of this equation is as follows: for $z < 0$, i.e. on the extension of the wave-guide surface, the potential is continuous.

The boundary condition will be noted with the use of Fourier transform. The inverse Fourier transform of expression (34) is

$$\int_{-\infty}^{\infty} g_l(z')\,l(z-z')\,dz' = \int_{-\infty}^{\infty} F_l(w)\,L(w)\,e^{iwz}dw \tag{37}$$

hence the boundary condition (28) becomes

$$\int_{-\infty}^{\infty} F_l(w)\,L(w)\,e^{iwz}dw = 0, \quad z > 0. \tag{38}$$

This equation will be satisfied when the product of functions $F_l(w)$ and $L(w)$ is an analytic function in the upper half-plane and tends uniformly to zero on the infinite semicircle on the half-plane.

Both equations, (37) and (38), can be written in the form of homogeneous equations

$$\int_C F_l(w)\,e^{iwz}dw = 0, \quad z < 0 \tag{39}$$

$$\int_C F_l(w)\,L(w)\,e^{iwz}dw = 0, \quad z > 0 \tag{40}$$

where C is the integration contour, consisting of a line parallel to the real axis (it can be the axis itself in particular) and a loop around point $w = -\gamma_l$ (Fig. 4).

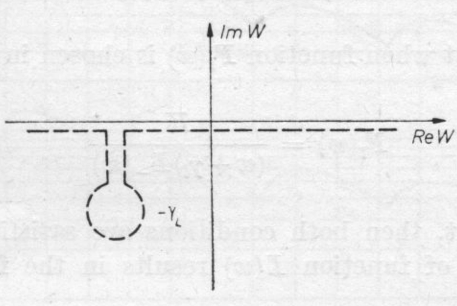

Fig. 4. Integration contour C in the plane of the complex variable w. Consists of a line parallel to the real axis and a loop around point $w = \gamma_l$

It follows from this paragraph that the condition of continuity of the potential on the extension of the wave-guide surface (36) and the condition of decay of the normal component of the vibration velocity on its wall (38) are fulfilled, when

1. Function $F_l(w)$ is analytic in the lower half-plane, $\mathrm{Im}\,w < \mathrm{Im}\,k$, excluding the point $w = -\gamma_l$, where it has a first order pole with a residuum equal to A_l/i, and it tends to zero on this half-plane for $|w| \to \infty$.

2. Product $F_l(w) \cdot L(w)$ is an analytical function in the upper half-plane, $\mathrm{Im}\,w > -\mathrm{Im}\,k$, and it uniformly tends to zero for $|w| \to \infty$ in this range.

Thus, the problem of the acoustic field of a semi-infinite cylindrical waveguide can be solved by determining function $F_l(w)$, which satisfies conditions (24) and (25), i.e. by solving the pair of integral equations (36) and (38) or (39) and (40).

Integral equations derived in this paragraph are identical with WAJNSZTEJN'S equations for electromagnetic waves [5]. Therefore, his methods and results can be applied in further considerations.

4. Application of the Wiener-Hopf method in solving obtained integral equations

As it has been mentioned in the preceeding paragraph, the integral form (28) of the boundary condition (10) is a Wiener-Hopf type equation, so it can be solved with the factorization method. In this paper only a short outline of the solution will be presented, because of the applied complicated calculation methods [11].

An assumption is made at present that the distribution of function $L(w)$ onto analytical factors, respectively in the upper and lower half-plane of the complex variable w is known

$$L(w) = L_+(w) L_-(w). \tag{41}$$

It can be seen that when function $F_l(w)$ is chosen in the form

$$F_l(w) = \frac{K}{(w+\gamma_l) L_-(w)} \tag{42}$$

where K is a constant, then both conditions are satisfied.

The factorization of function $L(w)$ results in the following expressions

$$L_+(w) = (k+w)\left(H_1^{(1)}(va) J_1(va) \prod_{i=1}^{N} \frac{\gamma_i + w}{\gamma_i - w}\right)^{1/2} e^{S(w)/2} \tag{43}$$

$$L_-(w) = (k-w)\left(H_1^{(1)}(va) J_1(va) \prod_{i=1}^{N} \frac{\gamma_i - w}{\gamma_i + w}\right)^{1/2} e^{-S(w)/2} \tag{44}$$

