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ACOUSTICAL WAVE PROPAGATION IN A CYLINDRICAL LAYER SYSTEM IN VISCOUS MEDIUM

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A problem is considered of an acoustical wave propagating along a hollow, infinite elastic cylinder filled with air and surrounded by a viscoelastic tissue. Such a case approximately corresponds to a biopsy performed with the application of a needle introduced to such tissues as liver, kidney, muscles, and the like.

In the problem under consideration it was proved, that the volume viscosity is significant, whereas shear viscosity can be neglected.

Basic equations were formulated in terms of displacement potentials, as well as the boundary conditions. This led to a characteristic equation of the problem which were solved numerically.

It was proved that a boundary wave propagates along the needle with a velocity and attenuation not much smaller than in the surrounding tissue. Part of the energy is transfered from the needle into the tissue where the energy is dissipated. Distributions of the radial and axial stress components and radial displacement components were found.

Introduction

The numerical solution of a problem of acoustical wave propagation along a hollow cylinder filled with air and submerged in a absorbing liquid concerns an effect observed during the conduction of a biopsy controlled ultrasonically. It was observed, that under certain physical conditions a wave is produced which propagates along the needle, reaches the point of the needle and returns, giving an image of the needle point on the screen of the echoscope. This problem has been worked on under certain physical limitations in papers [1], [2], [3]. These papers proved, that the velocity of the propagating wave is close to the wave velocity in the biological structure surrounding the needle. Previous papers concerned the wave propagation in perfectly elastic media. Now we assume, that the biological structure surrounding the needle is an viscoelastic medium.

As it results from paper [4], such biological structures as muscle, kidney, liver, on which a biopsy is performed, have viscous properties. The propagation of an acoustical wave in a viscoelastic medium was considered in work [5]. It analysed a plane case. Now a needle used for puncturing a given biological structure will be approximated by a perfectly elastic hollow cylinder, infinitly long (Fig. 1).

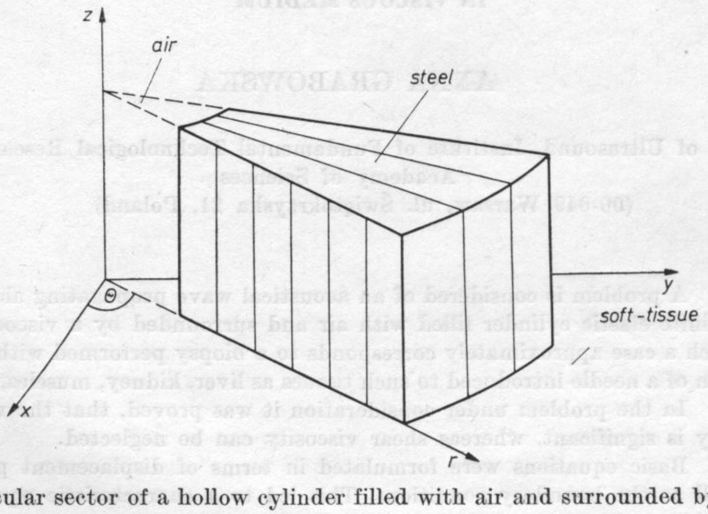

Fig. 1. A circular sector of a hollow cylinder filled with air and surrounded by a viscoelastic medium

The analyzed wave is a progressive one. We also assume, that the viscoelastic biological medium surrounding from the outside the hollow cylinder is an infinitely extended medium.

The aim of the paper was to develop basic equations, to solve numerically the characteristic equation, and to determine the parameters characterizing the wave motion in this system.

Basic equations

The theory of elasticity assumes, that the components of the stress tensor are linear functions of the strain. This assumption (Hook's law) is in force, when the purely elastic forces are significantly greater then the forces depending on the strain velocity (viscous forces).

In a case when these forces are comparable and the stress components are also linear functions of the components of the strain velocity, we can say, that the given body has also viscous properties and we call it a viscoelastic body (Voigt's body for example). We approximate a biological medium by a mode of the Voigt body. Such a body, in a case of isotropy, is characterized by four material constants $\lambda', \mu', \lambda'', \mu''$, where λ' and μ' determine the elastic properties of the body, and λ'' and μ'' — the viscous properties.

The constitutive equation for a viscoelastic body is:

$$\tau_{ij} = \left(2\mu' + 2\mu'' \frac{\partial}{\partial t}\right) \varepsilon_{ij} + \left(\lambda' + \lambda'' \frac{\partial}{\partial t}\right) \varepsilon_{ij} \varepsilon_{kk} \quad i, j = x, y, z \tag{1}$$

where τ and ν are the stress and strain, respectively.

