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INFLUENCE OF RIBS ON THE ACOUSTIC BEHAVIOUR OF PIANO RESONANT PLATES

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The eigenfrequencies and vibrational modes of a simply supported stiffened rectangular plate are determined by means of perturbation method. Additional eigenmodes are excited by the presence of ribs and the eigenfrequencies are shifted. A qualitative evaluation of the influence of these additionally excited modes on the sound radiated shows that under certain conditions the ribs can improve the sound radiation of a resonant plate.

Glossary of symbols

E – modulus of elasticity
h – plate thickness
σ – Poisson's ratio
Δ^2 – biharmonic operator
ξ – vibrational displacement of plate
ϱ – density of plate material
t – time
ω – angular frequency
$\vec{n}$ – unit vector in a normal direction
x, y – Cartesian coordinates
a, b – plate dimensions in the x- and y- directions respectively
ψ_{mn} – eigenfunctions of the biharmonic operator
ω_{mn} – eigenfrequencies of the non-stiffened plate
i, k, m, n, r, s – integers
ξ_{ik} – eigenfunctions of the stiffened plate
V, L – perturbation quantities
$Sch(x,y)$ – switch function
d – thickness of the ribs
J – second order moment of the rib area
S_R – cross-sectional area of the ribs
a_{mnik} – coefficient
$\omega_{ik}^{(N)}$ – N-th approximation of the eigenfrequency of the stiffened plate vibrating in the (i, k) mode

$\omega_R(k)$ – eigenfrequency of the ribs vibrating in the (k) mode
h_R – rib height
k_S – wave number of the acoustic field
j – $j^2 = -1$
p – sound pressure
u'_{ik} – velocity of the stiffened plate vibration, (i, k) mode
u_{mn} – velocity of the unstiffened plate vibration, (m, n) mode
p'_{ik} – sound pressure generated by the stiffened plate, vibrating in the (i, k) mode
p_{mn} – sound pressure generated by the unstiffened plate, vibrating in the (m, n) mode
r, ϑ, φ – polar coordinates
ϱ_L – density of air
c – sound velocity
k_B – plate bending wave number
P – acoustic power radiated
S – radiation efficiency

1. Introduction

A large number of methods for calculating the vibrational modes and eigenfrequencies of stiffened plates have been published previously. Two papers which are of special interest for the present purpose will be mentioned here: KIRK [1] determined the natural frequencies of the first symmetric and the first antisymmetric modes of a simply supported rectangular plate which is reinforced by a single integral stiffener placed along one of its centre lines; while KOVINSKAJA and NIKOFOROV [2] investigated the field of flexural waves on an infinite point excited plate with two or three ribs.

There are also a large number of papers dealing with the interaction of flexural waves on a stiffened plate with the acoustic field in the surrounding room. MAIDANIK [3] used a statistical method for estimating the response of ribbed panels to acoustic excitation. ROMANOV [4] and [5] calculated the sound radiation from an infinite stiffened plate which is excited by a stochastic force between the ribs, and EVSEEV [6] discussed the sound radiation from an infinite plate excited by a harmonic force. All these papers show the immense mathematical difficulties which are connected with the theoretical treatment of the vibrations of a stiffened plate, and the sound field generated by these vibrations.

For this reason, simplifications have to be made, and these are determined by the intended practical application of the results. In order to investigate the acoustic behaviour of resonant plates in musical instruments, the eigenmodes and eigenfrequencies of a rectangular simply supported stiffened plate will be determined. The ribs are assumed to be parallel to one boundary of the plate. Both assumptions: the simple boundary conditions and a simple arrangement of the ribs, are necessary to make the problem mathematically manageable, without lengthy digital computation.

