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THE BOWED STRING AS THE TWO-TERMINAL OSCILLATOR

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The paper presents a method of the evaluation of the shape of bowed string oscillations. The method employs the analogy between the bowed string and the electrical, two-terminal oscillator. The velocity-dependent friction force between string and bow, investigated experimentally many years ago, was applied as a stimulating function of the oscillator. The evaluated shapes and amplitudes are in good agreement with experimental results.

1. Introduction

The analysis of the generating process of bow-sustained string vibrations is a classical, but still valid subject in the field of both musical acoustics and the theory of oscillations. The reality of this problem results from the fact that the successive investigations yield merely approximate solutions because the string vibration maintenance mechanism is highly complicated.

It should be remembered that the bow simultaneously generates transverse, longitudinal and torsional vibrations of the string as well as of its fixed points, with the consequent so-called octave vibrations [11]. All these vibrations interact by superposition at the incommensurate frequencies of their individual modes [8].

It is to be noticed, moreover, that the string corresponds to a distributed-constant system where propagation conditions affect the shape of vibrations. Moreover, string vibrations are highly nonlinear due to nonlinear relation between transverse, longitudinal and torsional elastic forces and their corresponding displacements. Also the bow, vibrating jointly with the string, brings other nonlinear effects into string motion [7]. Thus, many papers devoted to the vibration theory of strings, deepening the analysis of various types of nonlinearities, disregard the nonlinear problems which arise from the excitation mechanism, i.e. from the bow-string excitation [14, 15].

The complexity and difficulty of the bowed string vibration maintenance analysis justifies the interest which accompanies subsequent trials of finding approximate solutions of the problem.

Although the nature of bowed string oscillations has been thoroughly investigated already in the XIXth century by HELMHOLTZ [13] the mechanism of its excitation into self-sustained oscillations remained an ambiguous problem for a long time. Successful trials of solving it up began about sixty years ago with CHARRON'S thesis [3].

The author investigated with a special attention the dependence of the friction force between bow and string upon their mutual velocity acting as the so-called stimulating function in the process of vibrations generation [5].

Giving up to present here the results of numerous contributions to the problem of the self-oscillating bowed string, it should be mentioned that relatively frequent misinterpretations of observed phenomena may be found in the literature of the subject. An interpretation of a typical oscillogram of the string velocity versus time, where parts of the vibration period corresponding to intervals of the steady and of the varying velocity are obviously interchanged, may serve as an example [6]. This interpretation, given by authors of many outstanding papers on the vibration theory of the violin, proves that some important problems here are still to be studied and explained.

A friction force versus velocity diagram, presented in a JASA paper [12], provides another example. A continuous curve showing zero friction force at zero relative velocity is given there, which is discrepant from all typical diagrams of this kind and from measured characteristics of string-bow systems [3, 6].

Friction characteristics at zero velocity have necessarily a discontinuity called *dead zone*. It plays an essential role in vibrations maintenance, providing proper conditions for a stimulating function exist.

So, it seemed reasonable to examine once more the problem of string excitation by a bow, even in a simplified way.

Attempting an analytical formulation of the problem the following assumptions have been accepted:

— only transverse vibrations are considered, as energetically prevailing,

— lumped-constant mechanical system is assumed as a string model,

— the damping of the free vibrating string is provided to be negligible in comparison with the bowed string one.

2. Bow-string system as a two-terminal oscillator

String vibrations maintained by permanent bow motion are typical self-sustained oscillations. Therefore, a bow-string system may be analysed as a mechanical oscillator. Although description methods of mechanical systems, based on the theory of electrical oscillators, are not frequently applied, such a procedure may be easily adopted by the use of electromechanical analogies.

The considered case of a string-bow oscillator may be presented as the electrical circuit shown in Fig. 1. This circuit permits the formulation of mechanical string motion equations (2) as an equivalent of the electrical voltage-

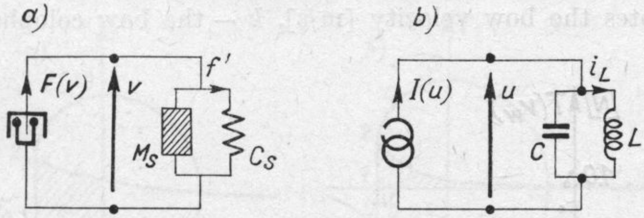

Fig. 1. Two-terminal oscillating system: a) mechanical system, b) its electrical motional analogue