Placing (1) in the equations of motion:

$$\varrho_c \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ij}}{\partial x_j} \quad i, j = x, y, z \tag{2}$$

where $\vec{u}$ — displacement we obtain the displacement equation for an isotropic viscoelastic body:

$$\varrho_c \frac{\partial^2 \vec{u}}{\partial t^2} = \left[(\lambda' + \mu') + (\lambda'' + \mu'') \frac{\partial}{\partial t}\right] \operatorname{grad} \operatorname{div} \vec{u} + \left(\mu' + \mu'' \frac{\partial}{\partial t}\right) \nabla^2 \vec{u} \tag{3}$$

In a case of harmonic motion with a frequency f, which we assume in the paper as:

$$\vec{u}(x, y, z, t) = \vec{u}(x, y, z) e^{j\omega t} \tag{4}$$

and applying equation (3) we reach:

$$-\varrho_c \omega^2 \vec{u} = (\lambda_c + \mu_c) \operatorname{graddiv} \vec{u} + \mu_c \nabla^2 \vec{u} \tag{5}$$

where

$$\lambda_c = \lambda' + j\omega\lambda'', \quad \mu_c = \mu' + j\omega\mu'', \quad \omega = 2\pi f. \tag{6}$$

The displacement equation (5) has the same form as the displacement equation in the theory of elasticity. The difference is only in two parameters, λ_c and μ_c, which now are complex and depend on frequency according to equation (6), while in an perfectly elastic medium these parameters were real numbers. So a perfectly elastic isotropic body is characterized by two Lamé constants, while a viscoelastic isotropic body — by four constants. λ' and μ' determine the elasticity of volume and the shape elasticity, respectively. λ'' and μ'' — the volume and shape viscosity, respectively. Equation (5) is solved in the same way as in the theory of elasticity [6]. A cylindrical coordinate system is chosen (r, θ, z) (Fig. 1). In our case of a deformation axially symmetrical in respect to axis z, the displacements, deformations and stresses are independent on the angle θ. For the vibrations of a viscoelastic medium we have a displacement equation:

$$(\lambda + \mu) \operatorname{grad} \operatorname{div} \vec{u} + \mu \nabla^2 \vec{u} = \varrho_c \ddot{\vec{u}}. \tag{7}$$

Coefficients λ and μ are:

$$\lambda = \lambda' + \lambda'' \frac{\partial}{\partial t}, \quad \mu = \mu' + \mu'' \frac{\partial}{\partial t} \tag{8}$$

where λ' and μ' determine the elastic properties, and λ'' and μ'' – the viscous properties of the body.

We assume, that the displacement vector, $\overline{u}_c$, in a viscoelastic medium has the following form:

$$\vec{u}_c(r, z, t) = \operatorname{grad} \Phi_c + \operatorname{rot} \vec{W}^c, \tag{9}$$

Putting the relationship (9) in the displacement equation (7), shows that equation (7) will be fulfilled if the scalar Φ_c and the vector $\vec{W}^c$ potentials are the solutions of equations:

$$\nabla^2 \Phi_c = \frac{\varrho_c}{\lambda + 2\mu} \frac{\partial^2 \Phi_c}{\partial t^2} \tag{10}$$

$$\nabla^2 \vec{W^c} - \frac{1}{r} \vec{W}^c = \frac{\varrho_c}{\mu} \frac{\partial^2 \vec{W}^c}{\partial t^2} \tag{11}$$

where

$$\lambda = \lambda' + \lambda'' \frac{\partial}{\partial t}, \quad \mu = \mu' \pm \mu'' \frac{\partial}{\partial t}. \tag{12}$$

In a case of a deformation axially symmetrical in respect to axis z, only one component, W_θ^c, of vector $\vec{W}^c$ differs from zero. Therefore, equation (11) can be rewritten:

$$\nabla^2 W_\theta^c - \frac{1}{r} \vec{W}_\theta^c = \frac{\varrho_c}{\mu} \frac{\partial^2 W_\theta^c}{\partial t^2}. \tag{13}$$

We bring equation (13) to a scalar wave equation, defining scalar quantity Ψ_c, in the following way:

$$\Psi_c \colon W_\theta^c = -\frac{\partial \Psi_c}{\partial r} \tag{14}$$

Placing dependence (14) in equation (13) we obtain:

$$\nabla^2 \Psi_c = \frac{\varrho_c}{\mu} \frac{\partial^2 \Psi_c}{\partial t^2} \quad \text{where } \mu = \mu' + \mu'' \frac{\partial}{\partial t} \tag{15}$$