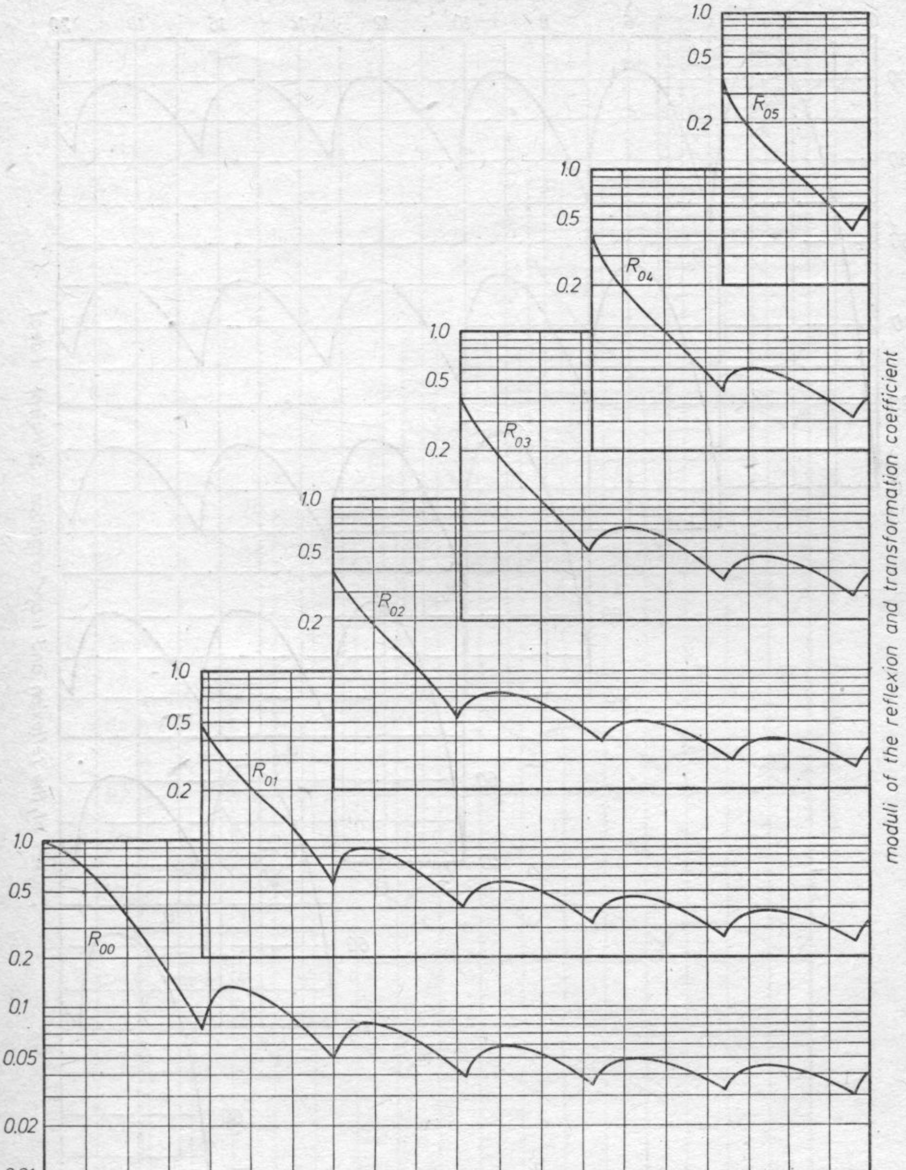

Fig. 5. Moduli of transformation and reflection coefficients of a plane wave at the wave-guide orfice in terms of the diffraction parameter ***ka***

