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THE EFFECT OF INPUT CROSS-SPECTRA ON THE ESTIMATION OF FREQUENCY RESPONSE IN CERTAIN MULTIPLE-INPUT SYSTEMS*

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The estimation of frequency response of multiple input systems is discussed from the standpoint of systems identification, with application to problems in noise control. The systems considered are assumed to have inputs, either random or deterministic, that are identical in form but shifted in time. Such inputs are found typically as force and pressure excitations in engines, pumps and compressors. It is shown that the input cross-spectra can be neglected for inputs of this type, providing proper frequency smoothing is used. If the time-shift between inputs is not equal or if the analysis bandwidth cannot be chosen arbitrarily, biased estimates of the frequency responses will result when the input cross-spectra are neglected. Expressions for this bias error are developed and several numerical examples are presented showing the effect of analysis bandwidth and timeshift on the bias error. This technique was applied to the problem of estimating the structural-acoustical frequency response of a diesel engine. By neglecting the cross-spectra between the combustion pressures the frequency responses were computed on-line with a small digital processor. As a result experimental and computer time were greatly reduced.

List of symbols

B_e – bandwidth [Hz]
f – frequency [Hz]
f_i – natural frequency of the i^{th} input
H_i – frequency response to the i^{th} input
k – frequency index of unsmoothed spectrum
m – frequency index of smoothed spectrum
K – number of frequency points smoothed per band
N – number of inputs
$x_i(t)$ – i^{th} input to system
$y'(t)$ – coherent output
$y(t)$ – total output
$z(t)$ – incoherent uncorrelated output
S_{ii} – auto-spectrum of the i^{th} input
S_{ij} – cross-spectrum between the i^{th} and j^{th} inputs
S_{iy} – cross-spectrum between the i^{th} input and total output

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$S_{iy'}$ – cross-spectrum between the i^{th} input and coherent output
S_{iz} – cross-spectrum between the i^{th} input and incoherent output
S_{zz} – auto-spectrum of incoherent output
S_{yy} – auto-spectrum of total output
$S_{y'y'}$ – auto-spectrum of coherent output
T – record length
Z_i – gain of the i^{th} system
ξ – damping ratio
τ_i – time delay between inputs i and 1
τ_{ij} – time delay between inputs i and j
φ_{ij} – phase angle between inputs i and j
$\hat{\eta}$ – estimate of any parameter η
$E\{\hat{\eta}\}$ – expected value of $\hat{\eta}$
$b\{\hat{\eta}\}$ – bias error of $\hat{\eta}$

Introduction

In modeling dynamic systems, one often requires information to complete the model that can only be accurately known from experiment. The process of determining this information is termed "parameter identification" since the needed information is in the form of one or more unknown parameters in the mathematical model [1]. For example, a second-order model has two unknown parameters: the system natural frequency and damping.

Traditionally, controlled laboratory experiments have been used to ascertain system properties. The system to be modeled is subjected to a known artificial excitation and the response is compared to the theoretical response as predicted by the mathematical model. The unknown parameters are then selected, by predetermined criteria, to yield the best agreement between predicted and experimental system response. The experimental techniques can be either time domain (such as step or pulse response) or frequency domain. For frequency domain testing the system excitation can be in the form of sinusoidal steady state, sine sweep, impulsive, step or stationary random.

Alternatively, it is possible to determine dynamic properties while a system is in its actual operating environment. This method is attractive for studying systems not suitable for laboratory testing, e.g. ships, buildings, and other structures. Parameter identification is accomplished by studying the system response due to normally occurring excitations, providing that the excitations and response are measurable. Since, in general, these quantities will be random, stochastic theory is involved in the measurement and analysis processes. A benefit of *in situ* testing is that the unknown parameter estimates are usually more realistic than those obtained by laboratory testing since in the latter case the system is removed from its operating environment. However, for *in situ* testing the measurement and analysis techniques must remove, or properly account for, the effect of extraneous information (e.g. ambient sound and vibration).

The estimate of the frequency response of a system is a useful information in determining unknown parameters. In the system control, frequency response data aids in the formulation of control strategy. Even in uncontrolled systems (e.g. climate) the frequency response data are useful for predicting the system behavior to sets of selected inputs.

Frequency response techniques can be applied to certain acoustic systems, particularly with respect to practical problems of noise control. When the system under study is a noise source of complex geometry, such as most machinery noise sources, simple acoustical models using idealized geometries (spheres, cylinders, panels, etc.) often do not yield detailed information about the system and its behavior. Frequency response data (providing it can be obtained) can give an insight into the system behavior and aid in formulating a more realistic acoustical model.

Chung et al. [2-3] have used frequency response techniques to determine the structural-acoustical behavior of diesel engines. Chung considered the engine as a set of N linear systems with N inputs x_i, $i = 1, 2, \ldots, N$ (cylinder pressures corresponding to the combustion excitation in the N cylinders of the engine) and a single output y (the engine noise measured at a point about 1 m from the side of the engine).

The uncorrelated output z is included in the model to account for any extraneous effects such as ambient sound or instrumentation noise. The frequency responses were calculated from measured time records of the system inputs and outputs using multiple input linear theory [4].

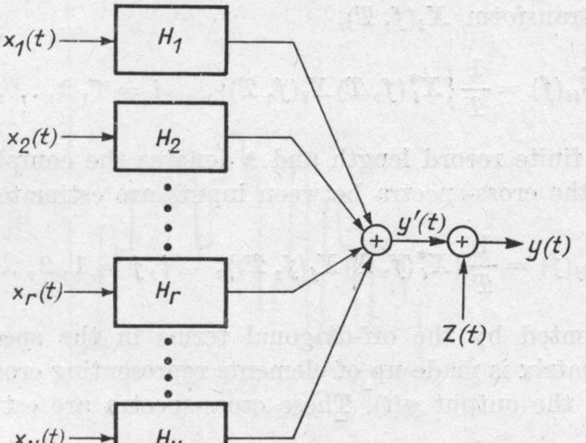

Fig. 1. A multiple input system with uncorrelated output z

Seybert and Crocker [5-6] used the diesel engine frequency responses to predict the effect on noise of engine operating conditions such as speed, load and injection timing.

