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ANALYSIS OF THE PROPAGATION OF A SPHERICAL WAVE WITH FINITE AMPLITUDE IN AN IDEAL GAS BY THE RENORMALISATION METHOD

CZESŁAW ADAM ROSZKOWSKI

Institute of Telecommunication and Acoustics, Wrocław Technical University
(50-317 Wrocław, ul. B. Prusa 53/55)

This paper gives an approximate solution of the hydrodynamic equations in the case of a spherical wave with finite amplitude. Using the perturbation renormalisation method it gives the desired description of the acoustic field of a spherical wave generated by a spherical source which pulsates monochromatically with finite amplitude in an infinite, lossless gaseous medium. The solutions obtained for the acoustic velocity and the acoustic pressure have the form of asymptotic expansion of the first order relative to a small perturbation parameter and are valid both for the near and the far field. The analysis of the acoustic field has for the first time been performed directly using the perturbation renormalisation method for a spherical wave.

Notation

$\hat{u}$ — acoustic velocity
u — normalised acoustic velocity
$\hat{p}$ — pressure
p — normalised acoustic pressure
p_0 — pressure in the unperturbed medium
$\hat{\varrho}$ — density of the medium
ϱ — relative density of the medium
ϱ_0 — density in the unperturbed medium
c_0 — weak-signal sound velocity
$\hat{t}$ — time
t — normalised time
t^* — formation time of the shock wave
$\hat{r}$ — length of the tracing radius of a given point
r — normalised length of the tracing radius of a given point in spherical coordinates
r^* — formation distance of the shock wave

R — static radius of the source
τ — normalised time
η — deformed coordinate
ϕ — potential of the acoustic velocity
Φ — parameter
γ — ratio of specific heats
O — large Landau symbol
δ — deformation parameter
ω — angular frequency
Ω — normalised angular frequency
ε — perturbation parameter
A — amplitude of the pulsating sphere
φ, φ_0, ν — constants

1. Introduction

An analytical solution of the hydrodynamic equations, which are the basis for consideration of such problems as the generation and propagation of acoustic waves with small but finite amplitude, is known only in the case of a plane wave propagating in an acoustically ideal medium. This solution has been given independently by EARNSHOW and RIEMANN (e.g. [11]). The hydrodynamic equations for spherical and cylindrical waves of finite amplitude have been considered in a relatively large number of papers, mainly concerned with the description of the propagation in the far field. Experimental work has also been performed, e.g. on spherical waves propagated in water [17] and air [4]. In general, two theoretical approaches to these problems can be distinguished. Some authors use a method which consists in approximating exact equations and seeking exact solutions (e.g. [1, 3]), others employ approximation methods (e.g. [5, 8, 9, 12, 16]. The solution of exact hydrodynamic equations for spherical and cylindrical waves of finite amplitude in an ideal medium has been given by AUGUSTYNIAK [2], with the assumption, however, that the velocity is of one sign. BLACKSTOCK [3] approximated a nonlinear wave equation which is valid for onedimensional travelling waves: plane, spherical and cylindrical, in a lossless medium to the form of the lossless Burgers equation and subsequently for the far field he reduced this equation to one analogous to the equation for plane waves. LOCKWOOD [12] carried out approximation of the second order of the hydrodynamic equations and subsequently solved these equations using the method of multiple scales [14] for a spherical source in a lossless medium, achieving a parametric description of the profile of the pressure wave valid for the far field. GINSBERG [8, 9] gave a description of the profile of the pressure wave and the acoustic wave generated by a monochromatic cylindrical source, taking into account a moving boundary condition, in the case of two and three-dimensional motion of the source. Using the renormalisation method he obtained asymptotic expansions of the first order of the expressions defining the pressure and the acoustic velocity in the far field and gave a matching procedure with which he achieved a description of the near field based on the description of the far field achieved previously. An extension of this analysis to the case when the motion of the source is a superposition of harmonic excitations was given by NAYFEH and KELLY [16].

