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MUTUAL ACOUSTIC IMPEDANCE OF CYLINDRICAL SOURCES FOR A SPECIFIC CASE

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The paper analyzes the imaginary component of the mutual impedence of two pulsating cylindrical rings placed on an infinitely long and stiff circular cylinder. The specific case of the radius of the cylinder being considerably shorter than the wave lengths of the acoustic waves radiated is considered. The imaginary component of the mutual impedence was calculated using the Hilbert transformation, based on the approximate expression for the real component of the mutual impedance given by Robey. Compared to earlier results the formulae have a simple form, and are thus convenient for numerical calculations. These calculations are illustrated graphically.

1. Introduction

In the consideration of the acoustical properties of a system of sources, significant interaction between sources should, in general be included. The theoretical calculation of these interactions consists in the evaluation of the mutual impedance of the sources.

It appears that the number of papers on the mutual acoustical interactions of cylindrical sources is relatively small.

An exact expression for the mutual acoustic impedance of a radiating system of cylindrical sources placed on the surface of an infinitely long and stiff circular cylinder, was given by Robey [10]. He assumed that the sources of the acoustic field consist of pulsating cylindrical rings of finite length. He used a Green's function for the calculation of the impedance. On account of the axial symmetry of the radiating sources he used a Green's function in a cylindrical coordinate system which is independent of the angular variation. For sources on a cylinder whose diameter is short compared to the wave length, he obtained an approximate formula for the mutual resistance.

The acoustic pressure and the mutual radiation impedance of rectangular pistons placed on the surface of a stiff and infinitely long circular cylinder has been investigated by GREENSPON and SHERMAN [2]. The pressure distribution on the baffle and the surface of the sources was obtained using the Neumann boundary condition problem in a cylindrical coordinate system. The knowledge of the pressure distribution was used to determine the mutual impedance of the two sources. They showed that the expression for the mutual impedance of two rectangular pistons becomes in the limiting case the same as the formula for the mutual impedance between two rings, which was given by Robey.

In paper [8] RDZANEK and WYRZYKOWSKI investigated the radiation of a system of slits of finite length placed symmetrically on the surface of a stiff circular cylinder. In order to derive the formulae for the mutual impedance of two slits they used the Green's function method, with dependence on all three cylindrical variables [7].

The problem of the mutual reactance of two pulsating cylindrical rings placed on a common cylindrical baffle was considered by Greenspon [1] and THOMPSON [12]. Referring to Robey's investigations, they calculated the mutual reactance of two sources with the assumption that the sources are placed on the surface of a cylinder whose diameter is shorter than the acoustic wave length. The final approximate formulae obtained for the mutual reactance were expressed by a single integral and an infinite series containing the axial function, which in the case of numerical calculations requires the use of a digital computer.

The use of the Hilbert transformation in the solution of the present problem permitted the final formulae in this paper to be obtained in a form more convenient for the performance of numerical calculations. The approximate expression for the real component of the mutual impedance, given by Robey, was assumed as the starting point. In addition to the trigonometric functions, only integral sines occur in this expression. Using the known real component of the mutual impedance, the imaginary component of the mutual impedance was calculated and expressed in terms of trigonometric functions and integral cosines. The numerical calculations are also illustrated graphically.

Notation

a — the radius of the cylinder
c — the wave propagation velocity
$Ci(x)$ — the integral cosine (21)
i — $\sqrt{-1}$
J_n — cylindrical Bessel function of the nth order
k — wave number
K_n — cylindrical MacDonald function of the nth order
l — half the source length
l_{ms} — the distance between the surface centres of the mth and the sth sources
N_n — cylindrical Neumann function of the nth order
p_{ms} — acoustic pressure from the sth source, acting on the mth source
v — the normal component of the particle velocity

v^x – the complex conjugate of the complex particle velocity v
v_0 – the amplitude of the particle velocity
$Si(x)$ – the integral sine (17)
Z_{ms} – the mutual mechanical impedence between the sth and the mth sources
θ_{ms} – relative acoustic resistance (4)
θ^0_{ms} – relative acoustic resistance of sources vibrating in phase
χ_{ms} – relative acoustic reactance (4)
χ^0_{ms} – relative acoustic reactance of sources vibrating in phase
ζ_{ms} – relative acoustic impedance (3)
δ_m – the initial phase of the particle velocity at the mth source
ϱ – the density of the medium
σ – the surface area