The frequency responses are calculated by solving the following set of algebraic equations, utilizing spectral estimates computed from measured time records [4], where it is assumed that the time records are stationary:

$$[S_{iy}(f)] = [S_{ij}(f)][H_i(f)], \tag{1}$$

where the frequency response matrix

$$[H_i(f)] = \begin{bmatrix} H_1(f) \\ H_2(f) \\ \cdots \\ H_N(f) \end{bmatrix},$$

the spectral matrix

$$[S_{ij}(f)] = \begin{bmatrix} S_{11}(f) & S_{12}(f) & \ldots & S_{1N}(f) \\ S_{21}(f) & S_{22}(f) & \ldots & S_{2N}(f) \\ \cdots & \cdots & \cdots & \cdots \\ S_{N1}(f) & S_{N2}(f) & \ldots & S_{NN}(f) \end{bmatrix},$$

the cross-spectral matrix

$$[S_{iy}(f)] = \begin{bmatrix} S_{1y}(f) \\ S_{2y}(f) \\ \cdots \\ S_{Ny}(f) \end{bmatrix}.$$

The diagonal elements of the spectral matrix are the auto-spectral densities of each input $x_i(t)$. These spectra can be estimated from computations of the finite Fourier transform $X_i(f, T)$,

$$\hat{S}_{ii}(f) = \frac{1}{T}\{X_i^*(f, T) X_i(f, T)\}, \quad i = 1, 2, \ldots, N, \tag{2}$$

where T is the finite record length and * denotes the complex conjugate.

Similarly, the cross-spectra between inputs are estimated by

$$\hat{S}_{ij}(f) = \frac{1}{T}\{X_i^*(f, T) X_j(f, T)\}, \quad i, j = 1, 2, \ldots, N \tag{3}$$

and are represented by the off-diagonal terms in the spectral matrix. The cross-spectral matrix is made up of elements representing cross-spectra between each input and the output $y(t)$. These cross-spectra are estimated by

$$\hat{S}_{iy}(f) = \frac{1}{T}\{X_i^*(f, T) Y(f, T)\}, \quad i = 1, 2, \ldots, N. \tag{4}$$

It should be noted that the spectral estimates computed by using equations (2)-(4) are inconsistent estimates and some form of smoothing must be used to reduce the variance of the estimates. Smoothing also removes the effect of the extraneous output $z(t)$, providing it is uncorrelated with respect to all inputs ($S_{iz}(f) = 0$, $i = 1, 2, \ldots, N$). If this is true, then it can be shown [4] that

$$S_{iy}(f) = S_{iy'}(f), \quad i = 1, 2, \ldots, N,$$

where $S_{iy'}(f)$ is the cross-spectrum between each input and the coherent output $y(t)$.

Using the quantities defined above, the total output spectral density $S_{yy}(f)$ can be computed by

$$S_{y'y'}(f) = S_{zz}(f) + S_{yy}(f), \tag{5}$$

where $S_{zz}(f)$ is the spectral density of the uncorrelated output $z(t)$, and

$$S_{y'y'}(f) = \sum_{i=1}^{N}\sum_{j=1}^{N} S_{ij}(f) H_i^*(f) H_j(f) \tag{6}$$

is the spectral density of the coherent output $y'(t)$.

A typical frequency response for the diesel engine is shown [5] in Fig. 2. It was computed using equation (1) with measured time records for the combustion pressures (measured by quartz pressure transducers in each cylinder) and the sound pressure as measured by a condenser microphone located about 1 m from the side of the engine. The engine was operating in a free-field environment. The frequency response in Fig. 2 describes the structural-acoustical behavior of the combustion induced noise; regions where the frequency response is high correspond to high dynamic response and/or high radiation efficiency of the engine structure.

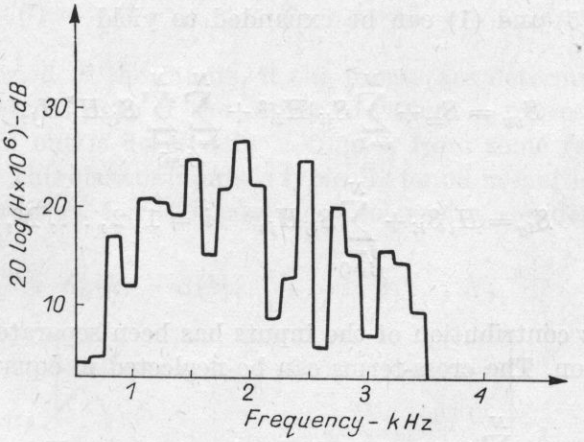

Fig. 2. Typical frequency response (magnitude) for one cylinder of a diesel engine

A problem associated with estimating the frequency responses of multiple input systems is the large number of cross-spectra between inputs that must be estimated for equation (1). For an N input system, $N(N-1)/2$ input cross-spectra must be estimated (the divisor 2 appears since the lower triangle cross-spectra in the spectral matrix are computed from the complex conjugate of the upper triangle spectra – $S_{ij}(f) = S_{ji}^*(f)$ – see equation (3)). Since the total number of spectral estimates needed for equation (1) is $N(N+3)/2$, the input cross-spectra represent a fraction of $(N-1)/(N-3)$ of the total spectra.

Many analog-to-digital converters and digital processors do not have sufficient capability to sample and compute more than one cross-spectral estimate at a time, thus making experimental time quite long for systems with many inputs. In addition, most mini-computer systems cannot perform matrix inversion (necessary to solve equation (1)) for large values of N due to computer core limitations.

The remainder of this paper will discuss a class of inputs normally occurring in many physical systems, where the input cross-spectra can be neglected. With such systems experimental time is reduced and the frequency responses can be computed from a set of uncoupled equations (1) (with input cross-spectra neglected). The computations are then suitable for mini-computers with small memories since matrix inversion is unnecessary. That is, equation (1) reduces to a set of equations

$$H_i(f) = S_{iy}(f)/S_{ii}(f), \quad i = 1, 2, \ldots, N. \tag{7}$$

Conditions for neglecting cross-spectral terms

Equations (5) and (1) can be expanded to yield

$$S_{yy} = S_{zz} + \sum_{i=1}^{N} S_{ii}|H_i|^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ (j\neq i)}}^{N} S_{ij} H_i^* H_j, \tag{8}$$

$$S_{iy} = H_i S_{ii} + \sum_{\substack{j=1 \\ (j\neq i)}}^{N} S_{ij} H_j, \quad i = 1, 2, \ldots, N, \tag{9}$$

where the direct contribution of the inputs has been separated from the cross--term contribution. The cross-terms can be neglected in equation (9) providing

$$\sum_{\substack{j=1 \\ (j\neq i)}}^{N} S_{ij} H_j \ll H_i S_{ii}, \quad i = 1, 2, \ldots, N, \tag{10}$$

or

$$\sum_{\substack{j=1 \\ (j\neq i)}}^{N} S_{ij} H_i^* H_j \ll |H_i|^2 S_{ii}, \quad i = 1, 2, \ldots, N,$$

thus

$$\sum_{i=1}^{N} \sum_{\substack{j=1 \\ (j\neq i)}}^{N} S_{ij} H_i^* H_j \ll \sum_{i=1}^{N} S_{ii}|H_i|^2. \tag{11}$$