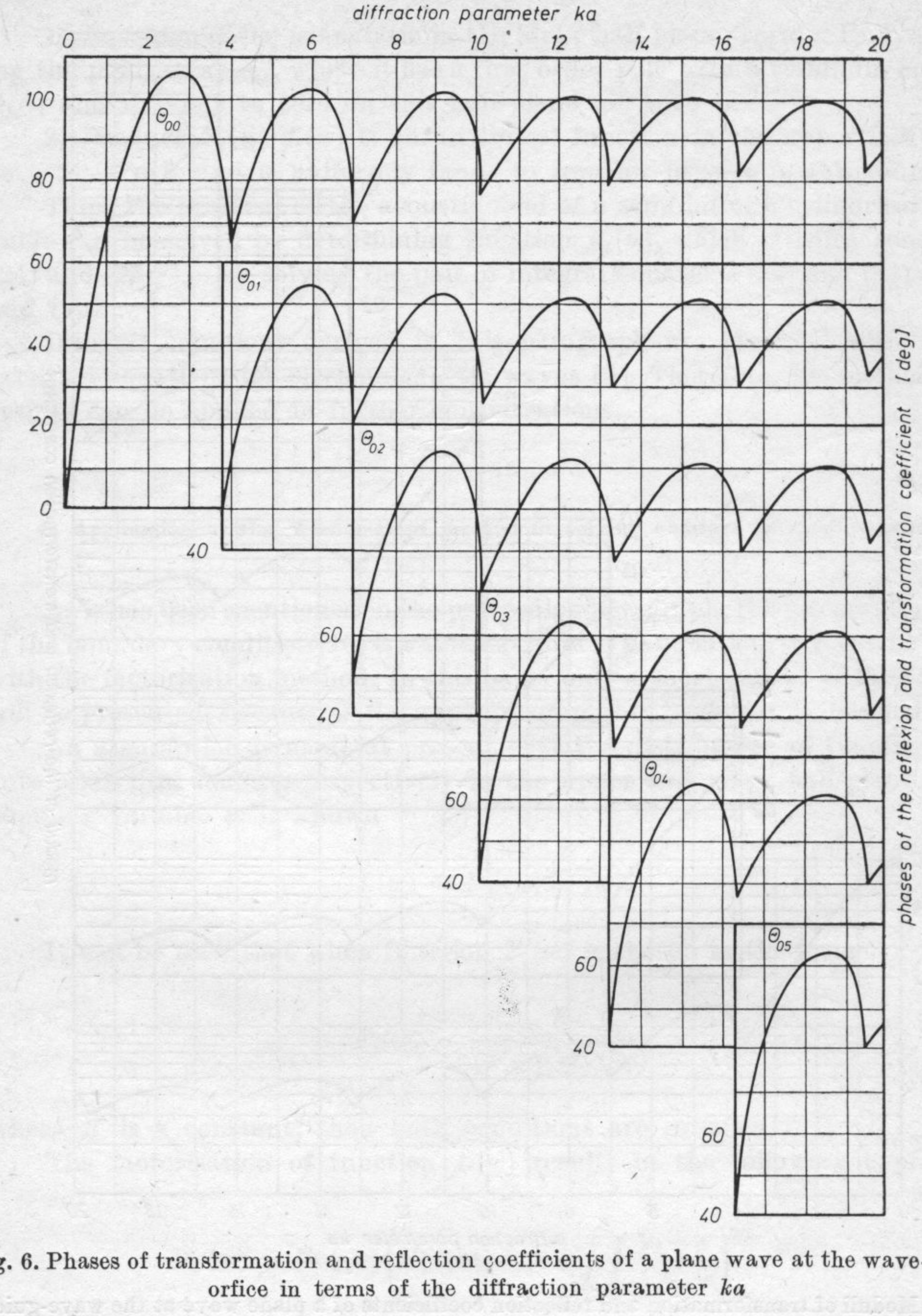

Fig. 6. Phases of transformation and reflection coefficients of a plane wave at the wave-guide orfice in terms of the diffraction parameter ka

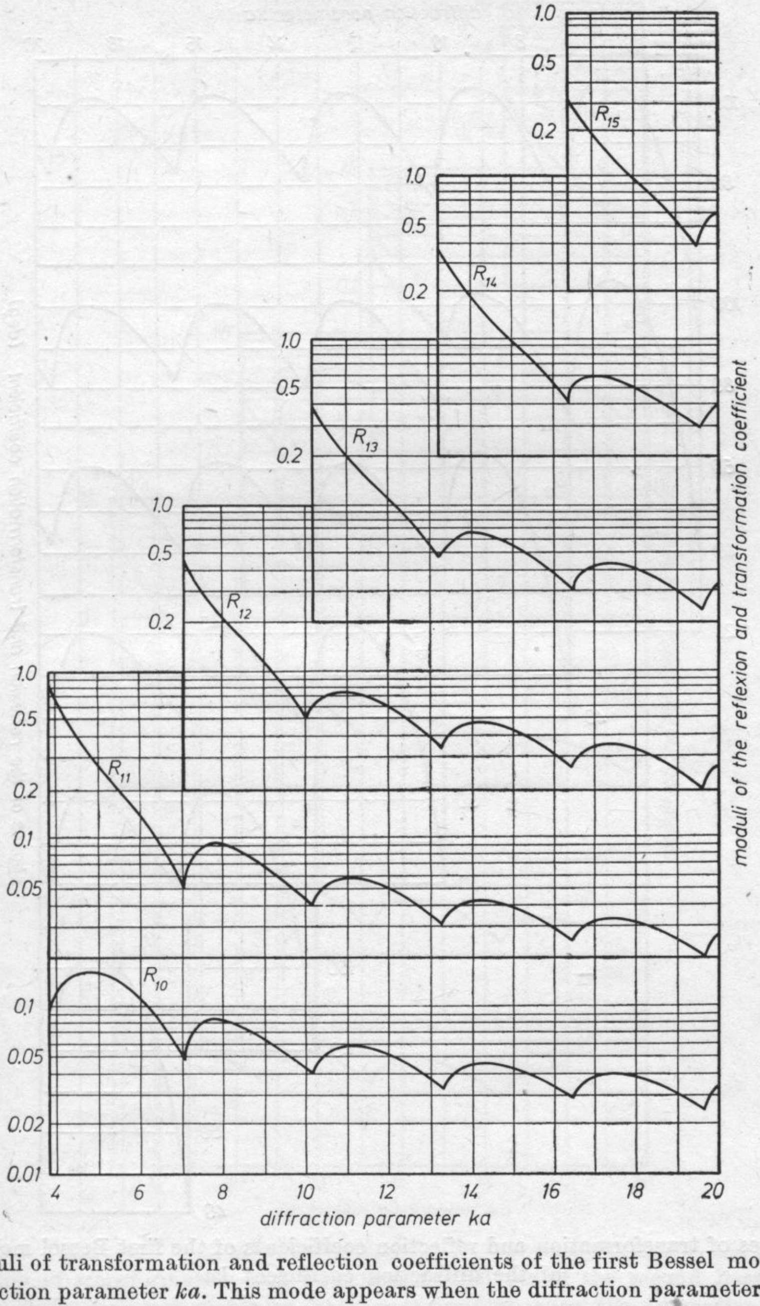

Fig. 7. Moduli of transformation and reflection coefficients of the first Bessel mode in terms of the diffraction parameter ka. This mode appears when the diffraction parameter ka exceeds the first zero of the Bessel function $J_1(z)$, $(ka > 3.83)$