-current equations (1). The corresponding quantities of (1) and (2) are coupled by the motional analogy:

$$\frac{du}{d\tau} = -\sqrt{\frac{L}{C}}\, I(u) - i, \qquad \frac{di}{d\tau} = u, \tag{1}$$

where

$$\tau = t/\sqrt{LC}\,, \qquad i = i_L\sqrt{L/C} = \Phi/\sqrt{LC},$$

i – variable quantity proportional to coil current i_L, Φ – magnetic flux in the coil;

$$\frac{dv}{d\tau} = -\sqrt{\frac{C_s}{M_s}}\, F(v) - f, \qquad \frac{df}{d\tau} = v, \tag{2}$$

where

$$\tau = t/\sqrt{C_s M_s}, \qquad f = f'\sqrt{C_s/M_s} = x/\sqrt{C_s M_s}\,,$$

f – variable quantity proportional to force f' applied to string compliance (see Fig. 1), x – string displacement.

The nonlinear function $F(v)$ of equation (2) represents the friction force between the bow and the string as a function of the relative string velocity. The function values found empirically by Charron [3] are presented in Fig. 2 given in their original form. It is obvious that this function does not depend on the sense of the bow movement, hence the curve $T(v_w)$ can be completed antisymmetrically to the left.

Assuming the string absolute velocity as an independant variable, i.e. transferring the curve $T(v_w)$ by the value of bow velocity V_0, we obtain formula (3). It performs the role of a stimulating function (see Fig. 3), like similar negative resistance functions do in typical two-terminal electrical oscillators [1],

$$F(v) = \frac{T_0 \operatorname{sign}(v+V_0)}{1+k|v+V_0|}, \tag{3}$$

where V_0 denotes the bow velocity [m/s], k — the bow colophanying factor.

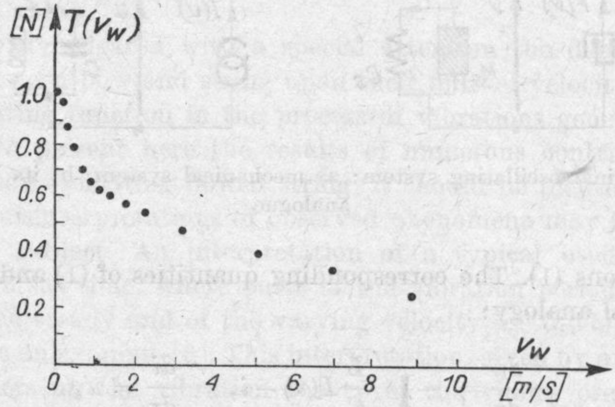

Fig. 2. Friction force between bow and string as a function of its relative velocity [3]. The dependence is given by the formula $t(v_w) = T_0/(1+kv_w)$, where T_0 — static friction force [N], k — bow colophanying factor

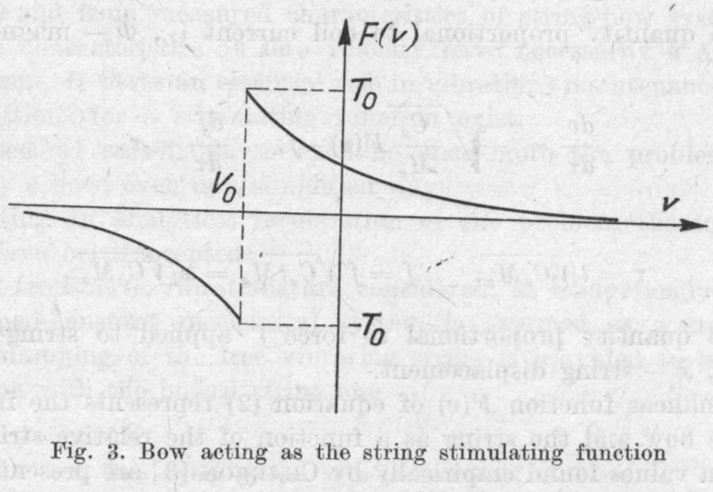

Fig. 3. Bow acting as the string stimulating function

3. Solution of the string motion equation

Equation (1), with nonlinearity given by formula (3), describes self-sustained oscillations. The equation variables $f(\tau)$ multiplied by a constant factor $(C_s M_s)^{1/2}$ and $v(\tau)$ represent the string displacement and the absolute string velocity. Since analytical solutions of this equation type are not available, a numerical method has been used. The method enables to calculate [10] steady-state values of $v(\tau)$ and $f(\tau)$. It is also possible to find limit cycles of the equation by means of the widely known Lienard's graphical construction, as shown in Fig. 4.