Hence, if equation (10) can be established, then equation (11) follows and equations (8) and (9) reduce to

$$S_{yy} = S_{zz} + \sum_{i=1}^{N} S_{ii}\,|H_i|^2, \tag{12}$$

$$S_{iy} = H_i S_{ii}. \tag{13}$$

There are several conditions under which equation (10) is fulfilled, however, most are trivial. For example, if the inputs are independent random processes so that $S_{ij} = 0$ for $i \neq j$ or if one-frequency response is dominant, see equation (9) (in which case the multiple input system reduces to a single input system), equation (10) is satisfied. There may also be certain symmetries between the frequency responses that would satisfy equation (10), but in general this will not be the case. In the case of the diesel engine, symmetry does exist between the arrangement of the cylinders and the microphone position, but measurements of the frequency responses do not reflect this.

One general condition that satisfies equation (10) occurs when the inputs are of the form

$$x_i(t) = x_1(t+\tau_i),\ \tau_i < T, \quad i = 1, 2, \ldots, N, \tag{14}$$

where T is the period of the inputs, if the inputs are deterministic processes, or the sample record length if the inputs are random processes. Each input has the same form but is delayed by a time τ_i from some reference (here *1*) input, as in Fig. 3. This class of inputs is typically found in multicylinder engines, pumps, and compressors. For this case the auto-spectra are identical (see equation (2)),

$$S_{ii}(k) = S(k), \quad i = 1, 2, \ldots, N, \tag{15}$$

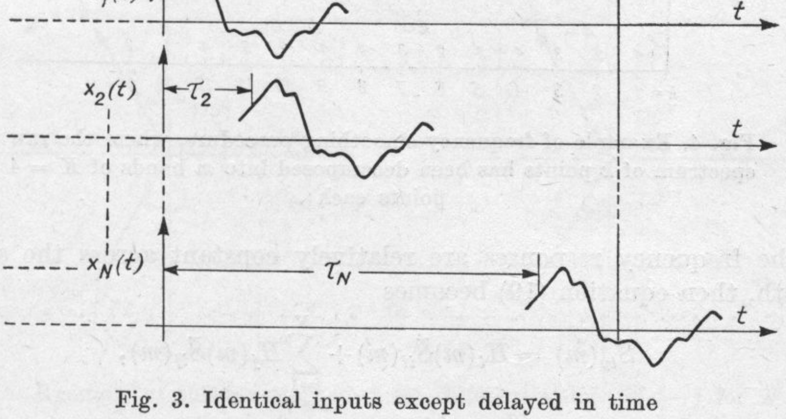

Fig. 3. Identical inputs except delayed in time

and the cross-spectra are given by (see equation (3))

$$S_{ij}(k) = S(k)e^{i\varphi_{ij}(k)}, \quad j \neq i;\ i, j = 1, 2, \ldots, N, \tag{16}$$

where

$$\varphi_{ij}(k) = \frac{\tau_{ij}(2\pi k)}{T}, \quad \tau_{ij} = \tau_i - \tau_j, \tag{17}$$

and $k = 1, 2, \ldots$ is the harmonic multiple of the fundamental frequency $1/T$.

Frequency smoothing is often used to reduce the variance of a spectral estimate and is accomplished by averaging the spectral values of the raw spectrum over a selected bandwidth. The disadvantage is a loss of frequency resolution since the bandwidth is increased from $1/T$ to K/T, where K is the number of spectral values averaged. That is, a smoothed spectrum is given by

$$\overline{S}(m) = \frac{1}{K}\sum_{l=1}^{K} S(l, m), \tag{18}$$

where the raw spectra $S(k)$ of k points have been decomposed into m bands of K points each (Fig. 4). Smoothing equation (9) yields

$$S_{iy}(m) = \frac{1}{K}\sum_{l=1}^{K} S_{iy}(l, m) = \frac{1}{K}\sum_{l=1}^{K}\Big\{H_i(l, m)S_{ii}(l, m) + \sum_{\substack{j=1\\(j\neq i)}}^{N} S_{ij}(l, m)H_j(l, m)\Big\}. \tag{19}$$

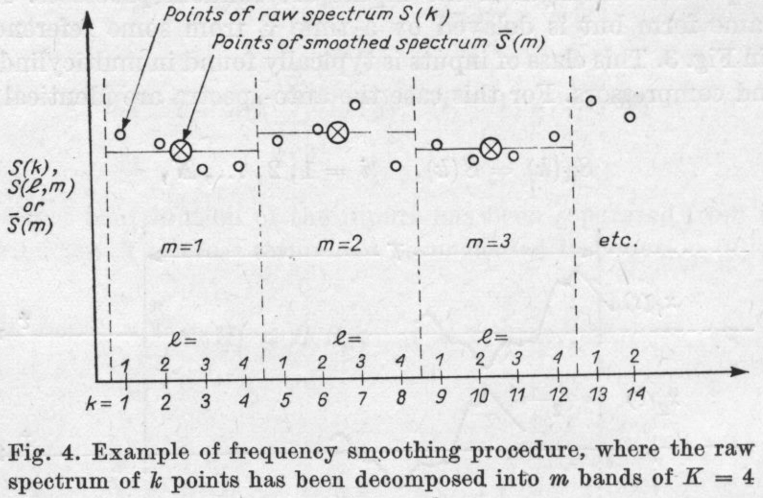

Fig. 4. Example of frequency smoothing procedure, where the raw spectrum of k points has been decomposed into m bands of $K = 4$ points each

If the frequency responses are relatively constant across the smoothing bandwidth, then equation (19) becomes

$$\overline{S}_{iy}(m) = H_i(m)\overline{S}_{ii}(m) + \sum_{\substack{j=1\\(j\neq i)}}^{N} H_j(m)\overline{S}_{ij}(m), \tag{20}$$

where

$$\bar{S}_{ii}(m) = \frac{1}{K}\sum_{l=1}^{K} S_{ii}(l, m) \quad \text{and} \quad \bar{S}_{ij}(m) = \frac{1}{K}\sum_{l=1}^{K} S_{ij}(l, m).$$

Examining $\bar{S}_{ij}(m)$ for the inputs described by (14) we get

$$\bar{S}_{ij}(m) = \frac{1}{K}\sum_{l=1}^{K} S(l, m)\, e^{i\varphi_{ij}(k)}. \tag{21}$$