2. Formulation of the problem

The equations of motion and the equation of state of the lossless gaseous medium will be given with dimensionless variables defined by the relations

$$r = \frac{\hat{r}}{R}, \quad t = \frac{c_0}{R}\hat{t}, \quad p = \frac{\hat{p}-p_0}{p_0}, \quad u = \frac{\hat{u}}{c_0}, \quad \varrho = \frac{\hat{\varrho}}{\varrho_0}, \quad \Omega = \frac{\omega R}{c_0}, \tag{1}$$

where $\hat{u}$ is the radial component of the acoustic velocity, $\hat{p}$ is the pressure, $\hat{\varrho}$ is the density of the medium, $\hat{r}$ is the distance from the centre of the sphere, p_0 and ϱ_0 are respectively the pressure and density in the unperturbed medium, c_0 is the weak – signal sound velocity, γ is the exponent of the adiabate, ω is the angular frequency and R is the static radius of the spherical source.

The hydrodynamic equations for the spherical wave in dimensionless variables are respectively

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\varrho\gamma}\frac{\partial p}{\partial r} = 0; \tag{2}$$

$$\frac{\partial \varrho}{\partial t} + u\frac{\partial \varrho}{\partial r} + \varrho\frac{\partial u}{\partial r} + \frac{2}{r}\varrho u = 0; \tag{3}$$

$$1 + p = \varrho^{\gamma}. \tag{4}$$

The motion of the lossless gaseous medium, under the assumption of irrotationality of the field, can be described with the dimensionless velocity potential function $\phi(r, t)$ such that $u = \partial\phi/\partial r$. The equations describing the motion of the gas, the equation of the potential and the equation of the pressure p, as derived from equations (2)-(4) [19] are given in the following form

$$\frac{\partial^2\phi}{\partial r^2} - \frac{\partial^2\phi}{\partial t^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} = 2\frac{\partial\phi}{\partial r}\frac{\partial^2\phi}{\partial r\partial t} + \left[\frac{1}{2}(\gamma-1)+1\right]\left(\frac{\partial\phi}{\partial r}\right)^2\frac{\partial^2\phi}{\partial r^2} + $$
$$+ (\gamma-1)\frac{\partial\phi}{\partial t}\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}(\gamma-1)\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial t} + \frac{1}{2}(\gamma-1)\frac{2}{r}\left(\frac{\partial\phi}{\partial r}\right)^3; \tag{5}$$

$$(1+p)^{(\gamma-1)/\gamma} = 1 - (\gamma-1)\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2\right]. \tag{6}$$

In turn, the motion of an oscillating spherical source with finite amplitude which generates a wave in an infinite medium, written in dimensionless variables, is defined by the relation

$$r_k(t) = 1 + \varepsilon\cos(\Omega t + \varphi); \tag{7}$$

where $\varepsilon = |A/R| \ll 1$ is a small perturbation parameter and A is the amplitude of the pulsating sphere.

The desired moving boundary condition is such that at each moment the normal component of the velocity of the medium (in the present case only this component of velocity occurs) is equal to the normal component of the velocity of the surface of the source, for all points of the surface of the source [13].

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=r_k=1+\varepsilon\cos(\Omega t+\varphi)} = \frac{dr_k(t)}{dt}. \tag{8}$$

3. Description of the potential of the acoustic velocity

In the problems which involve parametric perturbations the quantities to be expanded can depend on one or more independent variables, apart from the perturbation parameter. Construction of the asymptotic representation of the function $f(x;\varepsilon)$, where x is a scalar or vector variable, independent from the parameter ε in the terms of the asymptotic sequence $\delta_m(\varepsilon)$, gives [14]

$$f(x;\varepsilon) \sim \sum_{m=0}^{\infty} a_m(x)\,\delta_m(\varepsilon), \quad \varepsilon\to 0, \tag{9}$$

where $a_m(x)$ are terms which depend only on x.