2. Eigenmodes and eigenfrequencies of a stiffened plate

Flexural waves on a thin plate may be described by the differential equation

$$\frac{Eh^3}{12(1-\sigma^2)}\,\Delta^2\xi + \varrho h\frac{\partial^2\xi}{\partial t^2} = 0.$$

This equation is derived for instance in [7] and [8]. We confine the discussion to sinusoidal time dependence. Thus differentiation with respect to time may be replaced by $+j\omega$. This leads to

$$\frac{Eh^3}{12(1-\sigma^2)}\,\Delta^2\xi - \omega^2\varrho h\xi = 0.$$

In order to determine the eigenmodes of a finite plate this differential equation has to be solved with consideration of the boundary conditions. In the case of a simply supported plate, the displacement and bending moment are zero at the boundaries:

$$\xi = 0 \quad\text{and}\quad \frac{\partial^2\xi}{\partial\vec{n}^2} = 0.$$

For a rectangular plate with boundaries at $x = 0$ and $x = \mathrm{a}$ and $y = 0$ and $y = \mathrm{b}$ we can write these equations as

$$\frac{Eh^3}{12(1-\sigma^2)}\left[\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}\right]\xi - \omega^2\varrho h\xi = 0,$$

$\xi = 0$ if $x = 0$, or $x = a$, or $y = 0$, or $y = b$; $\partial^2\xi/\partial x^2 = 0$ if $y = 0$, or $y = b$; $\partial^2\xi/\partial y^2 = 0$ if $x = 0$, or $y = a$.

The solution of this eigenvalue equation is

$$\psi_{mn} = \frac{2}{\sqrt{ab}}\sin\frac{m\pi}{b}x\sin\frac{n\pi}{b}y$$

$$(0 \leqslant x \leqslant a;\ 0 \leqslant y \leqslant b;\ m, n = 1, 2, 3, \ldots) \tag{2}$$

with eigenvalues

$$\omega_{mn} = \sqrt{\frac{Eh^2}{12\varrho(1-\sigma^2)}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]. \tag{3}$$

The ribs influence the plate vibrations by virtue of their stiffness and inertial mass. Since, however, these are assumed to be small compared with the stiffness and inertia of the plate itself, the use of a perturbation method is justified. The differential equation for a stiffened plate may then be written as

$$\left[\frac{Eh^3}{12(1-\sigma^2)}\,\Delta^2 + V - (1+L)\,\varrho h\omega_{ik}\right]\xi_{ik} = 0. \tag{4}$$

$V\xi_{ik}$ is the stiffness force and $L\varrho h\omega_{ik}\xi_{ik}$ — the inertial mass force of the ribs which act on the plate. For ribs parallel to the y-axis we have

$$V = \frac{EJ}{d}\,\mathrm{Sch}(x)\,\frac{\partial^4}{\partial y^4} \quad \text{and} \quad L = \frac{S_R}{\varrho h}\,\mathrm{Sch}(x).$$

$\mathrm{Sch}(x)$ is a switch function with a values of 1 at the places where there are ribs and 0 elsewhere.

The eigenfunctions ψ_{mn} of the plate differential equation (1) form a system of normalized orthogonal functions. It is therefore possible to find the solution of equation (4) in terms of a series:

$$\xi_{ik} = \sum_{m,n} \psi_{mn} a_{mnik}. \tag{5}$$

According to the assumptions of perturbation method all the terms except that with ψ_{ik} are small. This is identical with the statement that a stiffened plate vibrates essentially like the corresponding unstiffened plate, but with some small amplitude vibration of additionally excited eigenmodes, i.e. for every approximation N, $a_{ikik}^{(N)} = 1$. The other coefficients a_{mnik} are calculated by means of an iteration method which is performed by inserting the series (5) into the differential equation (4) of the stiffened plate:

$$\omega_{ik}^{(N)} = \frac{\omega_{ik}^{(0)2} + \sum_{rs} V_{ikrs} a_{rsik}^{(N-1)}}{1 + \sum_{rs} L_{ikrs} a_{rsik}^{(N-1)}}, \tag{6}$$

$$a_{mnik}^{(N)} = \frac{\sum_{rs} V_{mnrs} a_{rsik}^{(N-1)} - \omega_{ik}^{(N)2} \sum_{rs} L_{mnrs} a_{rsik}^{(N-1)(N-1)}}{\omega_{mn}^{(0)2} - \omega_{ik}^{(N)2}}, \tag{7}$$

with

$$V_{mnrs} = \int_0^a \int_0^b \psi_{mn} V \psi_{rs}\, dx\, dy, \qquad L_{mnrs} = \int_0^a \int_0^b \psi_{mn} L \psi_{rs}\, dx\, dy.$$