If the auto-spectrum of the inputs $S(l, m)$ is relatively constant across the smoothing bandwidth, then

$$\bar{S}_{ij}(m) = \frac{S(m)}{K}\sum_{l=1}^{K} e^{i\varphi_{ij}(l,m)}. \tag{22}$$

It can be shown mathematically that

$$\sum_{l=1}^{K} e^{i\varphi_{ij}(l,m)} = 0, \quad j \neq i;\ i, j = 1, 2, \ldots, N. \tag{23}$$

For the case where the time delay between any two consecutively numbered inputs is T/N we have

$$\tau_{ij} = \frac{(i-j)T}{N}. \tag{24}$$

For this case equation (20) gives

$$\bar{S}_{iy}(m) = H_i(m)\,\bar{S}_{ii}(m), \quad i = 1, 2, \ldots, N. \tag{25}$$

In equation (23) the quantity $e^{i\varphi_{ij}(l,m)}$ can be interpreted as the "roots of unity" and represented geometrically in Figs. 5 and 6 for $N = 3$ and $N = 6$, respectively. An interesting property of these roots is that for any specified values N and $i-j$ the sum of the corresponding roots is zero, a fact stated in equation (23) and shown in Figs. 5 and 6.

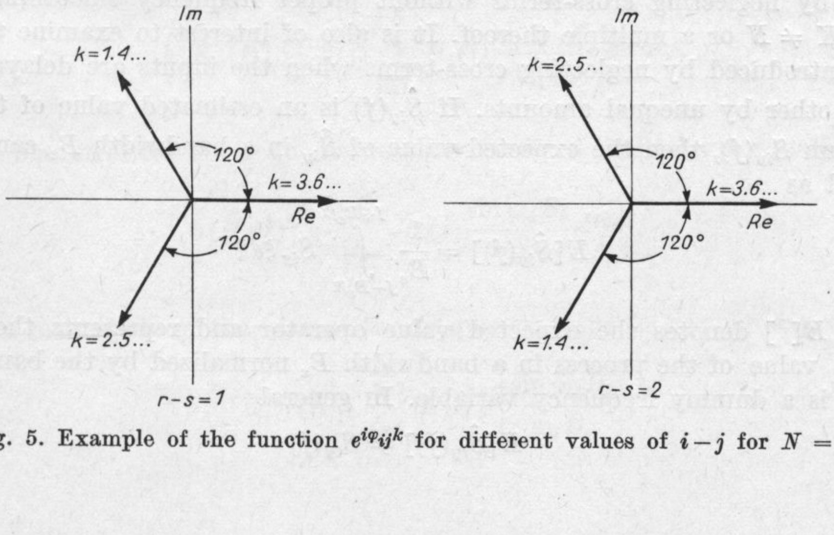

Fig. 5. Example of the function $e^{i\varphi_{ij}k}$ for different values of $i-j$ for $N = 3$

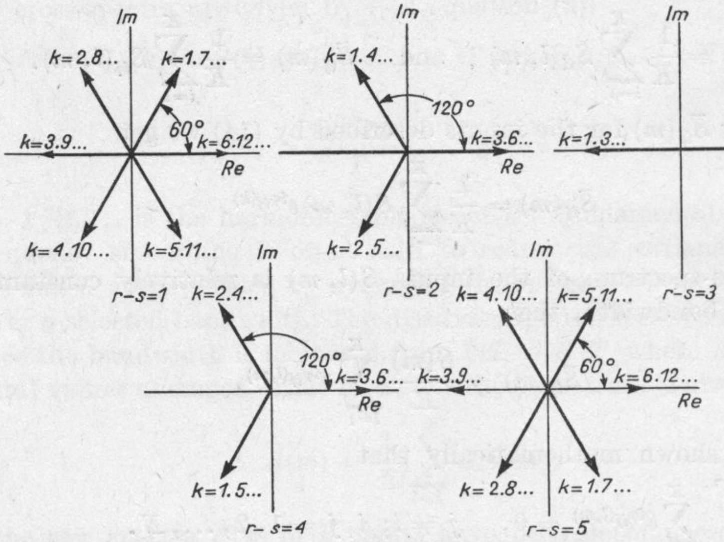

Fig. 6. Example of the function $e^{i\varphi_{ij}k}$ for different values of $i-j$ for $N = 6$

Consequently, there exists an optimum smoothing index K, equal to the number of mutually coherent inputs N, or a multiple thereof, that will allow the cross-spectral terms between inputs to be ignored in calculating frequency responses and in predicting the system output, when the inputs are given by equation (14).

Bias errors

In many practical cases one cannot restrict the analysis bandwidth to a multiple of N harmonics necessary to establish equation (25) or the time delay may not be as in equation (24). The question arises: "What bias error is introduced by neglecting cross-terms without proper frequency smoothing," i.e., when $K \neq N$ or a multiple thereof. It is also of interest to examine the bias error introduced by neglecting cross-terms when the inputs are delayed from one another by unequal amounts. If $\hat{S}_{iy}(f)$ is an estimated value of the true spectrum $S_{iy}(f)$, then the expected value of $\hat{S}_{iy}$ in a bandwidth B_e can be expressed as

$$E[\hat{S}_{iy}(f)] = \frac{1}{B_e} \int_{f-B_e/2}^{f+B_e/2} S_{iy}\,\xi d\xi \tag{26}$$

where $E[\ ]$ denotes the expected value operator and represents the mean-square value of the process in a bandwidth B_e normalized by the bandwidth, and ξ is a dummy frequency variable. In general

$$E[\hat{S}_{iy}(f)] \neq S_{iy}(f)$$

due to bias errors associated with the form of the actual spectrum S_{iy} or the nature of the estimation process. The bias error $b[\hat{S}_{iy}(f)]$ is defined as

$$b[\hat{S}_{iy}(f)] = E[\hat{S}_{iy}(f) - S_{iy}(f)]. \tag{27}$$

Stationary random inputs

Using equations (9), (15), (16) and (26) one obtaines

$$E[\hat{S}_{iy}(f)] = \frac{1}{B_e} \int_{f-B_e/2}^{f+B_e/2} \left\{ H_i(\xi) S(\xi) + S(\xi) \sum_{\substack{j=1 \\ (j \neq i)}}^{N} H_j(\xi) e^{i\varphi_{ij}(\xi)} \right\} d\xi \quad i = 1, 2, \ldots, N, \tag{28}$$

where the spectral quantities have been expressed as continuous functions of frequency rather than discrete values at harmonic numbers k.

