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DIRECTIONAL CHARACTERISTIC OF A CIRCULAR MEMBRANE VIBRATING UNDER THE EFFECT OF A FORCE WITH UNIFORM SURFACE DISTRIBUTION

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In this paper, the acoustic pressure of a circular membrane is analysed on the assumption that the distance between the field point and the membrane is much longer, both in terms of its linear dimensions and the wavelengths radiated. It was assumed that the membrane was excited to induced harmonic vibration – including nonresonance one – by a force with uniform surface distribution. The membrane was placed in a rigid, planar baffle, and the gas medium, into which it radiated, was lossless. Numerical examples of the directional characteristic were represented graphically.

1. Notation

a – membrane radius
a_0, b – radii of the annular surface of the membrane, on which the normal component of the force inducing vibration, different from zero, acts
c_M – wave propagation velocity on the membrane
f – surface density of the force inducing the vibration
f_0 – constant density, independent of time, of the force inducing vibration (1)
J_m – Bessel function of the mth order
N_m – Neumann function of the mth order
$k = \omega\sqrt{\eta/T}$
$k_0 = 2\pi/\lambda$
K – directionality coefficient (17)
K' – directionality coefficient (25)
p – acoustic pressure (10)
p' – acoustic pressure in the main direction (23)
p_0 – acoustic pressure in the main direction (22)
R – radial variable of the field point in the spherical coordinate system

r, r_0 — radial variable of the point of the membrane surface in the polar coordinate system
T — force stretching the membrane, referred to unit length
t — time
v — normal component of the vibration velocity of points of the membrane surface
β_{0n} — nth root of the equation $J_0(\beta_{0n}) = 0$
ξ — transverse displacement of the points of the membrane surface
η — surface density of the membrane
λ — wavelength in a gaseous medium
ϱ_0 — rest density of the gaseous medium
σ_0 — membrane surface area (circular field)
ω — angular frequency of the force inducing vibration

2. Introduction

In considering practical applications, membranes being sources or receivers of acoustic energy, excited to resonance vibration, are most often analysed. In order to investigate fully and in detail the acoustic properties of such vibrating systems, investigations should also be carried out for nonresonance frequencies.

In the case of nonresonance vibration, the velocity distribution is much more complex than that for resonance vibration and depends significantly on the factor forcing the vibration, e.g. on the distribution of the inducing force [1], [2].

In paper [2], analysing the forced vibration of the circular membrane, expressions were given for the vibration velocity in the form of the expansion into a Fourier series, and the distribution of the force inducing the vibration and the distribution of deviations, as the resultants of the series of sinusoidal vibrations with frequencies equal to the eigenfrequencies of the membrane, were given.

Two methods for the calculation of the vibration velocity distribution of the membrane were given in paper [1]. The first of these methods is based on the use of the eigenfunctions for a circular membrane performing free vibration. This method is convenient for the analysis of the physical properties of the vibrating membrane, but hardly useful for numerical results to be obtained, since the solutions given in it are slowly converging series. The other method leads to the establishment of the vibration distribution in the form of the sum of the general solution of a homogeneous vibration equation and of the specific solution for a heterogeneous vibration equation. This method was used by HAJASAKA to analyse theoretically the forced vibration of the circular membrane, induced to harmonic vibration by an electric force, by means of two circular electrodes parallel to the surface of the membrane.

The directional characteristic of the circular membrane, strained with the same force over the circumference, excited to resonance vibration, is known, e.g. from SKUDRZYK's papers [4], [5].

Referring to communique [3], the present paper also considered the directional characteristic of the circular membrane, however, on the assumption that it is excited to forced vibration — including nonresonance one — by a force with uniform surface distribution. The use was made of the expression for the vibration velocity, obtained by solving a heterogeneous vibration equation for undamped phenomena harmonic in time. The vibration distribution was given in the form of the sum of the general solution of the homogeneous equation and the specific solution of the heterogeneous vibration equation. It was also assumed that the membrane was placed in an ideal rigid, planar baffle, and the baffle — into which it radiated — was lossless. An expression was obtained for the directionality coefficient in a form convenient for numerical calculations, which were represented graphically.