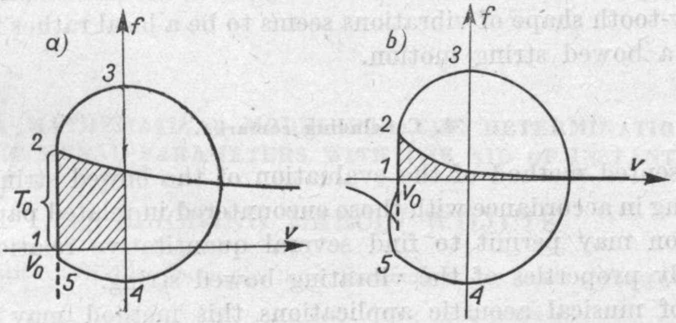

Fig. 4. Limit cycle of string vibrations. Exciting in intervals 1-2, 2-3, 4-5; damping in intervals 3-4, 5-1

String velocity: a) does not exceed bow velocity, b) exceeds bow velocity

Both numerical calculations and graphical method confirm the possibility of an increase of the string velocity above the bow velocity [6]. Really, at appropriately chosen movement parameters, one can observe an apparently paradoxical phenomenon of the string preceding the bow during a part of a period (Fig. 4b, 5a).

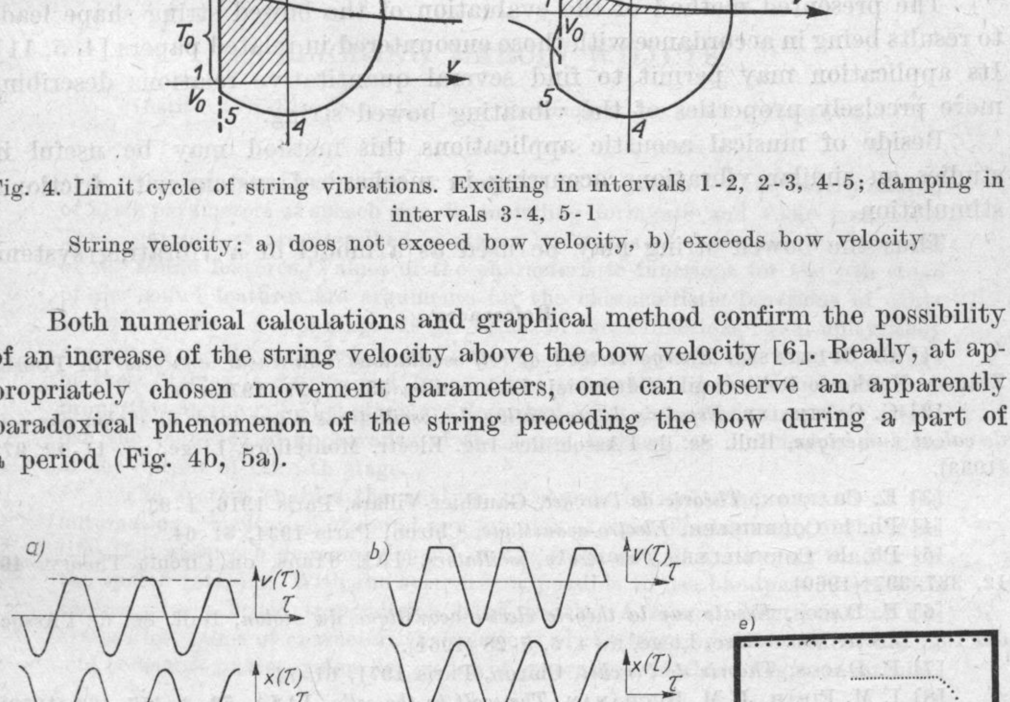

Fig. 5. Shapes of solutions of string motion equations. The following values of the parameters V_0, T_0, k of equation (3) are taken: a) 0.8; 0.1; 0.9; b) 0.4; 0.53; 0.415 (empirical values [2, 3]); c) 0.2; 0.9; 0.9; d) 0.15; 1.8; 1.8; e) computer printout from which the function $v(\tau)$ shown in Fig. 5c has been derived. Other functions have been formed in a similar way

Results of numerical calculations (see Fig. 5) show that the saw-tooth shape of a string displacement as a function of time is not only one possible, although it was just so presented in papers [8, 13].

Vibrations of the saw-tooth type can be produced only in a relatively narrow range of parameter values which condition the oscillation maintenance. Then the saw-tooth shape of vibrations seems to be a local rather than a general property of a bowed string motion.

4. Concluding remarks

The presented method of the evaluation of the bowed string shape leads to results being in accordance with those encountered in related papers [4, 5, 11]. Its application may permit to find several quantitative relations describing more precisely properties of the vibrating bowed string.

Beside of musical acoustic applications this method may be useful in studies on similar vibrations occurring in mechanical system with frictional stimulation.

Thus the bowed string may be used as a model of a vibrating system.

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