This expansion will be called asymptotic if

$$f(x;\varepsilon) = \sum_{m=0}^{N-1} a_m(x)\,\delta_m(\varepsilon) + R_N(x;\varepsilon); \tag{10}$$

$$R_N(x;\varepsilon) = O[\delta_N(\varepsilon)], \quad \lim_{\delta_N(\varepsilon)\to 0} \{O[\delta_N(\varepsilon)]/\delta_N(\varepsilon)\} = 0 \tag{11}$$

for all the considered values of x.

In the contrary case it is said that the expansion is singular.

For small but finite pulsation amplitude of the source the potential of the acoustic field generated (and also such quantities as u, p, ϱ) is a quantity of low value and can therefore be expanded in a power series with respect to the small parameter ε,

$$\phi(r,t;\varepsilon) = \varepsilon\phi_1(r,t) + \varepsilon^2\phi_2(r,t) + \ldots \tag{12}$$

and similarly

$$u(r, t; \varepsilon) = \varepsilon u_1(r, t) + \varepsilon^2 u_2(r, t) + \ldots; \tag{13}$$

$$p(r, t; \varepsilon) = \varepsilon p_1(r, t) + \varepsilon^2 p_2(r, t) + \ldots. \tag{14}$$

Substitution of the relation of ϕ in the form of expansion (12) into equation (8) with the right side expanded into a Taylor series and comparison of the terms with the same powers of ε give linear equations for $\phi_1(r, t)$ and $\phi_2(r, t)$ and the boundary conditions

order ε:

$$\frac{\partial^2\phi_1}{\partial r^2} - \frac{\partial^2\phi_1}{\partial t^2} + \frac{2}{r}\frac{\partial\phi_1}{\partial r} = 0; \tag{15}$$

$$\left.\frac{\partial\phi_1}{\partial r}\right|_{r=1} = -\Omega\sin(\Omega t + \varphi); \tag{16}$$

order ε^2:

$$\frac{\partial^2\phi_2}{\partial r^2} - \frac{\partial^2\phi_2}{\partial t^2} + \frac{2}{r}\frac{\partial\phi_2}{\partial r} = 2\,\frac{\partial\phi_1}{\partial r}\frac{\partial^2\phi_1}{\partial r\partial t} + (\gamma-1)\frac{\partial\phi_1}{\partial t}\frac{\partial^2\phi_1}{\partial r^2} + \frac{2}{r}(\gamma-1)\frac{\partial\phi_1}{\partial r}\frac{\partial\phi_1}{\partial t}; \tag{17}$$

$$\left.\frac{\partial\phi_2}{\partial r}\right|_{r=1} = -\cos(\Omega t + \varphi)\left.\frac{\partial^2\phi_1}{\partial r^2}\right|_{r=1}. \tag{18}$$

Equations (15) and (17) are linear equations. The first is a linearized equation of the velocity potential for the spherical wave, whereas the second is a linear, heterogeneous equation which describes a nonlinear correction for the potential function.

Solution of these equations, with relevant boundary conditions, gave the sought expansion of the velocity potential according to the powers of the perturbation parameter.

$$\begin{aligned}
\phi(r, t; \varepsilon) = {} & -\varepsilon\Omega(\Omega^2+1)^{-1/2}r^{-1}\cos\{\Omega[t-(r-1)]+\varphi+\varphi_0\} + \\
& +\varepsilon^2[\Omega(\Omega^2+1)^{-1/2}[(\Omega^2-2)\cos\varphi_0 + 2\Omega\sin\varphi_0]r^{-1}\cos^2\{\Omega[t-(r-1)]+\varphi\} + \\
& +\Omega[(\Omega^2+1)(4\Omega^2+1)^{-1/2}\left[\Omega(\Omega^2-1)\cos\varphi_0 + \left(\frac{3}{2}\,\Omega^2+1\right)\sin\varphi_0\right]r^{-1}\sin 2\{\Omega[t- \\
& \qquad -(r-1)]+\varphi+\nu\} + \\
& +\frac{1}{2}\,\Omega^3(\Omega^2+1)^{-1}(1+4\Omega^2)^{-1/2}r^{-1}\sin 2\{\Omega[t-(r-1)]+\varphi+\varphi_0+\nu\} + \\
& -\frac{1}{2}\,\Omega^3(\Omega^2+1)^{-1}r^{-2}\sin 2\{\Omega[t-(r-1)+\varphi+\varphi_0\} +
\end{aligned}$$