3. Vibration velocity

On a plane, which is an ideal rigid baffle, there is a circular membrane stretched by the same force round the circumference with the radius a. The membrane is induced to transverse vibration under the effect of an axially-symmetric inducing force, e.g. by means of two planar annular electrodes parallel

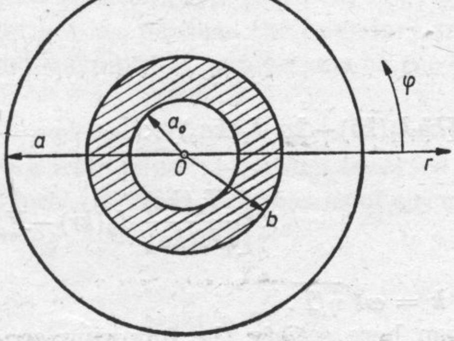

Fig. 1. The vibrating system: a – membrane radius; a_0, b – radii of the annular surface of the membrane on which the inducing force, different from zero, acts; $\{a \leqslant r < \infty,\ 0 \leqslant \varphi \leqslant 2\pi\}$ – the region of the rigid baffle

to the surface of the membrane, with the external radii b and the internal ones a_0 (Fig. 1). In this case, the factor inducing harmonic vibration can be an electric force with the surface density

$$f(r, t) = \begin{cases} 0 & \text{for } 0 < r < a_0 \\ f_0 \exp(i\omega t) & \text{for } a_0 < r < b \\ 0 & \text{for } b < r < a. \end{cases} \tag{1}$$

The vibration equation [2] of the circular membrane under the effect of an axially symmetric inducing force, is the following:

$$T\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \xi(r,t)}{\partial r}\right)-\frac{\partial^2 \xi(r,t)}{\partial t^2} = -f(r,t), \tag{2}$$

where $T = c_M^2\eta$.

The solution of equation (2) for a membrane excited to forced vibration by force (1) has the form

$$\xi_1(r,t) = \frac{\pi}{2}\frac{f_0}{\eta\omega^2}\left\{\frac{N_0(ka)}{J_0(ka)}\left[kbJ_1(kb)-ka_0J_1(ka_0)\right]- -kbN_1(kb)+ka_0N_1(ka_0)\right\}J_0(kr)\exp(i\omega t) \tag{3}$$

for $0 < r < a_0$;

$$\xi_2(r,t) = \frac{\pi}{2}\frac{f_0}{\eta\omega^2}\left\{\left[\frac{N_0(ka)}{J_0(ka)}\left(kbJ_1(kb)-ka_0J_1(ka_0)\right)- -kbN_1(kb)\right]J_0(kr)+ka_0J_1(ka_0)N_0(kr)-\frac{2}{\pi}\right\}\exp(i\omega t) \tag{4}$$

for $a_0 < r < b$; and

$$\xi_3(r,t) = \frac{\pi}{2}\frac{f_0}{\eta\omega^2}\left[kbJ_1(kb)-ka_0J_1(ka_0)\right]\times \times\left[\frac{N_0(ka)}{J_0(ka)}J_0(kr)-N_0(kr)\right]\exp(i\omega t) \tag{5}$$

for $b < r < a$, where $k = \omega\sqrt{\eta/T}$.

The solutions given here satisfy the following conditions:

a) The functions $\xi_1(r,t)$, $\xi_2(r,t)$ and $\xi_3(r,t)$ take finite values in the respective regions: $\{0 \leqslant r \leqslant a_0,\ 0 \leqslant \varphi \leqslant 2\pi\}$; $\{a_0 \leqslant r \leqslant b,\ 0 \leqslant \varphi \leqslant 2\pi\}$ and $\{b \leqslant r \leqslant a,\ 0 \leqslant \varphi \leqslant 2\pi\}$, and, thus, the solution of $\xi_1(r,t)$ also has a finite value for $r = 0$.

b) The boundary condition $\xi_3(r,t) = 0$ for $r = a$.

c) The agreement conditions

$$\xi_1(r,t) = \xi_2(r,t), \qquad \partial\xi_1(r,t)/\partial r = \partial\xi_2(r,t)/\partial r \quad \text{for } r = a_0$$

and

$$\xi_2(r,t) = \xi_3(r,t), \qquad \partial\xi_2(r,t)/\partial r = \partial\xi_3(r,t)/\partial r \quad \text{for } r = b.$$

The normal component of the vibration velocity of the points of the membrane surface, is obtained after taking into account that $v(r, t) = \partial\xi(r, t)/\partial t$, where $\xi(r, t) = \xi_0(r)\exp(i\omega t)$.