Also

$$\varphi_{ij} = 2\pi f \tau_{ij}, \tag{29}$$

where τ_{ij} is the delay between any two inputs i and j.

If $S(f)H_j(f)$ $(j = 1, 2, \ldots, N)$ is not a strong function of frequency across the smoothing bandwidth B_e, then equation (28) becomes

$$E[\hat{S}_{ij}(f)] = H_i(f)S(f) + \frac{1}{B_e} \sum_{\substack{j=1 \\ (j \neq i)}}^{N} S(f) H_j(f) \int_{f-B_e/2}^{f+B_e/2} e^{i\varphi_{ij}(\xi)} d\xi$$

or

$$E[\hat{S}_{iy}(f)] = S(f) \left\{ H_i(f) + \sum_{\substack{j=1 \\ (j \neq i)}}^{N} H_j(f) \frac{\sin \psi_{ij} B_e}{\psi_{ij} B_e} e^{i2\psi_{ij} f} \right\}, \tag{30}$$

where $\psi_{ij} = \pi \tau_{ij}$. Therefore the bias error in $\hat{S}_{iy}(f)$ is, from equation (27),

$$b[S_{iy}(f)] = S(f) \sum_{\substack{j=1 \\ (j \neq i)}}^{N} H_j(f) \frac{\sin \psi_{ij} B_e}{\psi_{ij} B_e} e^{i2\psi_{ij} f}, \tag{31}$$

and the bias error for the frequency response (defined similarly to equation (27))

$$b[\hat{H}_i(f)] = \sum_{\substack{j=1 \\ (j \neq i)}}^{N} H_j(f) \frac{\sin \psi_{ij} B_e}{\psi_{ij} B_e} e^{i2\psi_{ij} f}. \tag{32}$$

The magnitude of the bias of the frequency response can be expressed as

$$|b[H_i(f)]| \leqslant \sum_{\substack{j=1 \\ (j \neq i)}}^{N} |H_j(f)| \left| \frac{\sin \psi_{ij} B_e}{\psi_{ij} B_e} \right|, \tag{33}$$

so that the equality in equation (33) is the upper bound of the bias error for the frequency response magnitude. The zeros of equations (31)-(33) occur if

$$\frac{\sin \psi_{ij} B_e}{\psi_{ij} B_e} = 0,$$

that is

$$\psi_{ij} B_e = n\pi, \quad n = 1, 2, .., $$

or, using the definition of ψ_{ij},

$$B_e = \frac{n}{\tau_{ij}}, \quad j \neq i; \; i, j = 1, 2, \ldots, N. \tag{34}$$

Therefore, in general, there is no bandwidth that will result in zero bias error for all of the inputs under consideration. One exception is when the time delays are given by equation (24), in which case the largest bandwidth is

$$B_e = \frac{nN}{T}, \tag{35}$$

which is in agreement with the interpretation of equation (23). Although a special case, many physical systems have inputs of this form. In general though, an optimum bandwidth would be selected that would minimize the bias errors for all the frequency responses according to some predetermined criterion – for example that the sum of the bias errors for all the frequency responses should be a minimum.

Bias errors — periodic inputs

In the case of periodic inputs of the form of equation (14), where T is the period of the inputs, the bandwidth is restricted to a multiple of $1/T$, the fundamental frequency of the inputs. In this case, the expected value of the cross-spectrum between each input and the output is

$$E[\hat{S}_{iy}(m)] = \frac{1}{K} \sum_{l=1}^{K} S_{iy}(l, m), \tag{36}$$

and the bias error

$$b[\hat{S}_{iy}(m)] = E[\hat{S}_{iy}(m) - \hat{S}_{iy}(m)]. \tag{37}$$

Substituting equations (9) and (15) into (36) we get

$$E[\hat{S}_{iy}(m)] = \frac{1}{K} \sum_{l=1}^{K} H_i(l, m) S(l, m) + S(l, m) \sum_{\substack{j=1 \\ (j \neq i)}}^{N} H_j(l, m) e^{i\varphi_{ij}(l,m)}$$

$$(i = 1, 2, \ldots, N). \tag{38}$$

Again, assuming that the spectra and frequency responses are not highly variable across any band K/T, we have

$$E[\hat{S}_{iy}(m)] = S(m)\left\{H_i(m) + \sum_{\substack{j=1 \\ (j\neq i)}}^{N} H_j(m)\frac{1}{K}\sum_{l=1}^{K} e^{i\varphi_{ij}(l,m)}\right\}.$$

Since

$$E[\hat{S}_{iy}(m)] = S(m)\left\{H_i(m) + \sum_{\substack{j=1 \\ (j\neq i)}}^{N} H_j(m)\frac{e^{i2\pi\tau_{ij}Km/T}}{K}\left[\left(-\frac{1}{2} + \frac{\sin[(K+\frac{1}{2})\tau_{ij}2\pi/T]}{2\sin(\tau_{ij}\pi/T)} + \right.\right.\right.$$

$$\left.\left.\left. + i\left(\frac{1}{2}\cot\frac{\tau_{ij}\pi}{T} - \frac{\cos(K+\frac{1}{2})\tau_{ij}2\pi/T}{2\sin(\tau_{ij}\pi/T)}\right)\right]\right\},$$

the bias of the frequency response is

$$b[\hat{H}_i(m)] = \sum_{\substack{j=1 \\ (j\neq i)}}^{N} \frac{H_j(m)\,e^{i2\psi_{ij}Km/T}}{2K}\left[\left(-1 + \frac{\sin[(K+\frac{1}{2})2\psi_{ij}/T]}{\sin(\psi_{ij}/T)} + \right.\right.$$

$$\left.\left. + i(\cot\psi_{ij}/T) - \frac{\cos[(K+\frac{1}{2})2\psi_{ij}/T]}{\sin(\psi_{ij}/T)}\right)\right]. \qquad (39)$$

Numerical results

Equation (32) is an expression for the bias error in estimating the frequency response when the cross-spectra between inputs are neglected. Note that in order to estimate the bias error, estimates of the other frequency responses are needed. Several example will be discussed to show the form of the bias error and the effect, of the assumption that the product $S(f)H_j(f)$ is relatively constant in any one band.