We rearrange equation (15):

$$\left(\mu'+\mu''\frac{\partial}{\partial t}\right)\nabla^2\Psi_c = \varrho_c\frac{\partial^2\Psi_c}{\partial t^2},$$

$$\mu'\nabla^2\Psi_c+\mu''\nabla^2\frac{\partial}{\partial t}\Psi_c = \varrho_c\frac{\partial^2\Psi_c}{\partial t^2}. \tag{16}$$

Let:

$$\Psi_c(r,z,t) = \Psi_0(r)e^{-jpz+j\omega t}$$

$$(\mu'+j\omega\mu'')\nabla^2\Psi_c = -\varrho_c\omega^2\Psi_c$$

$$(\mu'+j\omega\mu'')\left[\frac{\partial^2\Psi_0}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi_0}{\partial r}-p^2\Psi_0\right] = -\varrho_c\omega^2\Psi_0$$

$$\frac{\partial^2\Psi_0}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi_0}{\partial r}+\left[\frac{\varrho_c\omega^2}{\mu'+j\omega\mu''}-p^2\right]\Psi_0 = 0 \tag{17}$$

We introduce a denotation:

$$\frac{\varrho_c\omega^2}{\mu'+j\omega\mu''}-p^2 = l_t^2 \tag{18}$$

We accept the solution of the Bessel equation (17) in the following form:

$$\Psi_c(r,z,t) = BH_0^{(2)}(l_t r)e^{-jpz+j\omega t} \tag{19}$$

Analogically we solve equation (10), so we assume the solution of eq. (10) to have the following form:

$$\Phi_c(r,z,t) = \Phi_0(r)e^{-jpz+j\omega t}$$

and we reach the Bessel equation:

$$\frac{\partial^2\Phi_0}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi_0}{\partial r}+\left[\frac{\varrho_c\omega^2}{\lambda'+2\mu'+j\omega(\lambda''+2\mu'')}-p^2\right]\Phi_0 = 0 \tag{20}$$

We denote:

$$\frac{\varrho_c\omega^2}{\lambda'+2\mu'+j\omega(\lambda''+2\mu'')}-p^2 = l_d^2. \tag{21}$$

Then the solution of equ. (20) is Φ_c in the form:

$$\Phi_c(r, z, t) = \mathrm{C}\mathrm{H}_0^{(2)}(l_d r)e^{-jpz+j\omega t}. \tag{22}$$

Equ. (10) and (15) describe the following wave:

$$\begin{aligned}\Phi_c(r, z, t) &= \mathrm{C}\mathrm{H}_0^{(2)}(l_d r)e^{-jpz+j\omega t}\\ \Psi_c(r, z, t) &= \mathrm{B}\mathrm{H}_0^{(2)}(l_t r)e^{-jpz+j\omega t}\end{aligned} \tag{23}$$

where

$$\begin{gathered} l_d^2 = \frac{\varrho_c\omega^2}{\lambda_c+2\mu_c} - p^2, \quad l_t^2 = \frac{\varrho_c\omega^2}{\mu_c} - p^2,\\ \lambda_c = \lambda' + j\omega\lambda'', \quad \mu_c = \mu' + j\omega\mu'',\\ H_0^{(2)}(z) = J_0(z) - jY_0(z)\end{gathered} \tag{24}$$

p — is the sought propagation constant. Factors $e^{j\omega t}e^{-jpz}$ characterize a harmonic, absorbed, plane wave propagating in direction z. Let $p = \mathrm{Re}(p) + j\,\mathrm{Im}(p)$. Then:

$$e^{j\omega t}e^{-jpz} = e^{j\omega t}e^{-j\,\mathrm{Re}(p)z}e^{-j^2\,\mathrm{Im}(p)z} = e^{j\omega t}e^{-j\,\mathrm{Re}(p)z}e^{\mathrm{Im}(p)z}$$

Hence we obtain $e^{j\omega t}e^{-jpz} = e^{j\omega t}e^{-j^{\omega/cz}-az}$, when

$$c = \omega/\mathrm{Re}(p), \quad a = -\mathrm{Im}(p) \tag{25}$$

where c — wave phase velocity, a — absorption coefficient.