In the boundary case, when the membrane is excited to transverse vibration by the force

$$f'(r, t) = \begin{cases} f_0\exp(i\omega t) & \text{for } 0 < r < b; \\ 0 & \text{for } b < r < a, \end{cases} \tag{6}$$

solutions (3), (4) and (5) are replaced, after assuming previously that $a_0 = 0$, by

$$\xi_2'(r, t) = \frac{f_0}{\eta\omega^2}\left\{\frac{\pi kb}{2}\left[\frac{N_0(ka)}{J_0(ka)}J_1(kb) - N_1(kb)\right]J_0(kr) - 1\right\}\exp(i\omega t) \tag{7}$$

for $0 < r < b$;

$$\xi_3'(r, t) = \frac{f_0}{\eta\omega^2}\frac{\pi kb}{2}J_1(kb)\left[\frac{N_0(ka)}{J_0(ka)}J_0(kr) - N_0(kr)\right]\exp(i\omega t) \tag{8}$$

for $b < r < a$; and

$$\xi_1'(0, t) = \xi_2'(0, t) = \frac{f_0}{\eta\omega^2}\left\{\frac{\pi kb}{2}\left[\frac{N_0(ka)}{J_1(ka)}J_1(kb) - N_1(kb)\right] - 1\right\}\exp(i\omega t). \tag{9}$$

In order to show that relation (9) is satisfied, it is necessary to assume in solutions (3) and (4) that $r = a_0$, perform the boundary transition $a_0 \to 0$, apply wronskian (14) and the asymptotic expressions $J_0(x) \approx 1$, $N_1(x) \approx -2/\pi x$ with $x \to 0$.

Solutions (7) and (8), which are specific cases of the more general solutions (3), (4) and (5), are known from paper [1], which analysed the membrane vibration under the effect of force inducing it by means of circular electrodes parallel to the surface of the membrane.

4. Acoustic pressure

The acoustic pressure distribution in the far field of a source vibrating in an ideal rigid and planar baffle can be calculated from the dependence [2]

$$p(R, \theta, \varphi, t) = \frac{i\varrho_0\omega}{2\pi}\frac{\exp(-ik_0R)}{R} \times$$
$$\times \int_{\sigma_0} v(r_0, \varphi_0, t)\exp[ik_0r_0\sin\theta\cos(\varphi - \varphi_0)]\,d\sigma_0 \tag{10}$$

for $1/2\ k_0r_0(r_0/R) \ll 1$, where R, θ, φ are the spherical coordinates of a point of the field; r_0 and ϕ_0 are the polar coordinates of a point of the source; $\sigma_0 = \pi a^2$ is the surface area of the circular membrane.

In the case of a circular membrane excited to axially symmetric vibration (3), (4) and (5), the vibration velocity of the points of its surface does not depend on the angular variable φ_0. Consideration also of the integral property [7].

$$\int_0^{2\pi} \exp[\pm ib\cos(\varphi-\varphi_0)]\,d\varphi_0 = 2\pi J_0(b) \tag{11}$$

leads to the following form of the expression for the acoustic pressure (10):

$$p(R,\theta,t) = i\varrho_0\omega\frac{\exp(-ik_0R)}{R}\left\{\int_0^{a_0}\frac{\partial\xi_1(r_0,t)}{\partial t}J_0(k_0r_0\sin\theta)r_0dr_0 + \right.$$

$$+\int_{a_0}^{b}\frac{\partial\xi_2(r_0,t)}{\partial t}J_0(k_0r_0\sin\theta)r_0dr_0 +$$

$$\left. +\int_{b}^{a}\frac{\partial\xi_3(r_0,t)}{\partial t}J_0(k_0r_0\sin\theta)r_0dr_0\right\}. \tag{12}$$