2. Assumptions of the analysis

It is assumed that there is a system of N harmonically vibrating sound sources in a liquid medium on the surface of an infinitely long circular cylinder of radius a, which acts as an ideal stiff baffle. The sound sources are pulsating rings of the cylinder (pistons in the shape of cylindrical rings), each having a length $2l$, and radius a (Fig. 1). The areas of the sources are equal and are

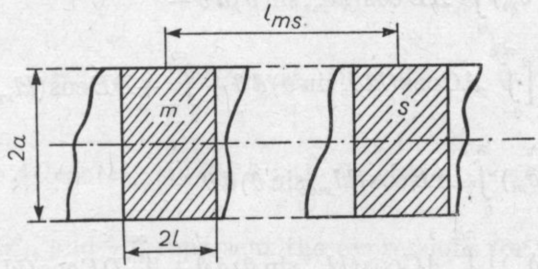

Fig. 1. Rings vibrating on the surface of a circular cylinder

$\sigma_m = \sigma_s = \sigma = 4\pi al$. The distance between the surface centres of the mth and sth sources is l_{ms}. It is assumed that all the points of the source vibrate in phase with a constant amplitude in the radial direction, with a shift of the phase of vibration occuring between individual extended sources. The normal component of the amplitude of the particle velocity (the radial component) of the mth source is equal to

$$v_m = v_0 e^{-i\delta_m} \tag{1}$$

where v_0 is the amplitude of the particle velocity for the initial phase on the mth source, σ_m, equal to zero.

The mechanical mutual impedance of the sth and mth sources is [6]

$$Z_{ms} = \frac{1}{v_{0s} v^*_{0m}} \int_{\delta m} p_{ms} v^*_m d\sigma, \tag{2}$$

where p_{ms} is the sound pressure from the sth source acting on the mth source, while v_m^x is the complex conjugate of the particle velocity v_m. From relation (1), $v_{0s} = v_{0m} = v_{0m}^x = v_0$.

Relating the mechanical impedance to the characteristic resistance of the medium, ϱc, and the area ofthe source, σ, we obtain the relative acoustic impedance

$$\zeta_{ms} = \theta_{ms} + i\,\chi_{ms}, \tag{3}$$

where quantities

$$\Theta_{ms} = \frac{\mathrm{Re}(Z_{ms})}{\varrho c\,\sigma}, \quad \chi_{ms} = \frac{\mathrm{Im}(Z_{ms})}{\varrho c\,\sigma} \tag{4}$$

are the mutual resistance and the mutual reactance, respectively.

3. Solution of the problem

Exact expressions for the mutual acoustic impedance of two sound sources under the given assumptions, can be written in the following way [2, 9, 10]:

$$\Theta_{ms} = \cos(\delta_s - \delta_m)\int\limits_0^{\pi/2} AB\cos(kl_{ms}\sin\vartheta)\,d\vartheta - $$
$$-\sin(\delta_s - d_m)\Big[\int\limits_0^{\pi/2} AC\cos(kl_{ms}\sin\vartheta)\,d\vartheta + \int\limits_0^{\infty} +DE\cos(kl_{ms}\cos h\psi)\,d\psi\Big], \tag{5}$$

$$\chi_{ms} = \sin(\delta_s - \delta_m)\int\limits_0^{\pi/2} AB\cos(kl_{ms}\sin\vartheta)\,d\vartheta +$$
$$+\cos(\delta_s - \delta_m)\Big[\int\limits_0^{\pi/2} AC\cos(kl_{ms}\sin\vartheta)\,d\vartheta + \int\limits_0^{\infty} DE\cos(kl_{ms}\cos h\psi)\,d\psi\Big]; \tag{6}$$

Using the following notation:

$$A = \frac{2}{\pi kl}\,\frac{\sin^2(kl\sin\vartheta)}{\sin^2\vartheta}, \tag{7}$$

$$B = \frac{2}{\pi ka}\,\frac{1}{\cos\vartheta\,[J_1^2(ka\cos\vartheta) + N_1^2(ka\cos\vartheta)]}, \tag{8}$$

$$C = \frac{J_0(ka\cos\vartheta)J_1(ka\cos\vartheta) + N_0(ka\cos\vartheta)N_1(ka\cos\vartheta)}{J_1^2(ka\cos\vartheta) + N_1^2(ka\cos\vartheta)}, \tag{9}$$

$$D = \frac{2}{\pi kl}\,\frac{\sin^2(kl\cos h\psi)}{\cos h^2\psi}, \tag{10}$$

$$E = \frac{K_0(ka\sin h\psi)}{K_1(ka\sin h\psi)}, \tag{11}$$

where $k = 2\pi/\lambda$ is the wave number, J_n represents a Bessel function, K_n — a MacDonald function, and N_n — a Neumann function (of the nth order in each case) [4, 13].