Example 1. Consider a six-input system where the inputs are mutually coherent band, limited white noise (constant spectral density) and the time delay between inputs is uniform and equal to 20 msec. The frequency responses are independent of frequency and have equal real and imaginary parts, as given by

$$H_i = Z_i(l+i),$$

where $Z_i = 1, 1, 2, 2, 3, 3$, for $i = 1, 2, \ldots, 6$, respectively. A band center-frequency of 208 Hz was chosen. Figure 7 shows the computed and actual bias error for $\hat{H}_1$ as a function of analysis bandwidth B_e, where the solid lines are the real and imaginary bias error calculated from equation (32) and the symbols are actual bias errors determined by estimating H_1 from equation (13), where cross-spectra between inputs have been neglected. The theoretical error

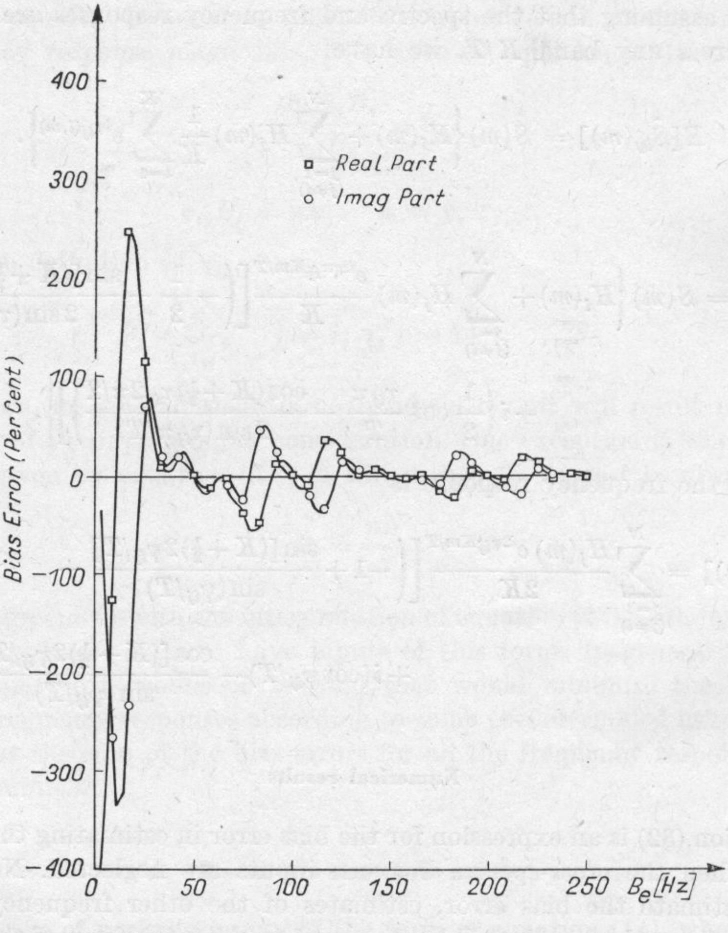

Fig. 7. Computed and theoretical bias error for Example 1

predicts exactly the actual error. The bias error is smaller for larger bandwidths due to the effect of frequency smoothing, which is really spectrum integration od $S_{iy'}$ (equation (26)). The cross-spectral terms of $S_{iy'}$ (equation (9)) are oscillatory in frequency, being both positive and negative, and tend to cancel when integrated, while the direct term, $H_1 S_{11}$, is positive across the entire band. As can be seen from Fig. 7, complete cancellation of the cross-spectral terms occurs at bandwidths that are multiples of 50 Hz. Since the delay between inputs is 20 msec, the fundamental frequency of the inputs is 8.333... Hz $(1/(6\times 0.020))$ and the cross-terms cancel at every bandwidth that is a multiple of six times the fundamental frequency or every 50 Hz. This is analogous to the examples in Figs. 5 and 6.

Example 2. *Non-uniform delay between inputs.* In this example the time delay between inputs 1 and 2, 3 and 4, 5 and 6 is 15 msec and the time delay between inputs 2 and 3, 3 and 4 and 6 and 1 is 25 msec, while the frequency responses and the input spectral densities are the same as in Example 1. Figure 8 shows the computed and actual bias error of $\hat{H}_1$ for this example at 208 Hz where it is again noted that equation (32) exactly predicts the actual bias error. In this case, however, a bandwidth of zero bias error for both real and imaginary parts of $\hat{H}_1$ does not occur until $B_e = 200$ Hz, although the error is quite small at 125 and 150 Hz.

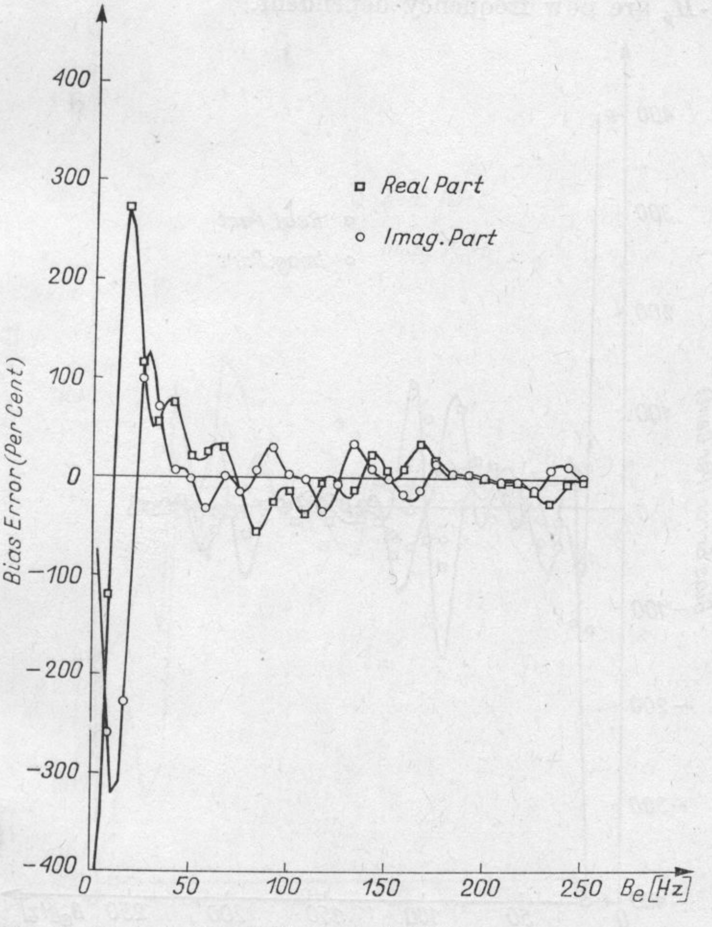

Fig. 8. Computed and theoretical bias error for Example 2

Example 3. *Frequency dependent response functions.* For this example H_1 is the same as in the previous examples, but H_1 - H_6 are chosen to be second-order systems with the relation

$$H_i(f) = \frac{Z_i}{1+2\xi\left(\frac{f}{f_i}\right)i-\left(\frac{f}{f_i}\right)^2}, \quad i = 2, 3, \ldots, 6,$$

where Z_i is as in the previous examples, the damping ratio $\xi = 0.1$, and the natural frequencies $f_i = 180$, 200, 220, 240 and 260 Hz for $i = 2, 3, \ldots, 6$. The time delay between inputs was as in Example 1. Figure 9 shows the computed and actual bias error for $\hat{H}_1$ in a band centered at 208 Hz. It can be seen from Fig. 9 that equation (32) has become inaccurate in predicting the bias error, particularly at certain frequencies. This is a result of the violation of the assumption that $H_i S_{ii}$ is relatively constant across the bandwidth, since the functions H_2-H_6 are now frequency dependent.