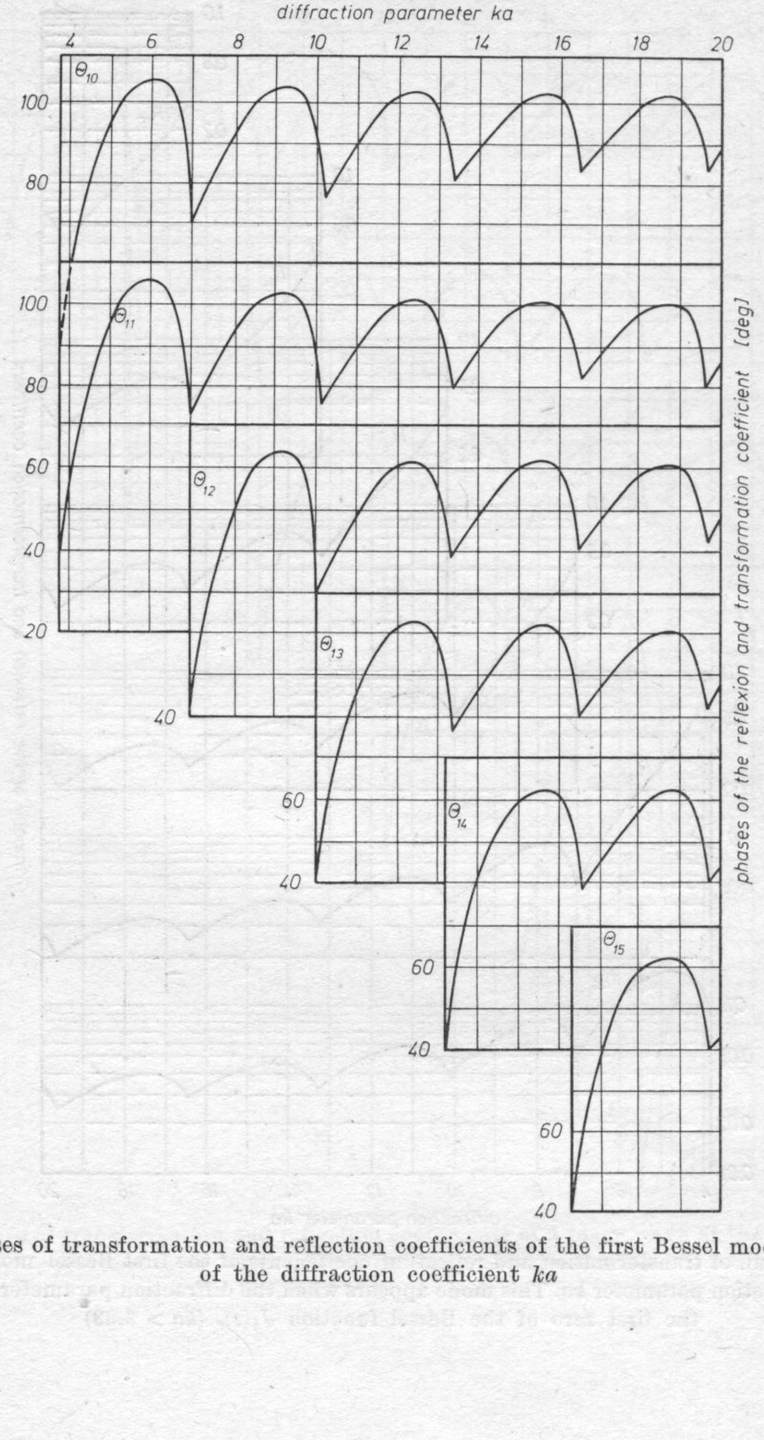

Fig. 8. Phases of transformation and reflection coefficients of the first Bessel mode in terms of the diffraction coefficient *ka*

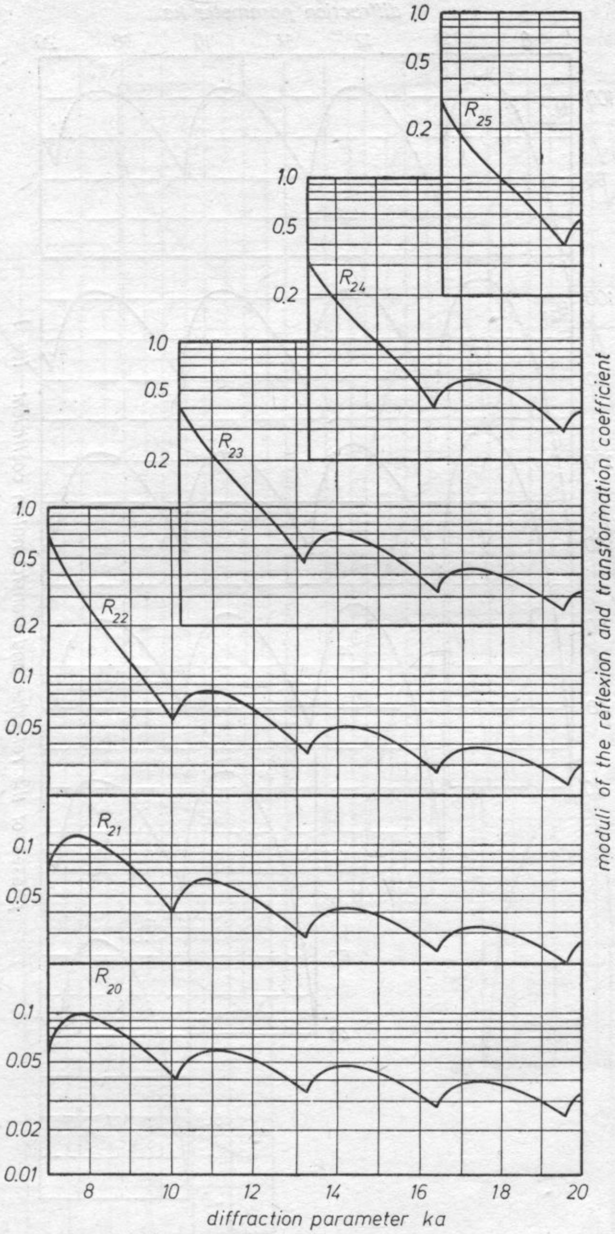

Fig. 9. Moduli of transformation and reflection coefficients of the second Bessel mode in terms of the diffraction parameter ka. This mode appears when the diffraction parameter ka exceeds the second zero of the Bessel function $J_1(z)$, $(ka > 7.01)$