In calculating the integrals occurring here, the following dependencies can be used [7]:

$$\int wJ_0(hw)Z_0(l\,w)\,dw = \frac{w}{h^2-l^2}\{hJ_1(hw)Z_0(lw)-lJ_0(hw)Z_1(lw)\}, \tag{13}$$

where Z_m is any cylindrical function of the mth order. Consideration also of the wronskian [7]

$$J_1(x)N_0(x)-J_0(x)N_1(x) = \frac{2}{\pi x} \tag{14}$$

leads to

$$p(R,\theta,t) = \frac{\varrho_0 f_0 b^2}{2\eta}\frac{\exp[i(\omega t-k_0R)]}{R}\frac{1}{1-\left(\frac{k_0}{k}\right)^2\sin^2\theta}\times$$

$$\times\left\{\frac{2J_1(k_0b\sin\theta)}{k_0b\sin\theta}-\left(\frac{a_0}{b}\right)^2\frac{2J_1(k_0a_0\sin\theta)}{k_0a_0\sin\theta}+\right.$$

$$\left.+\frac{2\left(\frac{a_0}{b}\right)J_1(ka_0)-2J_1(kb)}{kbJ_0(ka)}J_0(k_0a\sin\theta)\right\}. \tag{15}$$

In the main direction, i.e. for $\theta = 0$,

$$p_0(R, 0, t) = \frac{\varrho_0 f_0 b^2}{2\eta} \frac{\exp[i(\omega t - k_0 R)]}{R} \left[1 - \left(\frac{a_0}{b}\right)^2 + \right.$$

$$\left. + \frac{2\left(\frac{a_0}{b}\right) J_1(ka_0) - 2J_1(kb)}{kbJ_0(ka)}\right]. \tag{16}$$

The directionality coefficient [2]

$$K(\theta) = \frac{|p|}{|p_0|} \tag{17}$$

for $p_0 \neq 0$. When, in turn,

$$1 - \left(\frac{a_0}{b}\right)^2 + \frac{2\left(\frac{a_0}{b}\right) J_1(ka_0) - 2J_1(kb)}{kb\,J_0(ka)} = 0 \tag{18}$$

the phenomenon of antiresonance occurs, and the directionality coefficient must then be defined in another way. It can be achieved by referring the value of the pressure $p(R, \theta, t)$ to that of the pressure $p'(R, 0, t)$ in such a direction θ where it is maximum.

In the case of resonance vibration, $ka = \beta_{0n}$, where β_{0n} is the nth root of the equation $J_0(\beta_{0n}) = 0$. Then, calculation of the limit

$$\lim_{ka \to \beta_{0n}} K(\theta) = \lim_{ka \to \beta_{0n}} \frac{|p(R, \theta, t)|}{|p_0(R, 0, t)|} \tag{19}$$

gives the dependence

$$K_0(\theta) = \frac{|J_0(k_0 a \sin\theta)|}{\left|1 - \left(\frac{k_0 a}{\beta_{0n}}\right)^2 \sin^2\theta\right|}, \tag{20}$$

which is known, e.g. from papers [4], [5] and [6].

When, in turn, $a_0 = 0$, then instead of expressions (15) and (16), the following formulae are obtained for the acoustic pressure:

$$p(R, \theta, t) = \frac{\varrho_0 f_0 b^2}{2\eta} \frac{\exp[i(\omega t - k_0 R)]}{R} \times$$

$$\times \frac{1}{1 - \left(\frac{k_0}{k}\right)^2 \sin^2\theta} \left[\frac{2J_1(k_0 b \sin\theta)}{k_0 b \sin\theta} - \frac{2J_1(kb)}{kbJ_0(ka)} J_0(k_0 a \sin\theta)\right] \tag{21}$$

and

$$p_0(R, 0, t) = \frac{\varrho_0 f_0 b^2}{2\eta} \frac{\exp[i(\omega t - k_0 R)]}{R} \left[1 - \frac{2J_1(kb)}{kbJ_0(ka)}\right]. \qquad (22)$$

In this case, it is also possible to determine the directionality coefficient from formula (17), but on the assumption that $2J_1(kb) \neq kbJ_0(ka)$. When, in turn,

$$2J_1(kb) = kbJ_0(ka) \qquad (22a)$$

the phenomenon of antiresonance occurs. Relation (22a) can also be derived by assuming in dependence (18) that $a_0 = 0$.