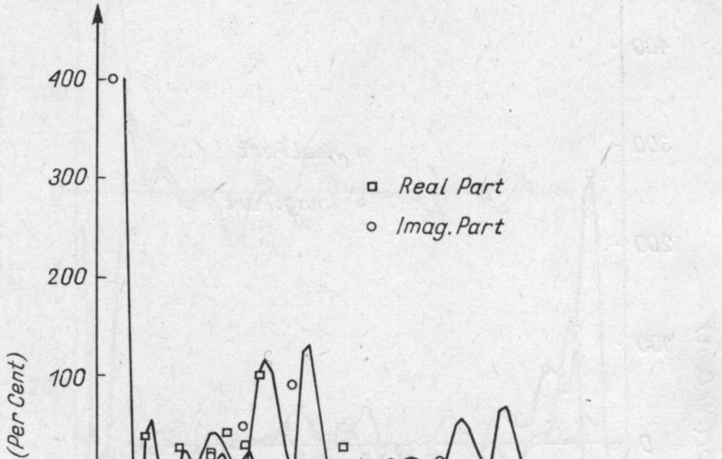

Fig. 9. Computed and theoretical bias error for Example 3

If the analysis band contains, or is near one or more of the natural frequencies, the bias error will not be accurately predicted by equation (32), particularly if the damping is low; the presence of natural modes in a band increases the frequency dependence of the frequency response.

To predict a more accurate bias error than equation (32) one would expand the estimates of the frequency responses in the Taylor series about the band center-frequency, place the series in equation (28), and proceed as before, omitting the assumption that $H_i S_{ii}$ is constant. This would yield an expression similar to equation (32), but including terms that would be functions of the first and higher derivatives of the frequency responses. Thus, it would then be necessary to have estimates of not only the frequency responses, but also the derivatives of the frequency responses. The number of terms in the Taylor series needed accurately to predict the error would depend on the variation of the frequency responses within the bandwidth.

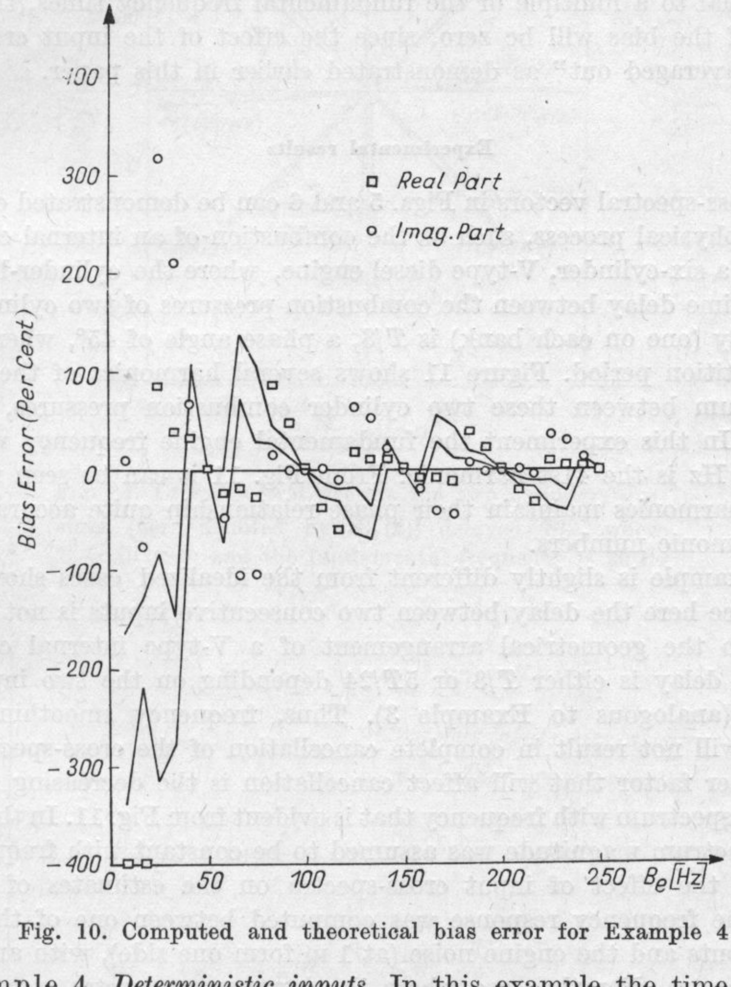

Fig. 10. Computed and theoretical bias error for Example 4

Example 4. *Deterministic inputs.* In this example the time delays and frequency responses are as in Example 1 while the inputs are delta functions in time (constant spectrum level). The only difference from the first example occurs in the analysis bandwidth; here bandwidth is restricted to a multiple of the fundamental frequency $1/T$, where $T = 120$ msec is the fundamental period of all inputs. Figure 10 is a plot of the computed bias error of $\hat{H}_1$ (equation (39)) and the actual error, showing that the error is accurately predicted only at the bandwidths of zero bias error. The failure of equation (39) to predict the bias is related to the difference in equations (26) and (36) where an integral has been replaced by a summation of a finite number of terms. The integrand S_{iy} is composed of a set of cross-terms which are highly frequency dependent (see equation (28)). Therefore, when the system inputs are deterministic and S_{iy} is estimated by smoothing in the frequency domain, severe bias can be introduced since the raw spectrum is not a continuous function. When the bandwidth is equal to a multiple of the fundamental frequency times, the number of inputs of the bias will be zero, since the effect of the input cross-spectra has been "averaged out" as demonstrated earlier in this paper.