$$+\frac{1}{2}\,\Omega^3(\Omega^2+1)^{-1}r^{-1}\sin 2\{\Omega[t-(r-1)]+\varphi+\varphi_0\}+$$

$$+\frac{1}{8}\,\Omega^4(\Omega^2+1)^{-1}(\gamma+1)r^{-1}\ln r\cos 2\{\Omega[t-(r-1)]+\varphi+\varphi_0\}+$$

$$-\frac{1}{8}\,\Omega^4(\Omega^2+1)^{-1}(\gamma+1)r^{-1}\sin 2\{\Omega[t-(r-1)]+\varphi+\varphi_0\}\cos 4\Omega r[Si(4\Omega r)-Si(4\Omega)]+$$

$$+\frac{1}{8}\Omega^4(\Omega^2+1)^{-1}(\gamma+1)r^{-1}\sin 2\{\Omega[t-(r-1)+\varphi+\varphi_0\}\sin 4\Omega r\,[Ci(4\Omega r)-Ci(4\Omega)]-$$

$$-\frac{1}{8}\,\Omega^4(\Omega^2+1)^{-1}(\gamma+1)r^{-1}\cos 2\{\Omega[t-(r-1)]+\varphi+\varphi_0\}\sin 4\Omega r[Si(4\Omega r)-Si(4\Omega)]-$$

$$-\frac{1}{8}\,\Omega^4(\Omega^2+1)^{-1}(\gamma+1)r^{-1}\cos 2\{\Omega[t-(r-1)]+\varphi+\varphi_0\}\cos 4\Omega r\,[Ci(4\Omega r)-$$

$$-Ci(4\Omega)]+\ldots, \tag{19}$$

where

$$\varphi_0=\tan^{-1}\Omega^{-1}, \qquad \nu=-\frac{1}{2}\tan^{-1}2\Omega.$$

The expansion of the dimensionless function of the potential of the acoustic velocity as defined by relation (19) is not asymptotic. The singularity of this expansion results from the presence in the second-order terms of the secular term (the sixth term in the second-order terms of the expansion) which causes the second-order terms of the expansion to take values of the same order or greater than those of the first-order terms with large distances from the source. This term occurs in the solution of equation (17). In turn, the nonlinear effects which result from the moving boundary condition are of the second order of magnitude. Therefore, in order to obtain the asymptotic expansion of the first order in the case of a linearized boundary condition, it is enough to consider the problem in the form

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=1}=-\varepsilon\Omega\sin(\Omega t+\varphi). \tag{20}$$

4. Description of the field of the acoustic pressure

NAYFEH and KLUWICK [15] and GINSBERG [7] have proved that in seeking the correct expressions of such physical quantities as the acoustic velocity or pressure it is necessary to eliminate the secular term from the expressionsy of these quantities and not from the expression of the acoustic velocity potential.

The acoustic velocity, defined as $u = \partial\phi/\partial r$ and derived from the expansion of the velocity potential, is given by the relation

$$u(r, t; \varepsilon) = \varepsilon[\Omega(\Omega^2+1)^{-1/2}r^{-2}\cos\{\Omega[t-(r-1)]+\varphi+\varphi_0\}-\Omega^2(\Omega^2+1)^{-1/2}\times$$
$$\times r^{-1}\sin\{\Omega[t-(r-1)]+\varphi+\varphi_0\}]+$$
$$+\varepsilon^2\frac{1}{4}\Omega^5(\Omega^2+1)^{-1}(\gamma+1)r^{-1}lnr\sin 2\{\Omega[t-(r-1)]+\varphi+\varphi_0\}+NST+\ldots, \quad (21)$$

where $\varphi_0 = \tan^{-1}\Omega^{-1}$ and NST are the nonsecular terms of the expansion.