The components of the displacement vector $\vec{u}^c$ can be noted with the application of the potentials Φ_c and Ψ_c:

$$u_r^c = \frac{\partial\Phi_c}{\partial r} + \frac{\partial^2\Psi_c}{\partial z\partial r}, \quad w_z^c = \frac{\partial\Phi_c}{\partial z} - \frac{\partial\Psi_c}{r\partial r} - \frac{\partial^2\Psi_c}{\partial r^2}. \tag{26}$$

The radial and axial stress are expressed by potentials Φ_c and Ψ_c in the following way:

$$\tau_{rr}^c = \lambda_c\left(\frac{1}{r}\frac{\partial\Phi_c}{\partial r} + \frac{\partial^2\Phi_c}{\partial r^2} + \frac{\partial^2\Phi_c}{\partial z^2}\right) + 2\mu_c\frac{\partial}{\partial r}\left(\frac{\partial\Phi_c}{\partial r} + \frac{\partial^2\Psi_c}{\partial r\partial z}\right), \tag{27}$$

$$\tau_{rz}^c = \mu_c\frac{\partial}{\partial r}\left(2\frac{\partial\Phi_c}{\partial z} + \frac{\partial^2\Psi_c}{\partial z^2} - \frac{\partial^2\Psi_c}{\partial r^2} - \frac{1}{r}\frac{\partial\Psi_c}{\partial r}\right). \tag{28}$$

The radial and axial stresses, as well as the components of the displacement vector $\vec{u}^s$ in an elastic cylinder were defined in paper [3] by scalar potentials

Φ_s and Ψ_s, defined by the following equations:

$$\begin{aligned}\Phi_s(r,z,t) &= [A_1 J_0(k_d r) + A_2 Y_0(k_d r)]e^{-jpz+j\omega t}\\ \Psi_s(r,z,t) &= [B_1 J_0(k_t r) + B_2 Y_0(k_t r)]e^{-jpz+j\omega t}\end{aligned} \tag{29}$$

where

$$k_d^2 = \frac{\omega^2}{c_d^2} - p^2, \quad k_t^2 = \frac{\omega^2}{c_t^2} - p^2 \tag{30}$$

and $J_0(z)$ and $Y_0(z)$ denote Bessel functions of the zero order first and second kind, respectively.

The components of the displacement vector $\vec{u}^s$, the radial and axial stresses are expressed by potentials Φ_s and Ψ_s as follows:

$$u_r^s = \frac{\partial \Phi_s}{\partial r} + \frac{\partial^2 \Psi_s}{\partial z \partial r}, \quad w_z^s = \frac{\partial \Phi_s}{\partial z} - \frac{\partial \Psi_s}{r \partial r} - \frac{\partial^2 \Psi_s}{\partial r^2},$$

$$\tau_{rr}^s = \lambda_s\left(\frac{1}{r}\frac{\partial \Phi_s}{\partial r} + \frac{\partial^2 \Phi_s}{\partial r^2} + \frac{\partial^2 \Phi_s}{\partial z^2}\right) + 2\mu_s \frac{\partial}{\partial r}\left(\frac{\partial \Phi_s}{\partial r} + \frac{\partial^2 \Psi_s}{\partial r \partial z}\right) \tag{30a}$$

$$\tau_{rz}^s = \mu_s \frac{\partial}{\partial r}\left(2\frac{\partial \Phi_s}{\partial z} + \frac{\partial^2 \Psi_s}{\partial z^2} - \frac{\partial^2 \Psi_s}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi_s}{\partial r}\right)$$

Material Constants of a Viscoelastic Biological Medium

The viscoelastic constants will be determined from the relationship between $\lambda', \mu', \lambda'', \mu''$ and the velocities of longitudinal, c_d, and transverse, c_t, waves in soft tissue as well as the absorption coefficients α_d, α_t, for given frequencies, $\omega = 2\pi f$.
Therefore:

$$c_d = \omega/\mathrm{Re}(h), \quad \alpha_d = -\mathrm{Im}(h) \tag{31}$$

where:

$$h = [\varrho_c \omega/(\lambda_c + 2\mu_c)]^{1/2}, \quad \lambda_c = \lambda' + j\omega\lambda'', \quad \mu_c = \mu' + j\omega\mu''$$

and

$$c_t = \omega/\mathrm{Re}(l), \ \alpha_t = -\mathrm{Im}(l) \tag{32}$$

where $l = (\varrho_c \omega/\mu_c)^{1/2}$.