The method is started in the zeroth approximation with

$$\omega_{ik}^{(0)} = \omega_{ik},$$

$$a_{mnik}^{(0)} = \begin{cases} 1 & \text{for } m = i \text{ and } n = k, \\ 0 & \text{otherwise}. \end{cases}$$

We note that in the zeroth approximation, the eigenfrequencies and vibrational modes of a stiffened and an unstiffened plate are the same. Insertion of the zeroth approximation into equation (6) yields the eigenfrequencies of the stiffened plate in the first approximation, and this inserted in turn into equation (7) yields the coefficients a_{mnik} in the first approximation. Using the first approximation and equations (6) and (7) we may obtain the second one and so on. If some of the eigenmodes of the differential equation (1) of the plate are degenerate, i.e. they have the same eigenfrequency, some of the coefficients a_{mnik} become infinite, and the series (5) diverges. The problem may be made manageable by using linear combinations of the degenerate modes in such a manner that the corresponding V_{mnik} and L_{mnik} become zero and the diverging terms are eliminated.

Let us assume that the ribs are situated at x_ν and their thickness is small compared with the flexural wave length, so that we have

$$V_{mnrs} = \frac{2EJ}{\varrho ah}\left(\frac{s\pi}{b}\right)^4 \sum_\nu \sin\frac{m\pi}{a} x_\nu \sin\frac{r\pi}{a} x_\nu \delta_{ns},$$

$$L_{mnrs} = \frac{2S_R}{ah} \sum_\nu \sin\frac{m\pi}{a} x_\nu \sin\frac{\nu\pi}{a} x_\nu \delta_{ns},$$

where

$$\delta_{ns} = \begin{cases} 1 & \text{for } n = s \\ 0 & \text{otherwise.} \end{cases}$$

The first approximation is then written as

$$\omega_{ik}^{(1)2} = \frac{\omega_{ik}^{(0)2} + \dfrac{2EJ}{ah}\left(\dfrac{k\pi}{b}\right)^4 \sum_\nu \left(\sin\dfrac{i\pi}{a} x_\nu\right)^2}{1 + \dfrac{2S_R}{ah} \sum_\nu \left(\sin\dfrac{i\pi}{a} x_\nu\right)^2},$$

$$a_{mnik}^{(1)} = \frac{\dfrac{2S_R}{ah}\left[\dfrac{EJ}{\varrho S_R}\left(\dfrac{k\pi}{b}\right)^4 - \omega_{ik}^{(1)2}\right] \sum_\nu \sin\dfrac{m\pi}{a} x_\nu \sin\dfrac{i\pi}{a} x_\nu \delta_{ns}}{\omega_{mn}^2 - \omega_{ik}^{(1)2}}.$$

Using the eigenfrequency equation of the ribs,

$$\omega_R^2(k) = \frac{EJ}{\varrho S_R}\left(\frac{k\pi}{b}\right)^4,$$

the first approximation is written as

$$\omega_{ik}^{(1)2} - \omega_{ik}^{(0)2} = \frac{2\dfrac{S_R}{ah} \sum_\nu \left(\sin\dfrac{i\pi}{a} x_\nu\right)^2}{1 + \dfrac{S_R}{ah} \sum_\nu \left(\sin\dfrac{i\pi}{a} x_\nu\right)^2} \left(\omega_R^2(k) - \omega_{ik}^{(0)2}\right),$$

$$a_{mnik}^{(1)} = \frac{2\dfrac{S_R}{dh} \sum_\nu \sin\dfrac{m\pi}{d} x_\nu \sin\dfrac{i\pi}{a} x_\nu}{\omega_{mn}^{(0)2} - \omega_{ik}^{(1)2}} \left(\omega_R^2(k) - \omega_{ik}^{(0)2}\right).$$

From the equations it can be seen that eigenfrequencies are shifted by the ribs and additional eigenmodes are excited. The vibration amplitude of these additionally excited modes is much smaller than that of the original mode (in which the plate would vibrate if it had no ribs).