In order to eliminate the secular term from expansion (21), according to the renormalisation method chosen, new independent variables of time and distance, τ and η, were introduced. GINSBERG [8] and NAYFEH [15] have shown that in the problems which involve travelling waves it is sufficient to transform only one variable. It is convenient to assume the following form of the transformation of r and t,

$$r = \eta+\varepsilon r_1(\eta, \tau)+\ldots, \quad (22)$$
$$t = \tau. \quad (23)$$

After insertion of the expressions of r and t given by relations (22) and (23) into equation (21) and expansion of the right side for small ε, with a definite value of the new coordinate η the velocity $u(\eta, \tau; \varepsilon)$ is given by expression (24):

$$u(\eta, \tau; \varepsilon) = \varepsilon\big[\Omega(\Omega^2+1)^{-1/2}\eta^{-2}\cos\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}-\Omega^2(\Omega^2+1)^{-1/2}\times$$
$$\times\eta^{-1}\sin\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}\big]+$$
$$+\varepsilon^2\big[2\Omega^2(\Omega^2+1)^{-1/2}\eta^{-2}\sin\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}-2\Omega(\Omega^2+1)^{-1/2}\times$$
$$\times\eta^{-3}\cos\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}+$$
$$+\Omega^3(\Omega^2+1)^{-1/2}\eta^{-1}\cos\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}\big]r_1(\eta, \tau)+$$
$$+\varepsilon^2\frac{1}{4}\Omega^5(\Omega^2+1)^{-1}(\gamma+1)\eta^{-1}\ln\eta\sin 2\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}+NST+\ldots \quad (24)$$

It would seem apparently that in order to eliminate the secular term from expansion (24) the function $r_1(\eta, \tau)$ should be chosen (according to the renormalisation method) so that the secular term and the terms containing the function $r_1(\eta, \tau)$ would zero one another. It can readily be shown that in such a case $r_1(\eta, \tau)\to\infty$ for given values of the variable η, which would lead to infinitely great deformation of the profile of the wave. It follows from the analyses carried out in the case of a plane travelling wave [6, 15] and a cylindrical wave [8, 9] that the function of the deformation of the profile is the product of the funciton of distance from the source and of the acoustic velocity $u_1(\eta, \tau)$

$$r_1(\eta, \tau) = h_1(\eta)u_1(\eta, \tau), \quad (25)$$

where $h_1(\eta)$ is a function which depends only on η.

It was assumed that in order to obtain $r_1(\eta, \tau)$ in the same form as occurs in relation (25), it is necessary to take also into consideration, apart from the secular term, other terms in the second-order terms of expansion (24), although they do not cause the singularity

$$r_1(\eta,\tau)[\Omega^3(\Omega^2+1)^{-1/2}\eta^{-1}\cos\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}+2\Omega^2(\Omega^2+1)^{-1/2}\times$$
$$\times\eta^{-2}\sin\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}-2\Omega(\Omega^2+1)^{-1/2}\eta^{-3}\cos\{\Omega[\tau-(\eta-1)+\varphi+\varphi_0)\}]=$$
$$=-\frac{1}{4}\Omega^5(\Omega^2+1)^{-1}(\gamma+1)\eta^{-1}\ln\eta\sin 2\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}+$$
$$+\sum_n f_n(\eta)g_n\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}, \tag{26}$$

where $f_n(\eta)g_n\{\Omega[\tau-(\eta-1)]+\varphi_0+\varphi\}$ are asymptotic terms in the second-order terms of the expansion.

On the basis of relations (24)-(26), the function $h_1(\eta)$,

$$h_1(\eta)=\frac{1}{2}(\gamma+1)\eta\ln\eta, \tag{27}$$

and the following elements of the second term of expansion (24), which should be taken into account together with the secular term,

$$f_1g_1=\frac{1}{2}\Omega^4(\Omega^2+1)^{-1}(\gamma+1)\eta^{-2}\ln\eta\cos^2\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}; \tag{28}$$

$$f_2g_2=-\Omega^4(\Omega^2+1)^{-1}(\gamma+1)\eta^{-2}\ln\eta\sin^2\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}; \tag{29}$$

$$f_3g_3=-\Omega^2(\Omega^2+1)^{-1}(\gamma+1)\eta^{-4}\ln\eta\cos^2\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}; \tag{30}$$

$$f_4g_4=\Omega^3(\Omega^2+1)^{-1}(\gamma+1)\eta^{-3}\ln\eta\sin 2\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}. \tag{31}$$

were obtained.