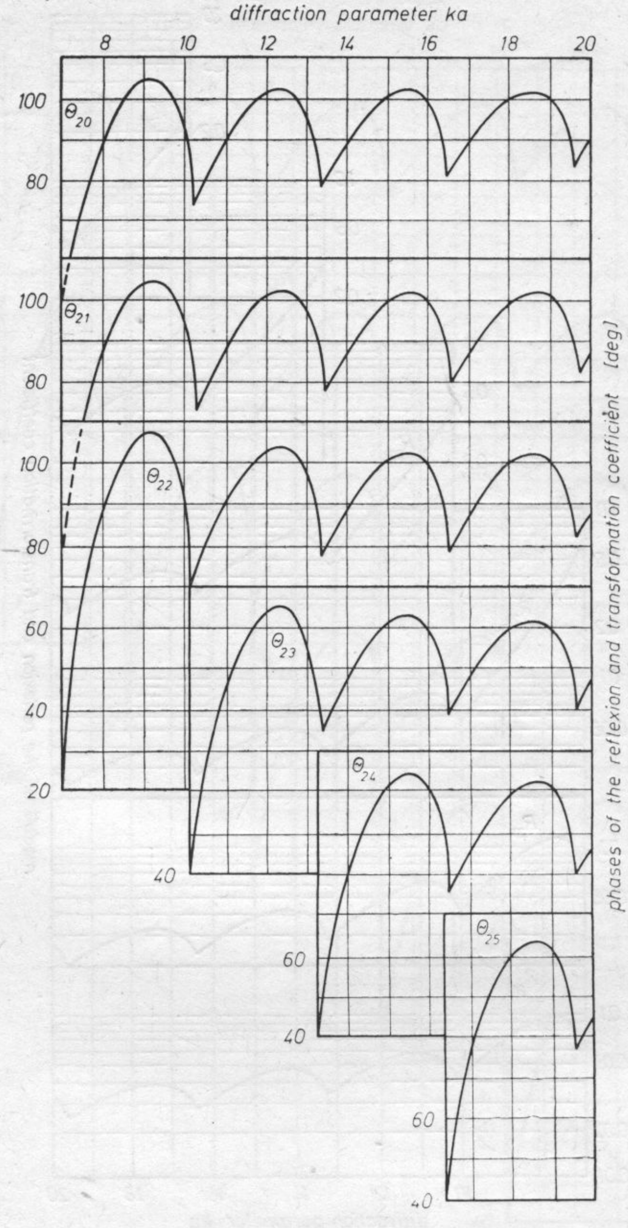

Fig. 10. Phases of transformation and reflection coefficients of the second Bessel mode in terms of the diffraction parameter *ka*