If the eigenfrequencies of the ribs for a given y-component $k\pi/b$ of the flexural wave number are smaller than the corresponding eigenfrequencies ω_{ik} of the unribbed plate, the eigenfrequencies $\omega_{ik}^{(N)}$ of the ribbed plates are smaller than those of the unribbed one, and the interaction of ribs and plate is mass controlled. For higher rib eigenfrequencies, the eigenfrequencies of the ribbed plate are higher than those of the unribbed one, and the interaction is stiffness controlled. If an eigenfrequency of the plate is equal to the corresponding eigen-

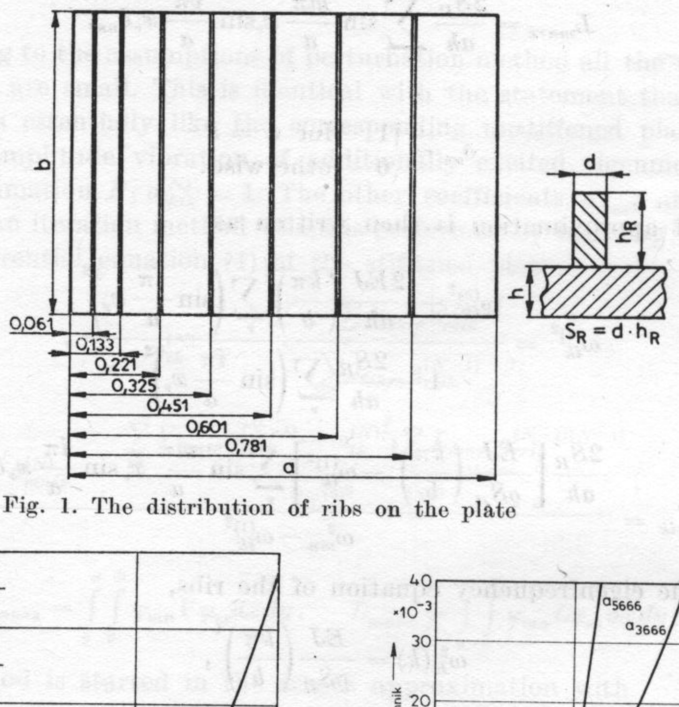

Fig. 1. The distribution of ribs on the plate

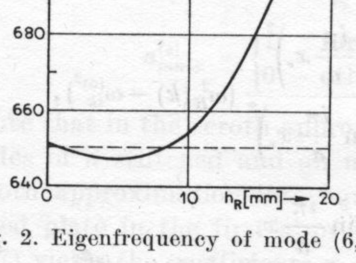

Fig. 2. Eigenfrequency of mode (6,6)

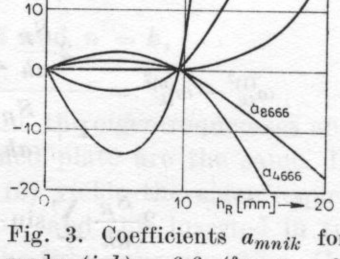

Fig. 3. Coefficients a_{mnik} for mode $(i, k) = 6{,}6$ (for $n \neq 6$, $a_{mnik} = 0$)

frequency of the ribs, the frequency shift (and the amplitudes of the additionally excited modes) become zero.

The calculation of the coefficients a_{mnik} shows that if the ribs are equidistant only a few modes are additionally excited, and most of the a_{mnik} become zero. In the case of nonequidistant ribs a large number of modes is excited.