5. A numerical example and final remarks

Figs 2, 3 and 4 show curves of the directionality coefficient of the radiation of the circular membrane excited to forced vibration. The value $p(R, \theta, t)$ of the acoustic pressure (21) was referred to the value $p'(R, 0, t)$ of pressure (22) on the main axis, where it was assumed that $ka = 2$, $b = a$, i.e.

$$p'(R, 0, t) = \frac{\varrho_0 f_0 a^2}{2\eta} \frac{\exp[i(\omega t - k_0 R)]}{R} \left[1 - \frac{J_1(2)}{J_0(2)}\right]. \qquad (23)$$

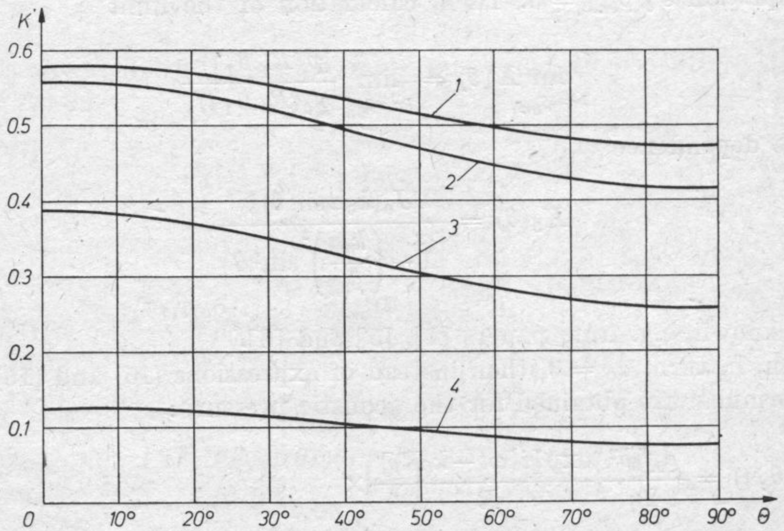

Fig. 2. The directionality coefficient of the radiation of a circular membrane vibrating under the effect of a force with uniform surface distribution, for the different values of b/a. Curve *1* – $b/a = 1$, curve *2* – $b/a = 3/4$, curve *3* – $b/a = 1/2$, curve *4* – $b/a = 1/4$. It was assumed that $ka = 4$, $k_0/k = 1/2$

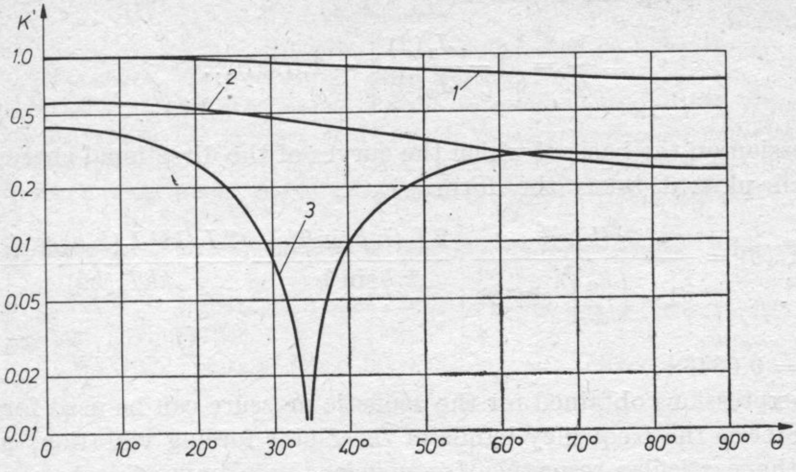

Fig. 3. The directionality coefficient of the radiation of a circular membrane vibrating under the effect of a force with uniform surface distribution, for the different values of the parameter *ka*. Curve *1* – $ka = 2$, curve *2* – $ka = 4$, curve *3* – $ka = 8$. It was assumed that $b = a$, $k_0/k = 1$