Experimental results

The cross-spectral vectors in Figs. 5 and 6 can be demonstrated experimentally for a physical process, such as the combustion of an internal combustion engine. For a six-cylinder, V-type diesel engine, where the cylinder-bank angle is 90°, the time delay between the combustion pressures of two cylinders firing consecutively (one on each bank) is $T/8$, a phase angle of 45°, where T is the engine repetition period. Figure 11 shows several harmonics of the measured cross-spectrum between these two cylinder combustion pressures, beginning at 820 Hz. In this experiment the fundamental engine frequency was 20 Hz, so that 820 Hz is the 41st harmonic. From Fig. 11 it can be seen that cross--spectrum harmonics maintain their phase relationship quite accurately, even at high harmonic numbers.

This example is slightly different from the idealized cases shown in Fig. 5 and 6 since here the delay between two consecutive inputs is not a uniform T/N due to the geometrical arrangement of a V-type internal combustion engine; the delay is either $T/8$ or $5T/24$ depending on the two inputs being considered (analogous to Example 3). Thus, frequency smoothing over N harmonics will not result in complete cancellation of the cross-spectra in this case. Another factor that will affect cancellation is the decreasing magnitude of the cross-spectrum with frequency that is evident from Fig. 11. In the idealized case the spectrum magnitude was assumed to be constant with frequency.

To see the effect of input cross-spectra on the estimates of frequency response, the frequency response was computed between one of the cylinder pressure inputs and the engine noise (at 1 m form one side), with and without the cross-spectra. For the case where the input cross-spectra were included, equation (1) was solved for the frequency response matrix. Equation (7) was used to estimate the frequency response for the case where the input cross--spectra were neglected. Figure 12 shows one of the frequency responses computed for both cases. The bandwidth used in the analysis was 140 Hz corresponding to a frequency smoothing of seven harmonics in each band (20 Hz

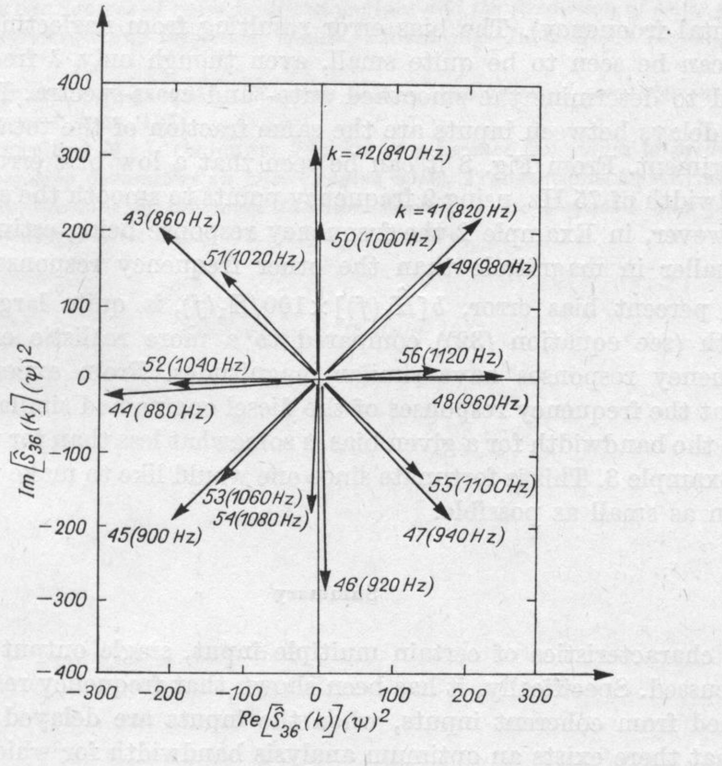

Fig. 11. Cross-spectral vectors for two cylinder pressures (here denoted by $\hat{S}_{36}(k)$) delayed $T/8$, where $T = 50$ msec and the fundamental frequency is 20 Hz

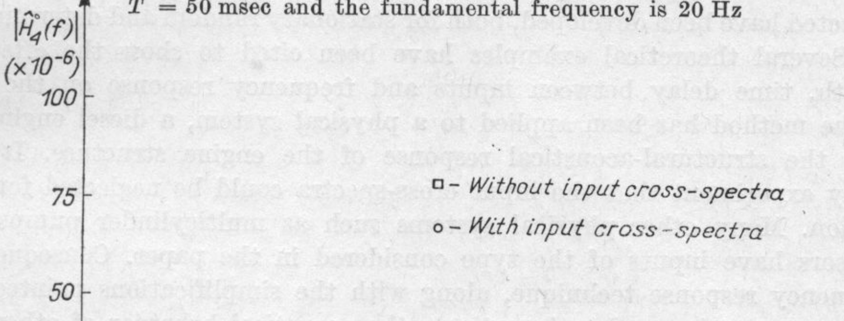

Fig. 12. Frequency response magnitude linear scale as a function of frequency, computed with and without considering input cross-spectra

fundamental frequency). The bias error resulting from neglecting input cross--spectra can be seen to be quite small, even though only 7 frequency points were used to determine the smoothed auto- and cross-spectra. In Example 2, the time delays between inputs are the same fraction of the total time T as in this experiment. From Fig. 8 it can be seen that a low bias error is predicted at a bandwidth of 75 Hz, using 9 frequency points to smooth the spectra instead of 7. However, in Example 2 the frequency response being estimated, H_1 was much smaller in magnitude than the other frequency responses (H_2-H_6) so that the percent bias error, $b[\hat{H}_1(f)]\times 100/H_1(f)$, is quite large for a given bandwidth (see equation (32)) compared to a more realistic example where the frequency responses have similar magnitudes. From experiment it was found that the frequency responses of the diesel engine had similar magnitudes; therefore the bandwidth for a given bias is somewhat less than for the theoretical case in Example 3. This is fortunate since one would like to make the bandwidth resolution as small as possible.

Summary

The characteristics of certain multiple input, single output systems have been discussed. Specifically, it has been shown that frequency responses can be determined from coherent inputs, when the inputs are delayed in time. It is shown that there exists an optimum analysis bandwidth for which the analysis is valid, and the bandwidth is related to the number of inputs and the time delay between the inputs. Expressions for bias error when input cross-spectra are neglected have been developed, both for stationary random and deterministic inputs. Several theoretical examples have been cited to show the effect of bandwidth, time delay between inputs and frequency response on the bias error. The method has been applied to a physical system, a diesel engine, to estimate the structural-acoustical response of the engine structure. It was shown by experiment that the input cross-spectra could be neglected for this application. Many other physical systems such as multicylinder pumps and compressors have inputs of the type considered in the paper. Consequently, the frequency response technique, along with the simplifications pointed out in this paper, can be used to investigate the acoustical behavior of other machinery noise source.

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