The eigenfrequency $\omega_{ik}^{(1)}$ and the coefficients $a_{mnik}^{(1)}$ of the plate shown in Fig. 1 are given in Figs. 2 and 3 for the (6,6) mode for different rib heights. It can be seen that for $h_R < 9$ mm the ribs act like an additional mass. The eigenfrequency is lower than that of the corresponding unribbed plate, the coefficients $a_{m6\,66}$ are positive for even m, and negative for odd m. For $h_R > 9$ mm the ribs act like an additional stiffness. The eigenfrequency is larger than that of the corresponding unribbed plate, and the coefficients $a_{m6\,66}$ are positive for odd m and negative for even m.

3. Sound radiation

The acoustic wave radiated by a baffled panel can be found from Rayleigh's integral. The acoustic pressure in the farfield produced by a harmonically vibrating plate can be obtained from the integral

$$P'_{ik}(r, \vartheta, \varphi) = -jks\varrho_L c \frac{e^{jksr}}{2\pi r} \int_0^b \int_0^a u'_{ik}(x, y) \exp\left[-j\left(\frac{\alpha x}{a}\right) - j\left(\frac{\beta y}{b}\right)\right] dx\,dy, \quad (8)$$

where

$$\alpha = k_S a \sin\vartheta \cos\varphi,$$

$$\beta = k_S b \sin\vartheta \sin\varphi,$$

r, ϑ and φ are the polar coordinates of the field point and u'_{ik} is the surface velocity distribution which, in our case, may be written in terms of the series

$$u'_{ik} = j\omega \sum_{m,n} \varphi_{mn} a_{mnik} = \sum u_{mn} a_{mnik}.$$

The sum represents the vibrational modes of the plate from which the velocity distribution is obtained by differentiation with respect to time (in case of harmonic vibration this means multiplication by $j\omega$). Thus the sound pressure may be expressed in terms of a series

$$p'_{ik} = \sum_{m,n} p_{mn} a_{mnik},$$

where p_{mn} is the sound pressure generated by the velocity distribution u_{mn}, which is the velocity distribution of a harmonically vibrating unstiffened plate in the (m, n) mode. The sound pressure field of a stiffened plate vibrating harmonically in the (i, k) mode is the superposition of the sound pressure fields of the corresponding unstiffened plate vibrating in several modes (m, n) with amplitudes a_{mnik} at the same frequency ω.

The radiated power may be calculated from the formula

$$P_{ik}=\int\limits_0^{2\pi}\int\limits_0^{\pi/2}\frac{|p_{ik}|^2}{\varrho_L c}\,r^2\sin\vartheta d\vartheta d\varphi,$$

which leads, for the stiffened plate, to

$$P_{ik}=\int\limits_0^{2\pi}\int\limits_0^{\pi/2}\frac{|p_{ik}|^2}{\varrho_L c}\,r^2\sin\vartheta d\vartheta d\varphi+$$

$$+\int\limits_0^{2\pi}\int\limits_0^{\pi/2}\frac{\sum\limits_{r,s\neq(m,n)}p_{mn}p^*_{r,s}a_{mnik}a_{rsik}}{\varrho_L c}\,r^2\sin\vartheta d\vartheta d\varphi+$$

$$+\int\limits_0^{2\pi}\int\limits_0^{\pi/2}\frac{\sum\limits_{m,n\neq(i,k)}|p_{m,n}|^2a^2_{mnik}}{\varrho_L c}\,r^2\sin\vartheta d\vartheta d\varphi, \tag{9}$$

where p^*_{rs} is the complex conjugate of p_{rs}. The first term represents the power which would be radiated by the corresponding unstiffened plate vibrating in the (i, k) mode, and the third term represents the power radiated by the additionally excited modes. The second term results from the interaction of the different modes.