From equations (22)-(27), the following asymptotic expansion of the first order of the acoustic velocity $u(r,t)$ was obtained in parametric form, valid for both the near and far field,

$$u(\eta,\tau)=\varepsilon\Omega(\Omega^2+1)^{-1/2}\eta^{-2}\cos\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}-$$
$$-\varepsilon\Omega^2(\Omega^2+1)^{-1/2}\eta^{-1}\sin\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}+O[\varepsilon^2\Omega^4(\Omega^2+1)^{-1}]; \tag{32}$$

$$r=\eta+\frac{1}{2}(\gamma+1)\eta\ln\eta u(\eta,\tau)+O[\varepsilon^2\Omega^4(\Omega^2+1)^{-1}]; \tag{33}$$

$$t=\tau. \tag{34}$$

5. Description of the field of the acoustic pressure

It is convenient to represent the dimensionless equation of the acoustic pressure (6) in the following form,

$$1+p=\left\{1-(\gamma-1)\left[\frac{\partial \phi}{\partial t}+\frac{1}{2}\left(\frac{\partial \phi}{\partial r}\right)^2\right]\right\}^{\gamma/\gamma-1}. \quad (35)$$

Expansion of the binomial on the right side of this equation and insertion of the expression of $\phi(r, t; \varepsilon)$, given by relation (12), gave the following form of the equation of the acoustic pressure

$$p(r, t; \varepsilon) = -\gamma\left[\varepsilon\frac{\partial \phi_1}{\partial t}+\varepsilon^2\frac{\partial \phi_2}{\partial t}+\frac{1}{2}\varepsilon^2\left(\frac{\partial \phi_1}{\partial r}\right)^2-\frac{1}{2}\varepsilon^2\left(\frac{\partial \phi_1}{\partial t}\right)^2+\ldots\right]. \quad (36)$$

In a way analogous to the expression defining the acoustic pressure, an asymptotic expansion of the first order of the acoustic pressure was derived in parametric form from equations (36) and (19),

$$p(\eta, \tau) = -\varepsilon\Omega^2(\Omega^2+1)^{-1/2}\gamma\eta^{-1}\sin\{\Omega[\tau-(\eta-1)]+\varphi+\varphi_0\}+O[\varepsilon^2\Omega^4(\Omega^2+1)^{-1}]; \quad (37)$$

$$r = \eta+\frac{1}{2}\frac{\gamma+1}{\gamma}\eta\ln\eta\, p(\eta, \tau)+O[\varepsilon^2\Omega^4(\Omega^2+1)^{-1}]; \quad (38)$$

$$t = \tau. \quad (39)$$

This description is valid for the near and far field up to to the place where the discontinuity (r^*, t^*) occurs in the profile of the wave. From equations (32)-(34) and (37)-(39), it is possible to determine the relation between the original coordinate and the deformed one. As an example, this dependence is shown graphically in Fig. 1.