In the calculation of the mutual impedance from (5) and (6), when $ka \ll 1$, it is convenient to use the properties of cylindrical functions which have simple expressions for values of the argument. Thus the factors (8) and (9) can be reduced to expressions containing elementary functions, and subsequently the integrals over the range $(0, \pi/2)$ which occur in formulae (5) and (6), (see [9, 10, 12]), can be calculated. This property cannot be applied to the factor (11) which contains MacDonald functions with the argument $ka \sin h\psi$, since integration over the variable ψ in formulae (5) and (6) is over the infinite limits $(0, \infty)$ (see [1]).

To avoid the difficulty of the calculation of the integral over infinite limits $(0, \infty)$, another method will be used. Formulae (5) and (6) can be written in the following way:

$$\Theta_{ms} = \cos(\delta_s - \delta_m)\Theta_{ms}^0 - \sin(\delta_s - \delta_m)\chi_{ms}^0, \tag{12}$$

$$\chi_{ms} = \sin(\delta_s - \delta_m)\Theta_{ms}^0 + \cos(\delta_s - \delta_m)\chi_{ms}^0, \tag{13}$$

where

$$\Theta_{ms}^0 = \int_0^{\pi/2} AB\cos(kl_{ms}\sin\vartheta)\,d\vartheta, \tag{14}$$

$$\chi_{ms}^0 = \int_0^{\pi/2} AC\cos(kl_{ms}\sin\vartheta)\,d\vartheta + \int_0^{\infty} DE\cos(kl_{ms}\cos h\psi)\,d\psi. \tag{15}$$

The quantities Θ_{ms}^0 and χ_{ms}^0 represent the expressions for the mutual impedance of the cylindrical sources obtained under the assumption that $\delta_s = \delta_m$, and thus that the sources are vibrating in phase.

The real component of the mutual impedance θ_{ms}^0, when $ka \ll 1$ and $l_{ms} \geqslant 2l$, is known and is [10].

$$\Theta_{ms}^0 = \frac{a}{l}\Big\{\frac{1}{4}[k(l_{ms}+2l)]\mathrm{Si}[k(l_{ms}+2l)] + \frac{1}{4}[k(l_{ms}-2l)]\mathrm{Si}[k(l_{ms}-2l)] - \\ - \frac{klms}{2}\mathrm{Si}(kl_{ms}) - \sin^2 kl\cos(kl_{ms})\Big\}, \tag{16}$$

where [3]

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt. \tag{17}$$

The imaginary component of the mutual impedance χ_{ms}^0 is found by a Hilbert transformation [5], [6], using the known real part of the mutual impedance (16).

The Hilbert's transform, in the notation assumed here, is written as

$$\chi^0_{ms}(k) = \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \frac{\Theta^0_{ms}(y)}{y-k}\, dy, \tag{18}$$

where we take the principal value of the integral. Observing that the real part of the mutual impedance Θ^0_{ms} in (16) is an even function of the wave number k, the transformation (18) can be reduced to the form

$$\chi^0_{ms}(k) = \frac{2k}{\pi} \int\limits_0^{\infty} \frac{\Theta^0_{ms}(y)}{y^2-k^2}\, dy. \tag{19}$$

To calculate the integral in formula (19), we refer to reference [11]; we have

$$\int\limits_0^{\infty} \frac{y\,\mathrm{Si}(ay)}{y^2-k^2}\, dy = \frac{\pi}{2}\,\mathrm{Ci}(ka) \tag{20}$$

for $a > 0$ and $k > 0$, where [3]

$$\mathrm{Ci}(x) = \int\limits_{\infty}^{x} \frac{\cos t}{t}\, dt. \tag{21}$$

To show that

$$\int\limits_0^{\infty} \frac{\sin^2(ay)\cos(by)}{k^2-y^2}\, dy = \frac{\pi}{2k} \sin^2(ka)\sin(kb), \tag{22}$$

the equality

$$\sin^2(ay)\cos(by) = \frac{1}{2}\cos(by) - \frac{1}{4}\cos(b+2a)y - \frac{1}{4}\cos(b-2a)y \tag{23}$$

and [11] formula

$$\int\limits_0^{\infty} \frac{\cos(by)}{k^2-y^2}\, dy = \frac{\pi}{2k}\sin(kb), \tag{24}$$