The influence of the ribs on the sound radiated by a stiffened plate may be understood by investigating the second and third terms of [9]. For this purpose the results obtained by WALLACE [9], who analyzed the radiation resistance of a rectangular panel, are very useful. Wallace studied the energy radiated by a mode of a harmonically vibrating simply supported panel, in the farfield.

For high frequencies, if $k_S/k_B \gg 1$, the radiation efficiency S is equals to one. For low frequencies, if $k_S/k_B \ll 1$, when m and n are both odd integers, the radiation efficiency is proportional to the square of the ratio of the corresponding wave numbers, $S \sim (k_S/k_B)^2$, when m is odd and n is even or, vice versa, $S \sim (k_S/k_B)^4$, and, when m and n are both even integers, $S \sim (k_S/k_B)^6$.

Since $a_{mnik} \ll 1$ for $n \neq i$ and $n \neq k$ the ribs can have a substantial influence only when the mode in which the unstiffened plate would vibrate radiates comparatively little energy and the additionally excited modes have a relatively high radiation efficiency. For example, for modes with both indices even, the first term of equation (9) is small, and the second has a dominating influence, for $k_S/k_B \ll 1$, if modes with one or both indices odd are excited. If, for odd m,

$$a_{mnik}a_{ikik}=a_{mnik}>0,$$

that is if the plate ribs interaction is stiffness controlled ($h_R > 9$ mm in our example), the ribs enhance the radiation of sound in modes which originally had a small radiation efficiency.

If, for odd m,

$$a_{mnik} a_{ikik} = a_{mnik} < 0,$$

i.e. if the plate ribs interaction is mass controlled ($h_R < 9$ mm in our example), the ribs diminish the radiation of sound in modes with a small radiation efficiency.

4. Conclusions

Ribs cause an eigenfrequency shift. Ribs of small height diminish the plate eigenfrequencies, while ribs of large height increase the eigenfrequencies. In the first case the plate ribs interaction is mass controlled, and in the second one stiffness controlled. In addition, the ribs modify the sound radiation of the plate, in particular for those modes which originally have a low radiation efficiency (modes with both indices even and to some extent modes with one index even and one index odd in the low-frequency region, $k_S/k_B \ll 1$). Ribs of small height diminish sound radiation, ribs of large height enlarge the sound radiation in these modes. However, this effect is significant only in the case of nonequidistant ribs. Thus it is possible, in principle, to equalize the frequency response of the plate by adjusting the heights and spacings of the ribs.

References

[1] C. L. Kirk, *Natural frequencies of stiffened rectangular plates*, J. Sound Vib., **13**, 4, 375–388 (1970).

[2] S. J. Kovinskaja, A. S. Nikiforov, *Pola izgubnych voln' v beskonečnych plastinach, podkreplennych rebrami žëstkosti pri točečnom vozbuždenii konstrukcii*, Ak. Žur., **XIX**, 1, 47–52 (1973).

[3] G. Maidanik, *Response of ribbed panels to reverberant acoustic fields*, JASA, **34**, 6, 809–826 (1962).

[4] V. N. Romanov, *Izlučenie zvuka beskonečnoj plastinoj pri naličii na nej reber žestkosti*, Ak. Žur., **XVII**, 1, 116–121 (1971).

[5] V. N. Romanov, *K voprosu ob izlučcenii zvuka beskonečnok plastinoj s rebrami zěstkosti*, Ak. Žur., **XVIII**, 4, 602–607 (1972).

[6] V. N. Evseev, *Izlučěnie zvuka beskonecnoj plastinoj s periodičeskimi neodnorodnostjami*, Ak. Žur., **XIX**, 3, 345–351 (1973).

[7] L. D. Landau, E. M. Lifschitz, *Lehrbuch der theoretischen Physik*, Bd. 7, Elastizitätstheorie, Berlin 1970.

[8] V. S. Gontkevič, *Sobstvennye kolebanija plastinok i oboloček*, Kiev 1968.

[9] C. E. Wallace, *Radiation resistance of a rectangular panel*, JASA, **51**, 312, 946–952 (1972).

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