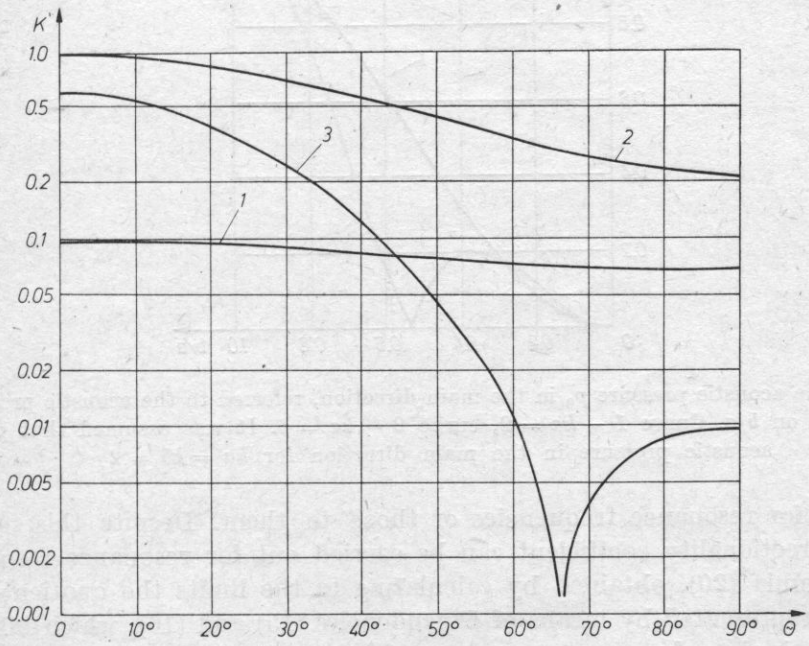

Fig. 4. The directionality coefficient of the radiation of a circular membrane vibrating under the effect of a force with uniform surface distribution, for the different values of the parameter *ka*. Curve *1* – $ka = 1$, curve *2* – $ka = 2$, surve *3* – $ka = 4$. It was assumed that $b = a$, $k_0/k = 2$

After introducing the notation

$$\frac{1}{\gamma} = \left|1 - \frac{J_1(2)}{J_0(2)}\right| = 1.5757 \ldots \tag{24}$$

the expression on the basis of which the curves of the directional characteristics have been plotted takes the form

$$K'(\theta) = \frac{\gamma(b/a)^2}{\left|1 - \left(\frac{k_0}{k}\right)^2 \sin^2\theta\right|} \left|\frac{2J_1(k_0 b \sin\theta)}{k_0 b \sin\theta} - \frac{2J_1(kb)J_0(k_0 a)\sin\theta}{kbJ_0(ka)}\right|, \tag{25}$$

where $\gamma = 0.6346 \ldots$

The expressions obtained for the acoustic pressure can be used for calculating in practice the frequency bands of the factor forcing vibration, contained between the particular resonance frequencies. It is impossible to analyse the

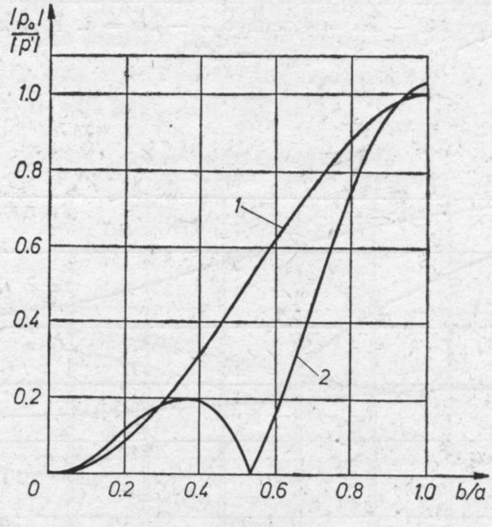

Fig. 5. The acoustic pressure p_0 in the main direction, referred to the acoustic pressure p', depending on b/a. Curve *1* – $ka = 2$, curve *2* – $ka = 6$. It was assumed that p' is the acoustic pressure in the main direction for $ka = kb = 2$

pressure for resonance frequencies or those to them. Despite this, analysis of the directionality coefficient can be carried out for resonance frequencies from formula (20), obtained by calculating in the limits the quotient of the pressure represented by means of dependencies (15) and (16), which take then infinitely high values.

Expressions (15) and (21) derived for the acoustic pressure can also be used in a more profound analysis of the physical properties of the radiating membrane as a vibrating system, e.g. to calculate the acoustic resistance and reactance.

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