Papers dealing with the propagation of ultrasonic waves in soft tissue give only quantities characteristic for a longitudinal wave. A transverse wave propagating in tissue is absorbed so quickly (1000 times quicker than a longitudinal wave), that practically it is not applied. Also the measurement of this wave is very difficult. Paper [4] states first results of experiments conducted on determing the velocity, c_t, of a transverse wave, the absorption coefficient a_t and the bulk elasticity μ' and viscosity μ'' coefficients of such structures as muscle, liver, kidney. For frequencies in a range from 2 to 14 MHz only the intervals of c_t, a_t, μ' and μ'' were determined [4]:

$$c_t \in [9 \div 10^2]\mathrm{m/s}, \quad a_t \in [2 \cdot 10^5 \div 3 \cdot 10^6]\,1/\mathrm{m},$$
$$\mu' < 10^6\ \mathrm{N/m^2}, \quad \mu'' \in [3 \cdot 10^{-3} \div 4 \cdot 10^{-3}]\ \mathrm{Ns/m^2}. \tag{33}$$

In order to determine μ' and μ'' from equations (31), (32) we assume that:

a) c_t is the geometric mean in the interval (33), therefore $c_t = 30$ m/s

b) a_t was measured in paper [4] in the frequency range 2–14 MHz. The absorption coefficient a_d for longitudinal waves in soft tissue rises in direct proportion to the frequency [7], the choice of the smallest coefficient a_t from the outerval (33) is accepted by analogy. Frequencies applied in biopsy are about 2.5 MHz, thus $a_t = 2 \cdot 10^3$ 1/m.

c) $c_d = 1.5\ 10^3$ m/s.

d) $a_d = 0.37$ 1/cm = 3,25 dB/cm.

With these assumptions, the following values of material constants (viscous and elastic) of soft tissue, were obtained for a frequency of $f = 2.5$MHz:

$$\begin{aligned} \lambda' &= 2.189\ 10^9\ \mathrm{N/m^2}, \quad \mu' = 5.854 \cdot 10^5\ \mathrm{N/m^2} \\ \lambda'' &= 0.9056\ \mathrm{Ns/m^2}, \quad \mu'' = 0.0033\ \mathrm{Ns/m^2} \end{aligned} \tag{34}$$

It can be seen, that the volume viscosity coefficient λ'' is nearly three orders of magnitude greater than μ''. O'Brien in paper [6] showed, that μ'' has the same value for tissue and for water (soft tissue contains is 70% of water) and for water he accepted an approximation $\mu'' \ll \lambda''$. The same assumption can be applied for tissue. Then, comparing in (34) the elasticity coefficients λ' and μ', we can observe that μ' is 4 orders of magnitude smaller than λ'. In such a case we make an assumption: $\mu' \ll \lambda'$.

Summarizing the above assumptions we can define the tissue as a liquid characterized by coefficients:

volume viscosity: $\lambda'' = 0.9056$ Ns/m²,

elasticity of volume: $\lambda' = 2.189 \cdot 10^9$ N/m².

In this liquid only a longitudinal wave propagates, because the absorption of a transverse wave in tissue is a 1000 times greater than the absorption of a longitudinal wave, what was proved in paper [4].

Therefore in the investigated case the displacement potential has the form:

$$\Phi^c(r, z, t) = \mathrm{C}\mathrm{H}_0^{(2)}(l_d r)e^{-jpz}e^{j\omega t} \tag{35}$$

where

$$l_d^2 = \frac{\omega^2 \varrho_c}{\lambda_c} - p^2. \tag{35a}$$

The radial component of the displacement vector, expressed by potential Φ^c, is:

$$u_r^c = -Cl_d H_1^{(2)}(l_d r) e^{-jpz} e^{j\omega t}. \tag{36}$$

The normal and shear stresses in a viscoelastic liquid with a volume viscosity are:

$$\tau_{rr}^c = -\lambda_c C H_0^{(2)}(l_d r)[l_d^2 + p^2] e^{-jpz} e^{j\omega t}, \quad \tau_{rz}^c = 0. \tag{37}$$

Taking advantage of equations (29) and (30a), the radial and axial stresses, and the radial component u_r^s of the displacement vector in a elastic solid body are expressed by:

$$\begin{aligned} \tau_{rr}^s = \Big[& -A_1\{J_0(k_d r)(\omega^2\varrho_s - 2\mu_s p^2) - \frac{k_d}{r} 2\mu_s J_1(k_d r)\} - \\ & - A_2\{Y_0(k_d r)(\omega^2\varrho_s - 2\mu_s p^2) - \frac{k_d}{r} 2\mu_s Y_1(k_d r)\} - \\ & - B_1(2\mu_s jpk_t/r)\{J_1(k_t r) - k_t r J_0(k_t r)\} - \\ & - B_2(2\mu_s jpk_t/r)\{Y_1(k_t r) - k_t r Y_0(k_t r)\}\Big] e^{-jpz} e^{j\omega t}, \end{aligned} \tag{38}$$