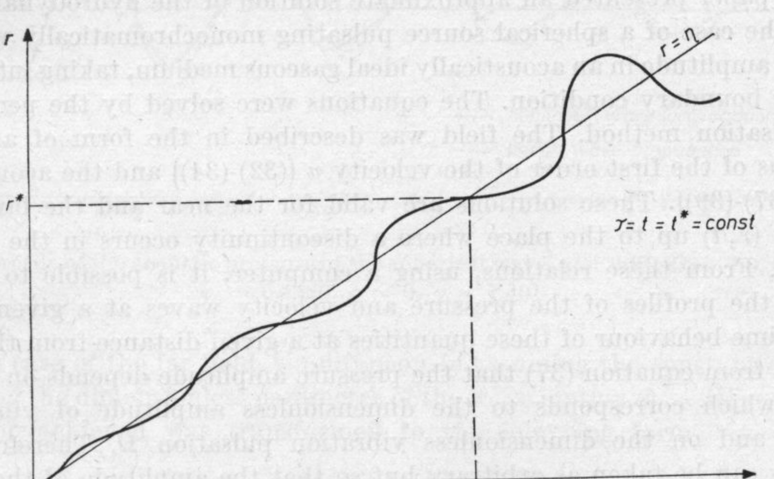

Fig. 1. The dependence of the coordinate r on the deformed η

When one value of the variable r corresponds to one value of the variable η, this then corresponds to one value of the profile of the wave (for $t = \text{const.}$). It follows from Fig. 1 that, from some value of the variable η, the transformation becomes singular, since the same value of the variable r corresponds to several values of the variable η. The profile of the wave becomes a multivalent function of the variable r, which corresponds to the formation of a shock wave. Determination of the shortest distance at which the discontinuity of the profile of the wave occurs can be reduced to the determination of the lowest value of r^* and of the time t^*, according to relation (40).

$$\left.\frac{\partial r}{\partial \eta}\right|_{r^*,t^*} = 0. \tag{40}$$

The use of relations (40) and (38), under the assumption that $\Omega r \gg 1$ (far field) gave

$$r^* = \eta^* \sim \exp \frac{2(\Omega^2+1)^{-1/2}}{\varepsilon(\gamma+1)\Omega^3}; \tag{41}$$

$$\cos\{\Omega[t^* - (r^* - 1)] + \varphi + \varphi_0\} = -1. \tag{42}$$

For another definite observation time $t \neq t^*$ the discontinuity forms at a farther distance from the source. Expressions (41) and (42), as defined for the pressure wave for the far field, are also valid for the velocity wave.

6. Conclusions

This paper presented an approximate solution of the hydrodynamic equations in the case of a spherical source pulsating monochromatically with finite vibration amplitude in an acoustically ideal gaseous medium, taking into account a moving boundary condition. The equations were solved by the perturbation renormalisation method. The field was described in the form of asymptotic expansions of the first order of the velocity u ((32)-(34)) and the acoustic pressure p ((37)-(39)). These solutions are valid for the near and the far field for all points (r, t) up to the place where a discontinuity occurs in the profile of the wave. From these relations, using a computer, it is possible to calculate and plot the profiles of the pressure and velocity waves at a given moment and the time behaviour of these quantities at a given distance from the source. It follows from equation (37) that the pressure amplitude depends on the parameter ε which corresponds to the dimensionless amplitude of vibration of a sphere and on the dimensionless vibration pulsation Ω. Therefore, these quantities can be taken as arbitrary but so that the amplitude of the acoustic pressure ε_1 satisfies near the source the condition $\varepsilon_1 = \varepsilon\Omega^2(\Omega^2+1)^{-1/2} \ll 1$ (similarly in the case of the amplitude of the acoustic velocity). This solution is valid for the amplitudes $\gamma\varepsilon_1 \leqslant 0.1$, which corresponds to a level of the acoustic pressure $\leqslant 174$ dB in air in normal conditions. As an example, Figs. 2 and 3 show profiles of the acoustic velocity and acoustic pressure in air for $\varepsilon = 0.01$ and $\Omega = 7$ and the observation time $t = t^*$.