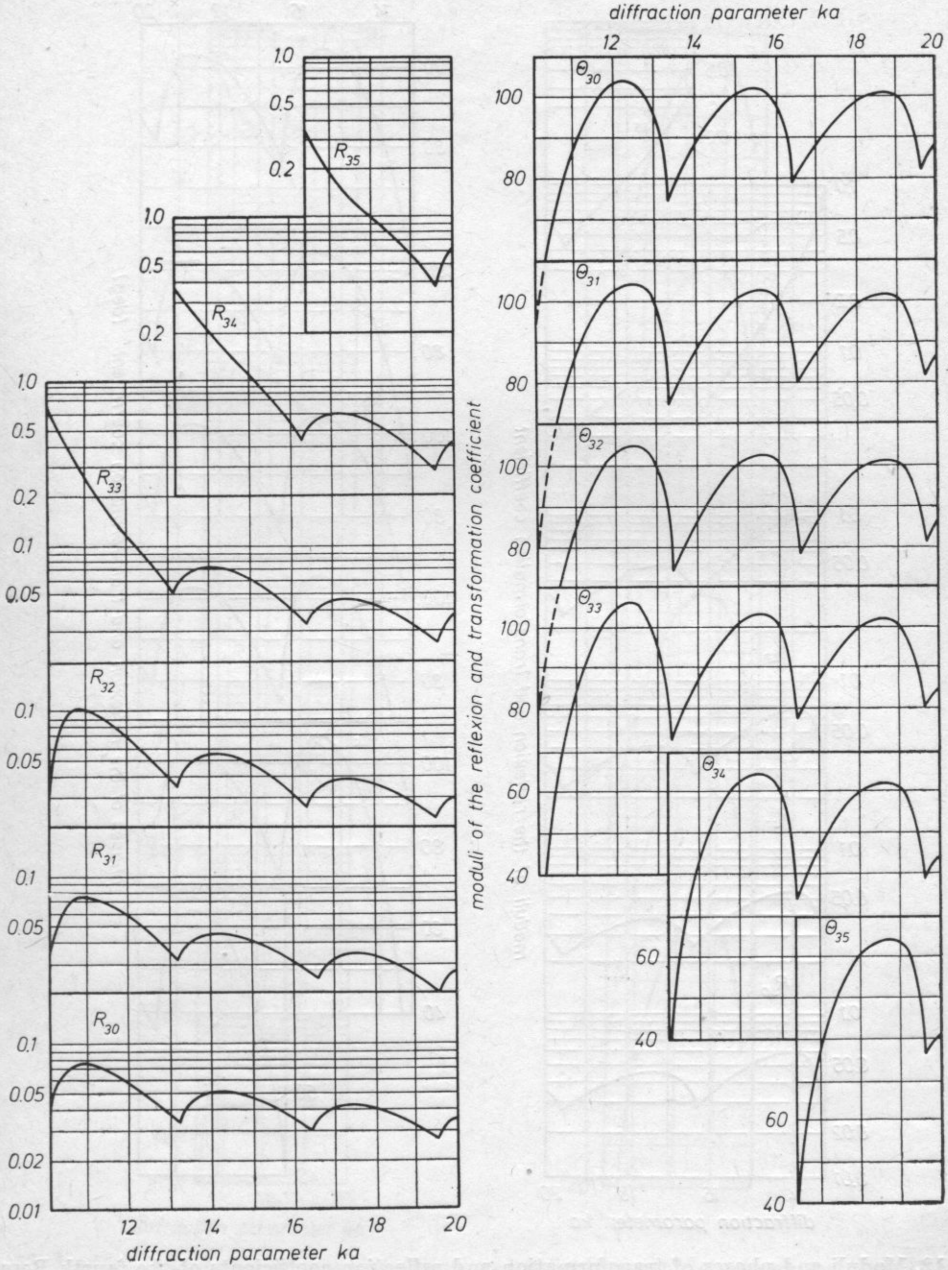

Fig. 11. Moduli and phases of transformation and reflection coefficients of the third Bessel mode in terms of the diffraction parameter ka. This mode appears when the diffraction parameter ka exceeds the third zero of the Bessel function $J_1(z)$, $(ka > 10.17)$

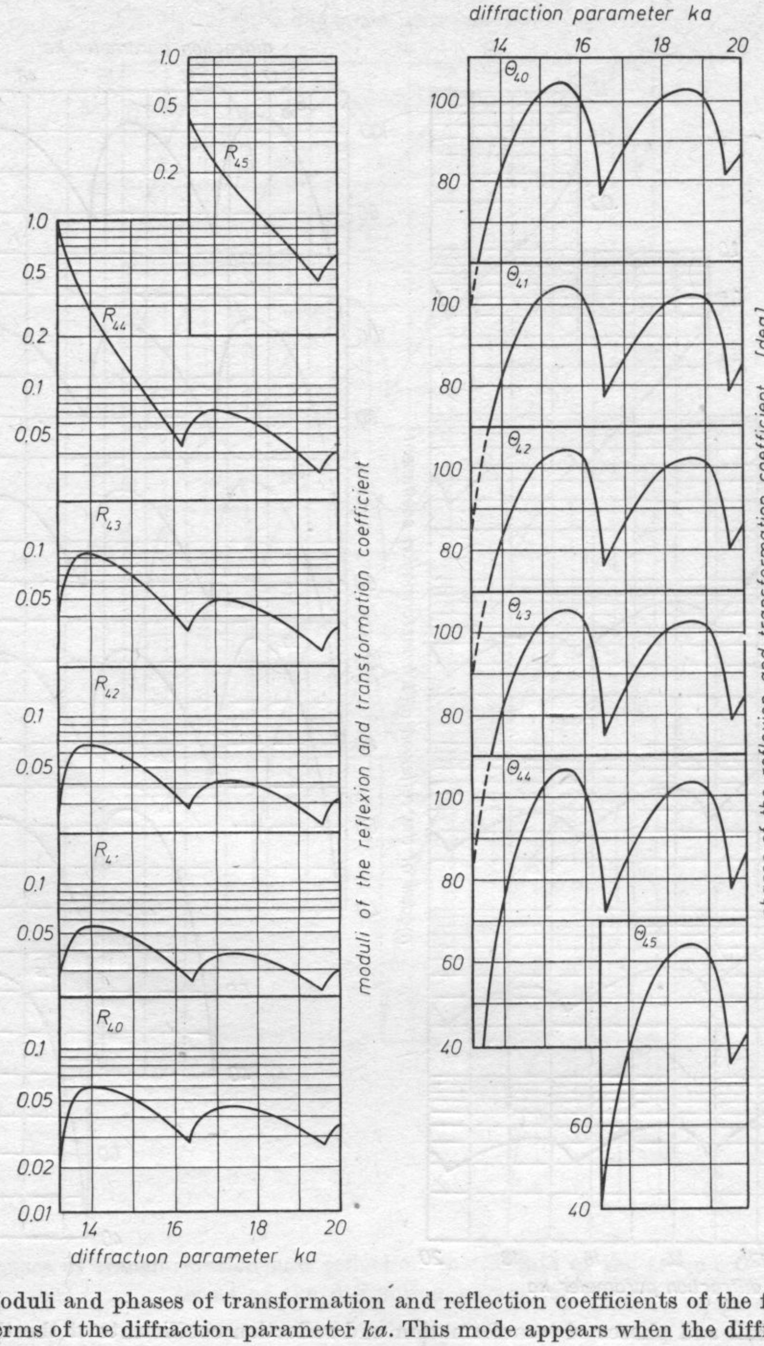

Fig. 12. Moduli and phases of transformation and reflection coefficients of the fourth Bessel mode in terms of the diffraction parameter ka. This mode appears when the diffraction parameter ka exceeds the fourth zero of the Bessel function $J_1(z)$, $(ka > 13.32)$