for $b > 0$ and $k > 0$, should be used. Inserting (20), (22), (23) and (24) into (19) we obtain

$$\chi^0_{ms} = \frac{a}{l}\left\{\frac{1}{4}[k(l_{ms}+2l)]\,\mathrm{Ci}[k(l_{ms}+2l)] + \frac{1}{4}[k(l_{ms}-2l)]\,\mathrm{Ci}[k(l_{ms}-2l)] - \right.$$

$$\left. -\frac{1}{2}kl_{ms}\,\mathrm{Ci}(kl_{ms}) + \sin^2(kl)\sin(kl_{ms})\right\} \tag{25}$$

for $kl_{ms} > 0$, $l_{ms} > 2l$, which is the expression for the imaginary part of the mutual impedance of the sources for small values of the parameter $ka\,(ka \ll 1)$.

If $l_{ms} = 2l$, i.e. when the sources are contiguous (see Fig. 1), we obtain

$$\Theta^0_{ms} = \frac{a}{l}\,[kl\,\mathrm{Si}(4kl) - kl\,\mathrm{Si}(2kl) - \sin^2(kl)\cos(2kl)] \tag{26}$$

and

$$\chi^0_{ms} = \frac{a}{l}\,[kl\,\mathrm{Ci}(4kl) - kl\,\mathrm{Ci}(2kl) + \sin^2(kl)\sin(2kl)]. \tag{27}$$

4. Conclusion

The calculation of the mutual acoustic reactance of two pulsating cylindrical rings on a stiff circular cylinder, performed using a Hilbert transform gives a formula which is simple in form. The formula permits the analysis of the acoustic interactions between cylindrical sources to be performed for small values of the parameter ka.

From formulae (16) and (25) the mutual impedance of cylindrical rings vibrating in phase can be calculated. The knowledge of the impedance permits the mutual impedance to be determined in the more general case when the sources do not vibrate in phase. Formulae (12) and (13) can be used for this purpose. The formulae for the mutual impedance take a particularly simple form when the sources are contiguous.

The dependencies of the mutual acoustic impedance on kl and kl_{ms}, as calculated from the formulae given in the present paper, are plotted in Figs. 2 and 3.

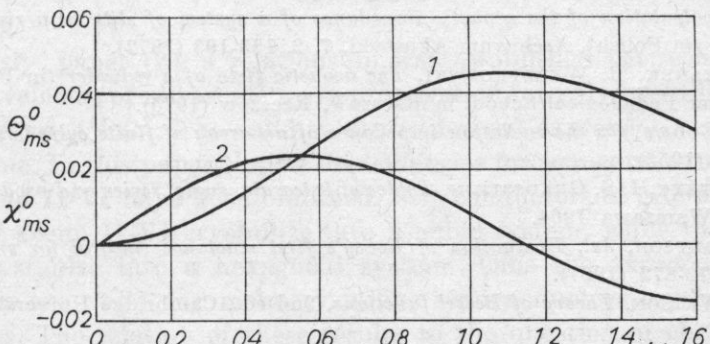

Fig. 2. The plot of the mutual impedance of two contiguous rings ($l_{ms} = 2l$), against the parameter $kl = 2\pi l/\lambda$

curve 1 – resistance, curve 2 – reactance. It is assumed that $a/l = 0.1$

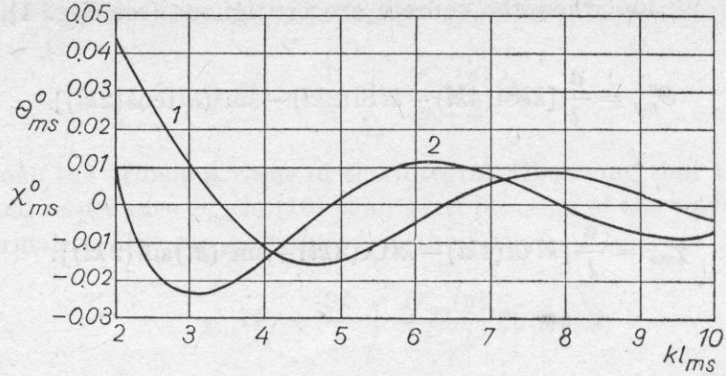

Fig. 3. The plot of the mutual impedance against the parameter $kl_{ms} = (2\pi/\lambda)l_{ms}$
curve *1* – resistance, curve *2* – reactance. It is assumed that $kl = 1$, $a/l = 0.1$

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