$$\begin{aligned} \tau_{rz}^s = \mu_s\{ & 2jpk_d A_1 J_1(k_d r) + 2jpk_d A_2 Y_1(k_d r) + B_1 k_t(p^2 - k_t^2) J_1(k_t r) \\ & + B_2 k_t Y_1(k_t r)(k_t^2 - p^2)\} e^{-jpz} e^{j\omega t}, \end{aligned}$$

$$u_r^s = [-A_1 k_d J_1(k_d r) - A_2 k_d Y_1(k_d r) + B_1 jpk_t J_1(k_t r) + B_2 k_t jp Y_1(k_t r)] e^{-jpz} e^{j\omega t}$$

where

$$k_d^2 = \frac{\omega^2}{c_d^2} - p^2, \quad k_t^2 = \frac{\omega^2}{c_t^2} - p^2,$$

c_d and c_t being the propagation velocities of the longitudinal and transversal waves, respectively, in the medium (issue).

Boundary conditions

The boundary conditions should be fulfilled, on the surface of the hollow cylinder with the internal and external radius, *a* and *b*, respectively. These conditions have the form of a continuity of radial and axial stresses, and the continuity of the radial component of the displacement vector, so:

$$\begin{aligned} \tau_{rr}^s = \tau_{rr}^c, \quad \tau_{rz}^s = \tau_{rz}^c, \quad u_r^s = u_r^c \quad \text{for } r = a \\ \tau_{rr}^s = 0, \quad \tau_{rz}^s = 0, \quad \text{for } r = b \end{aligned} \tag{39}$$

Placing relationships (36), (37), (38) in the system of equations (39) we obtain a system of 5 homogeneous equations with unknown complex amplitudes: A_1, A_2, B_1, B_2, C. If there is to be a non-trivial solution of the equation system, the determinant formed from coefficients standing by amplitudes A_1, A_2, B_1, B_2, C must disappear:

$$|b_{ij}| = 0 \quad i, j = 1, 5. \tag{40}$$

Determinant $|b_{ij}|$ is expressed by formula (40a). Terms b_{ij} $i, j = 1,5$ contain: the material constants characteristic for the needle and the biological structure, wave numbers k_d, k_t, l_d, the sought wave number p, which occurs explicitly in the equation, and is also included in k_d, k_t, l_d and in the arguments of the first and second kind Bessel functions.

The solution of the characteristic equation $|b_{ij}| = 0$ by means of analytical methods in order to obtain the complex wave number p — is impossible. The complicated form of the characteristic equation suggests the application of numerical methods. We look for such values of the wave number p, which corresponds to:

a) the zeroing of the determinant,

b) to a wave velocity close to the wave velocity in the surrounding liquid medium and absorption close to wave absorption in the surrounding unlimited liquid medium.

The signs of k_d, k_t and l_d in formulas (30) and (35a) have been chosen in such a way, that the wave propagating away from the media boundary is attenuated. The characteristic equation was solved numerically for the following data: $f = 2.5$ MHz, $a = 0.75$ mm, $b = 0.5$ mm.

The needle is made of steel with density $\varrho_s = 7.7$ g/cm^3, Lame constants: $\lambda_s = 1.07 \cdot 10^{12}$g/(cm s^2) and $\mu_s = 8.03 \cdot 10^{11}$g/(cm s^2). The velocities of the longitudinal and transverse waves are $c_d = 5.9$ km/s, $c_t = 3.23$ km/s, respectively. A viscoelastic liquid was accepted as tissue. It was characterized by the following parameters: volume viscocity $\lambda'' = 0.9056$ Ns/m^2, elasticity $\lambda' = 2.189 \cdot 10^9$ N/m^2, density $\varrho_c = 1$ g/cm^3. The velocity and absorption of the wave in this medium are $c = 1.5$ km/s, $\alpha = 0.37$ 1/cm, respectively.

Under these assumptions the acquired velocity of a wave propagating along a hollow cylinder immersed in a viscoelastic liquid was $c_x = 1.49741$ km/s, in other words samewhat smaller than the velocity of a longitudinal wave in an unlimited viscoelastic liquid, accepted at 1.5 km/s. The obtained absorption