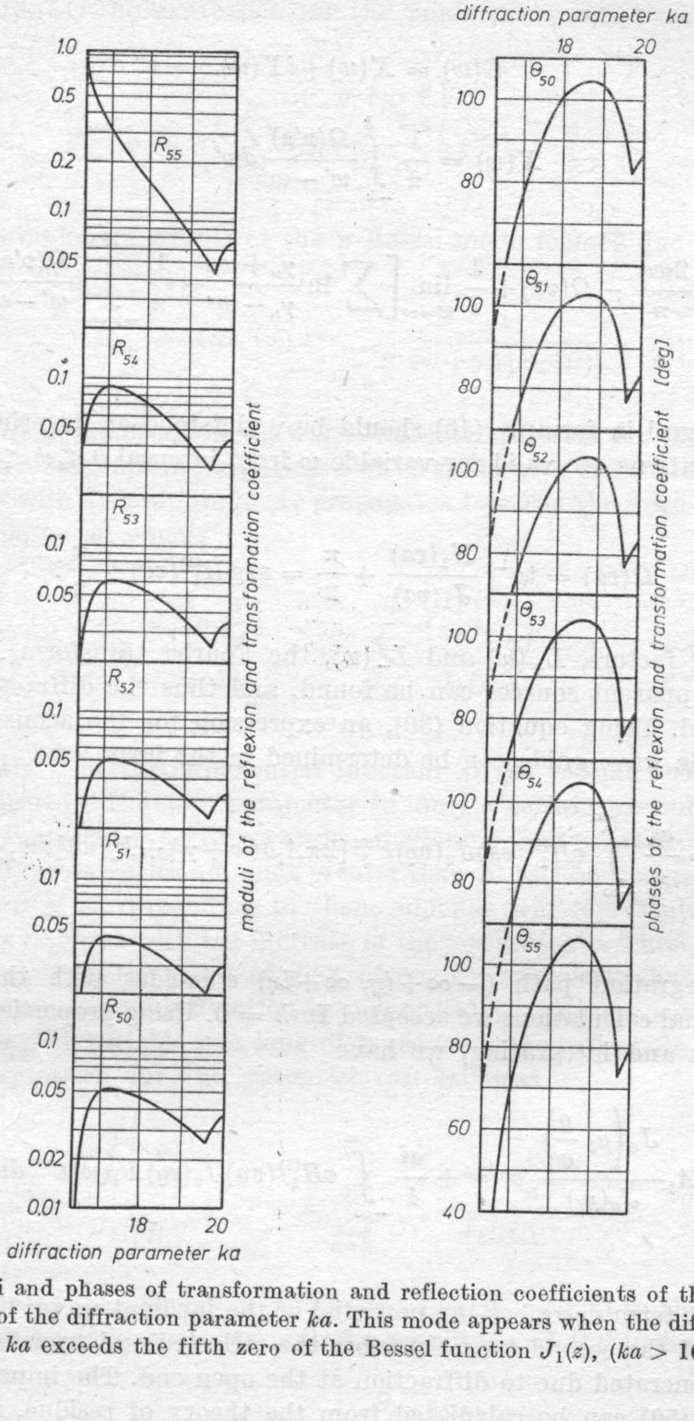

Fig. 13. Moduli and phases of transformation and reflection coefficients of the fifth Bessel mode in terms of the diffraction parameter ka. This mode appears when the diffraction parameter ka exceeds the fifth zero of the Bessel function $J_1(z)$, $(ka > 16.47)$

where

$$S(w) = X(w) + iY(w) \tag{45}$$

$$X(w) = \frac{1}{\pi} \int_{-k}^{k} \frac{\Omega(v'a)}{w'-w}\, dw', \tag{46}$$

$$Y(w) = \frac{2wa}{\pi} - \Omega(va) + \frac{1}{i} \lim_{M\to\infty} \left[\sum \ln \frac{\gamma_n + w}{\gamma_n - w} - \frac{1}{\pi} \int_{-\gamma_M}^{\gamma_M} \frac{\Omega(v'a)}{w'-w}\, dw' \right]. \tag{47}$$

The integral in formula (46) should be understood as its principal value. Two last equations are valid for variable w from interval $0 \leqslant w \leqslant k$. Function $\Omega(va)$ equals

$$\Omega(va) = \mathrm{tg}^{-1} \frac{N_1(va)}{J_1(va)} + \frac{\pi}{2} = \arg H_1^{(1)}(va) + \frac{\pi}{2}. \tag{48}$$

Knowing factors, $L_+(w)$ and $L_-(w)$, the Fourier transform $F_l(w)$ of the function of apparent sources can be found, and thus the diffraction problem can be solved. Using equation (30), an expression for the acoustic potential (31) inside the wave-guide can be determined in the form

$$\Phi_l(\varrho, z) = \frac{ai}{4} \int_{-\infty}^{\infty} vH_1^{(1)}(va) J_0(v\varrho) e^{iwz} [2\pi A_l \delta(w+\gamma_l) + F_l(w)] dw, \qquad \begin{matrix} \varrho > a \\ z > 0 \end{matrix}. \tag{49}$$