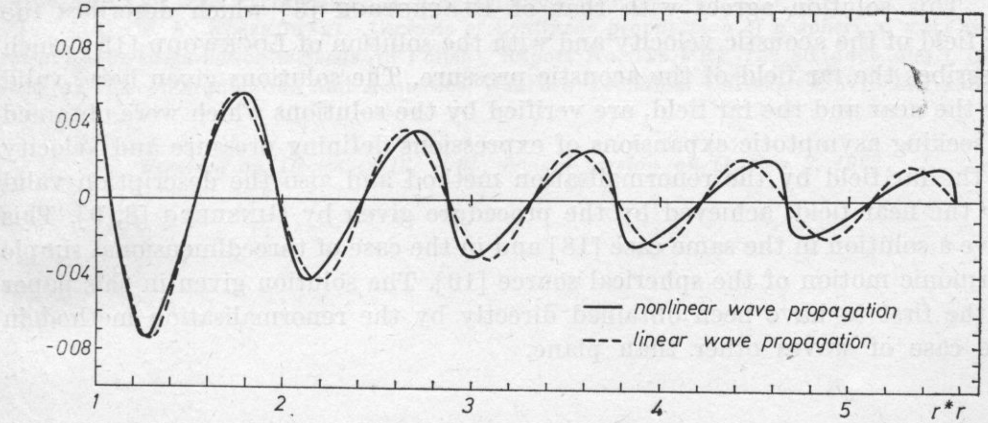

Fig. 2. The profile of the acoustic velocity of the spherical wave, $\varepsilon = 0.01$, $\Omega = 7.0$, $t = 4.996$ $\varphi_0 = 0.1420$, $\varphi = 0$, $\gamma = 1.401$

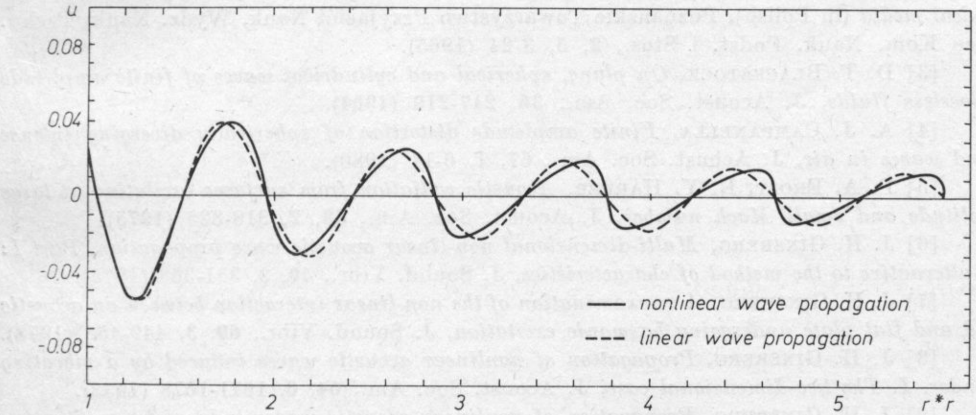

Fig. 3. The profile of the acoustic pressure of the spherical wave, $\varepsilon = 0.01$, $\Omega = 7.0$, $t = 4.996$, $\varphi_0 = 0.1420$, $\varphi = 0$, $\gamma = 1.401$

In the case of the far field, simultaneously assuming the constant $\varphi = -\varphi_0$ and defining the dimensionless parameter of the wave motion $\Phi = \Omega[\tau-(\eta-1)]$, the solution achieved was transformed to the following form

$$u = -\varepsilon\Omega^2(\Omega^2+1)^{-1/2}r^{-1}\sin\Phi; \tag{43}$$

$$p = -\varepsilon\Omega^2(\Omega^2+1)^{-1/2}\gamma r^{-1}\sin\Phi; \tag{44}$$

$$\Phi = \Omega[t-(r-1)]-\sigma\sin\Phi; \tag{45}$$

$$\sigma = \varepsilon\frac{1}{2}\Omega^3(\Omega^2+1)^{-1/2}(\gamma+1)\ln r. \tag{46}$$

This solution agrees with that of BLACKSTOCK [3] which describes the far field of the acoustic velocity and with the solution of LOCKWOOD [12] which describes the far field of the acoustic pressure. The solutions given here, valid for the near and the far field, are verified by the solutions which were obtained in seeking asymptotic expansions of expressions defining pressure and velocity in the far field by the renormalisation method and also the description valid for the near field, achieved by the procedure given by GINSBERG [8, 9]. This gave a solution in the same case [18] and in the case of threedimensional simple harmonic motion of the spherical source [10]. The solution given in this paper is the first to have been obtained directly by the renormalisation method in the case of waves other than plane.

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