$$\left|\begin{array}{lll}
\dfrac{2k_d\mu_s}{a}J_1(k_da) & \dfrac{2k_d\mu_s}{a}Y_1(k_da) & \dfrac{-2\mu_s jpk_t}{a}[J_1(k_ta)-k_taJ_0(k_ta)] \\
-\dfrac{2j\mu_spk_t}{a}[Y_1(k_ta)-k_taY_0(k_ta)] & \lambda_cH_0^{(2)}(l_da)(l_d^2+\rho^2) & \\
-J_0(k_da)(\omega^2\varrho_s-2\mu_sp^2) & -Y_0(k_da)(\omega^2\varrho_s-2\mu_sp^2) & \\
2j\mu_sk_dpJ_1(-k_da) & 2j\mu_sk_dpY_1(k_da) & \mu_sk_tJ_1(k_ta)(p^2-k_t^2) \\
\mu_sk_tY_1(k_ta)(p^2-k_t^2) & 0 & \\
-k_dJ_1(k_da) & -k_dY_1(k_da) & jk_tpJ_1(k_ta) \\
k_tpY_1(k_ta) & l_dH_1^{(2)}(l_da) & \\
\dfrac{2k_d\mu_s}{b}J_1(k_db) & \dfrac{2k_d\mu_s}{b}Y_1(k_db) & \dfrac{-2j\mu_spk_t}{b}[J_1(k_tb)-k_tbJ_0(k_tb)] \\
\dfrac{-2j\mu_spk_t}{b}[Y_1(k_tb)-k_tbY_0(k_tb)] & 0 & \\
-J_0(k_db)(\omega^2\varrho_s-2\mu_sp^2) & -Y_0(k_db)(\omega^2\varrho_s-2\mu_sp^2) & \\
2jp\mu_sk_dJ_1(k_db) & 2jp\mu_sk_dY_1(k_db) & \mu_sk_tJ_1(k_tb)(p^2-k_t^2) \\
\mu_sk_tY_1(k_tb)(p^2-k_t^2) & 0 &
\end{array}\right| = |b_{ij}| \qquad (40a)$$

coefficient $a_x = 0.349$ 1/cm is also lower from the absorption coefficient of a longitudinal wave in an unlimited viscoelastic medium, accepted at $\alpha = 0.370$ 1/cm.

As a consequence of solving the characteristic equation the following values of wave numbers k_d, k_t and l_d were obtained:

$$k_d = -0.36 - j\ 101,\ k_t = -0.39 - j\ 92.9,\ l_d = 16.3 - j\ 0.0037 \quad (41)$$

and the following values of the amplitudes of the displacement potentials:

$$A_2 = -j,\ B_1 = -j\ 0.0072,\ B_2 = -0.0072,\ C = -0.0011 - j\ 0.00073 \quad (42)$$

for an assumed amplitude value $A_1 = 1 + j0$.

Putting together the obtained numerically results (41) and (42), we receive the displacement potentials in the following form:

a) in a viscoelastic liquid:

$$\Phi^c = \mathrm{C}H_0^{(2)}\big((16.3 - j\ 0.0037)r\big) e^{-j104z} e^{-0.349z} e^{j5\pi 10^6 t}. \quad (43)$$

It results from formula (43), that the wave propagates also in direction z. The wave is attenuated along the z axis.

The second kind Hankel function $H_0^2(l_d r)$ represents a wave propagating in the direction of the increasing r. For $r \to \infty$ we have an asymptotic representation:

$$H_0^{(2)}(l_d r) \xrightarrow[r\to\infty]{} \left(\frac{2}{l_d r \pi}\right)^{1/2} \exp\left[-j\left(l_d r - \frac{\pi}{4}\right)\right]$$

$$= \left[\frac{2}{(16.3 - j\ 0.0037) r\pi}\right]^{1/2} e^{-j16.3r} \cdot e^{-0.0037r} e^{3\cdot 5\pi 10^6 t}. \quad (44)$$

From this representation it can be seen, that the wave propagates also in direction r and is attenuated with the increase of r, because $\mathrm{Re}(l_d) > 0$ and $\mathrm{Im}(l_d) < 0$. In an ideal case (absorption in liquid is equal to zero) the real part, l_d, is zeroed and l_d is an imaginary quantity. Then the wave is attenuated with the increase of r, [3].

b) in an elastic solid body:

$$\Phi_s = [A_1 J_0(-0.4\,r - j\ 101r) + A_2 Y_0(-0.4r - j\ 101r)] e^{-j101z} \cdot e^{-0.349z} \times$$
$$\times e^{j5\pi 10^6 t} \quad (45)$$

$$\Psi_s = [B_1 J_0(-0.4\,r - j\ 93r) + B_2 Y_0(-0.4\ r - j\ 93r)] e^{-j101z} \cdot e^{-0.349} \cdot e^{j5\pi 10^6 t} \quad (46)$$