The integration path $(-\infty + i\eta,\ \infty + i\eta)$ coincides with the real axis, because in final calculations we accepted $\operatorname{Im} k = 0$. Using properties of cylindrical functions and integrating, we have

$$\Phi_l(\varrho, z) = A_l \frac{J_0\left(\mu_l \dfrac{\varrho}{a}\right)}{J_0(\mu_l)} e^{-i\gamma_l z} + \frac{ai}{4} \int_{-\infty}^{\infty} vH_1^{(1)}(va) J_0(v\varrho) F_l(w) e^{iwz} dw, \qquad \begin{matrix} \varrho > a \\ z > 0 \end{matrix}. \tag{50}$$

The first term describes the potential of the incident wave; thus it can be accepted that the second term describes the reflected and transformed waves, which are generated due to diffraction at the open end. The improper integral in equation (50) can be calculated from the theory of residue, remembering that variable v is an elemental variable, so the integrand is not unique. From formulae (41)–(44) the expression for the potential of reflected waves is

$$\Phi_l^{\mathrm{ref}}(\varrho, z) = \sum_{n=0}^{\infty} B_{l,n} \frac{J_0\left(\mu_n \frac{\varrho}{a}\right)}{J_0(\mu_n)} e^{i\gamma_n z}, \qquad \begin{matrix} \varrho < a \\ z > 0 \end{matrix}. \tag{51}$$

$B_{l,n}$ is the complex amplitude of the n Bessel mode formed due to diffraction at the open end

$$B_{l,n} = A_l L_+(\gamma_l) \operatorname{res}_{w=\gamma_n} \frac{1}{(w+\gamma_l)\big(L_-(w)\big)}. \tag{52}$$

Summarizing, once again we will write the expression for the acoustic potential inside a semi-infinite cylindrical wave-guide with rigid walls, when a 1-st Bessel mode with an amplitude A_1 propagates towards the open end. In such a case the potential equals

$$\Phi_l(\varrho, z) = A_l \frac{J_0\left(\mu_l \frac{\varrho}{a}\right)}{J_0(\mu_l)} e^{-\gamma_l z} + \sum_{n=0}^{\infty} \frac{J_0\left(\mu_n \frac{\varrho}{a}\right)}{J_0(\mu_n)} e^{i\gamma_n z}, \qquad \begin{matrix} \varrho < a \\ z > 0 \end{matrix}. \tag{53}$$

The analysis of the exponential function in the second term shows that for an established diffraction parameter ka, only a certain part of the terms of the sum will represent progressive waves. Beginning from a certain N, so $\mu_N < ka \leqslant \mu_{N+1}$, coefficients γ with an index greater than N will be imaginary numbers and in that case corresponding to them addends will represent disturbances exponentially damped with the increase of the z coordinate. These disturbances are not waves from the point of view of energy transport, thus they do not have to be taken into consideration in energetical calculations as well as in case of great values of variable z (a long distance from the orfice). In this case the following expression for the potential can be used

$$\Phi_l(\varrho, z) = A_l \frac{J_0\left(\mu_l \frac{\varrho}{a}\right)}{J_0(\mu_l)} e^{-i\gamma_l z} + A_l \sum_{n=0}^{\infty} R_{l,n} \frac{J_0\left(\mu_n \frac{\varrho}{a}\right)}{J_0(\mu_n)} e^{i\gamma_n z}, \qquad \begin{matrix} \varrho < a \\ z > 0 \end{matrix}. \tag{54}$$

$R_{l,n}$ is a complex transformation coefficient and it is equal to the ratio of the amplitude of the induced mode and the amplitude of the incitent mode

$$R_{l,n} = \frac{B_{l,n}}{A_l}. \tag{55}$$

For $l = n$ it is simply a reflection coefficient. Diagrams present moduli and phases of these coefficients, with the diffraction parameter ka, varying from 0 to 20.

5. Conclusions

To recapitulate, effects taking place at the end of the wave-guide are as follows: one of the Bessel modes, which is allowed from the point of view of the diffraction parameter ka, propagates towards the orfice. In undergoes diffraction at the orfice — part of the energy is radiated outside, the rest returns to the wave-guide in the form of allowed wave modes — higher, lower and also the mode of the same order as the incident wave. In order to determine the acoustic field inside the wave-guide in such a case, N complex reflection and transformation coefficients have to be established. The number of these coefficients has to equal the number of modes which can propagate in the wave-guide at an assigned diffraction parameter. Furthermore, if we include that the incident wave can be a superposition of all allowed modes, then the number of coefficients, describing the field, increases to N^2. This may complicate the univocal interpretation of the results. This is a view shared by many authors occupied with this problem. They consider that the Wiener-Hopf method applied to diffraction problems is mathematically very complicated and the interpretation of the results is difficult due to their complicated form [8, 9]. This statement is only partly true, because the mathematical description of the theory is indeed difficult (it is not presented in this paper, because only final formulae have been used in paragraph 4), but the interpretation of the results can be carried out with the application of the analysis of energetic quantities, such as impedance for example [10].

Thus the problem of the acoustic field of a cylindrical wave-guide is important from the cognitive point of view, because it is one of the fundamental diffraction problems, as well as from the practical point of view, because elements which can be approximated by a long cylindrical pipe without a baffle, occur frequently in acoustic systems.

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