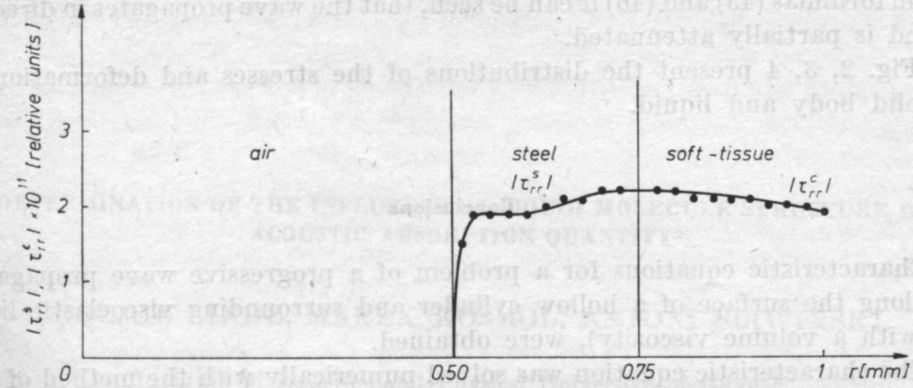

Fig. 2. Distributions of stress modules, τ_{rr}^s and τ_{rr}^c

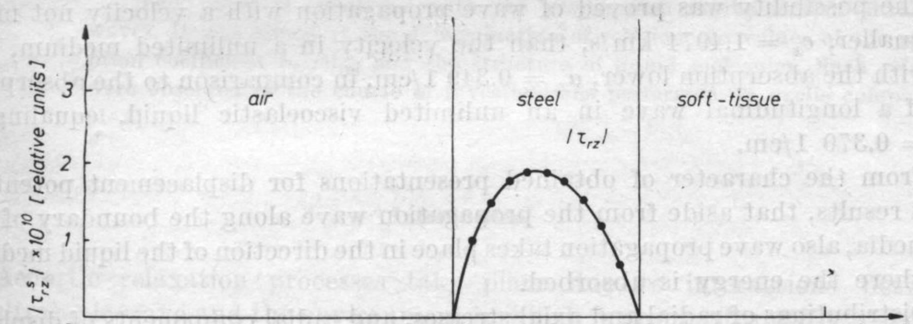

Fig. 3. Distributions of stress modulus τ_{rz}^s

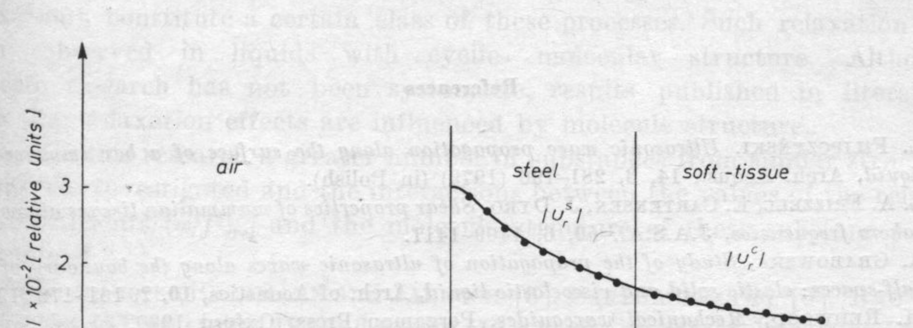

Fig. 4. Distributions of the displacement component modules u_r^s and u_r^c

From formulas (45) and (46) it can be seen, that the wave propagates in direction z and is partially attenuated.

Fig. 2, 3, 4 present the distributions of the stresses and deformations in a solid body and liquid.

Conclusions

1. Characteristic equations for a problem of a progressive wave propagating along the surface of a hollow cylinder and surrounding viscoelastic liquid (with a volume viscosity), were obtained.
2. The characteristic equation was solved numerically with the method of successive approximations and the zero crossing method. The wave velocity close to the velocity of a longitudinal wave characteristic for a viscoelastic liquid (1.5 km/s), was sought.
   The possibility was proved of wave propagation with a velocity not much smaller, $c_x = 1.4974$ km/s, than the velocity in a unlimited medium, and with the absorption lower, $\alpha_x = 0.349$ 1/cm, in comparison to the absorption of a longitudinal wave in an unlimited viscoelastic liquid, equaling $\alpha_d = 0.370$ 1/cm.
3. From the character of obtained presentations for displacement potentials it results, that aside from the propagation wave along the boundary of the media, also wave propagation takes place in the direction of the liquid medium where the energy is absorbed.
4. Distributions of radial and axial stresses, and radial components of displacement were obtained. The wave decays exponentially with the increase of the distance from the media boundary. The character of stresses and displacements is shown in Fig. 2, 3, 4.

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