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THE EQUATION OF ACOUSTIC RAYS IN AN INHOMOGENEOUS MOVING MEDIUM

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This paper is devoted to the derivation of the differential equations for rays in an inhomogeneous medium which moves with a velocity $\overline{W}(z)$. The velocity of sound propagation in this medium is assumed to be a function of one variable $a(z)$.

The discussion is based on the Snell's generalized law which has been derived by a new method.

1. Introduction

Let us assume that in a moving and inhomogeneous medium, an arbitrare wave surface is given by the equation $\varphi(\bar{r}, t) = 0$. This surface moves and changy its form. Then, for the time instants $t_3, t_2, t_1, \ldots$ we can write the respectve equations

$$\varphi(\bar{r}, t_1) = 0,\ \varphi(\bar{r}, t_2) = 0,\ \varphi(\bar{r}, t_3) = 0, \ldots$$

Then we have the relation of the form $t = \psi(\bar{r})$. If the sound velocity at a point determined by the vector $\bar{r}$ is $a(\bar{r})$ and the medium moves with a velocity $\overline{W}(\bar{r})$, then $\psi(\bar{r})$ satisfies the so-called generalized eikonal equation [3] which can be written in the form

$$H\left(x, y, z, \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial z}\right) = 0.$$

Solution of this equation takes the form of a set of partial differential equations describing a family of curves which are orthogonal to the surface $t = \psi(\bar{r})$ [14]. These curves represent rays. The same set of equations is obtainable either by starting from the fundamental equations of hydrodynamics [12, 13] or by constructing the Hamilton equations [5, 7], as is the case in optics.

The simple way for deriving the equations of rays is based on the generalized Snell's law. This paper presents a new method for deriving this law in the case of a layered medium.

The differential equations thus obtained can be integrated if the functions $a(z)$ and $W_x(z)$, $W_y(z)$, $W_z(z)$ are given. Such an integration has been accomplished for the case of medium at rest $\overline{W}(z) \equiv 0$, with the sound velocity being linearly dependent on the coordinate $a(z) = a_0(1-\beta z)$.

2. The Snellius generalized equation

Wilibrord SNELL (SNELLIUS, 1591-1626) established experimentally the direction of the way in which propagated wave varies with a discrete change in sound velocity.

The generalized SNELL'S law determines how the direction of propagation changes in a moving medium with the sound velocity being varied continuously.

There are several ways of deriving the generalized SNELL'S law [1, 2, 4, 6, 8, 9, 10]. The method presented in this paper seems to be simpler than the others.

WARREN [15] presents a number of works in which the SNELL'S law was derived wrongly.

In an immobile medium, the form of the wave surface is a function of the propagation velocity $\bar{a} = \bar{n}a(\bar{r})$ ($\bar{n}$ — unit vector perpendicular to the wave surface) and initial conditions. If we assume that the medium moves with velocity $\overline{W}(\bar{r})$, then the resultant propagation velocity of disturbance is

$$\overline{U} = \bar{n}a(\bar{r}) + \overline{W}(\bar{r}). \tag{1}$$

The form of the wave surface depends only on the velocity component $\overline{U}$ perpendicular to this surface

$$\overline{U}_n = \bar{n}[a(\bar{r}) + \bar{n}\overline{W}(\bar{r})].$$

The component $\overline{U}_{\|}$ parallel to the wave surface causes an arbitrary acoustic particle to move over this surface and contributes nothing to the physical pattern of the phenomenon (Fig. 1).

Let us assume that the medium is layered, that is, each of its parameters and thus the propagation velocity $a(z)$ and the velocity of the medium $\overline{W}(z)$ are

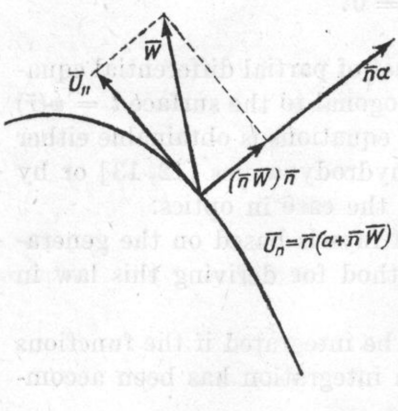

Fig. 1. Components $\overline{U}_{\|}$ and $\overline{U}_n$ efvelocity $\overline{U} = \bar{n}a + \overline{W}$

functions of only one variable. In order to derive the generalized SNELL'S law let us assume additionally that this medium is formed of layers Δz thick, within which the velocity of the moving medium, the velocity of sound and the unit vector $\bar{n}$ are constant, but differ (Fig. 2) from layer to layer.

The ray corresponding to the wave propagating in such a medium is a broken line and the phase in the k-th layer is

$$\psi = t + \left(\frac{\bar{n}\bar{r}}{a+\overline{W}\bar{n}}\right)_k$$

where (Fig. 2) $\bar{n}_k$ – unit vector perpendicular to the wave surface, a_k – sound

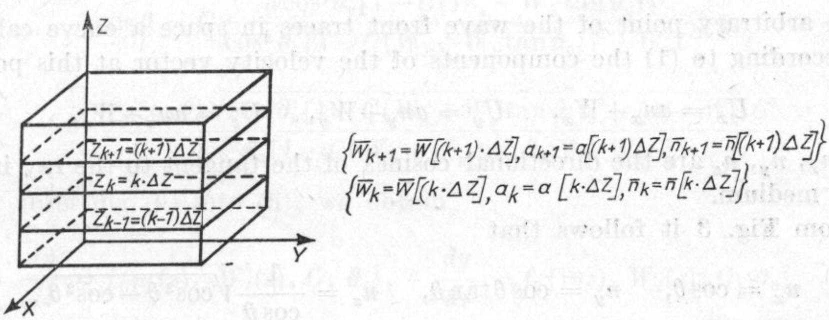

Fig. 2. Model of a medium consisting of layers

velocity, $\overline{W}_k$ – velocity of the moving medium in the k-th layer, and $\bar{r}_k$ – vector of the form

$$\bar{r}_k = (x - x_{k-1})\bar{i} + (y - y_{k-1})\bar{j} + (z - z_{k-1})\bar{k}.$$

The phases in the plane of contact of two layers at a point $\bar{r} = x\bar{i} + y\bar{j} + \Delta z\bar{k}$ must be equal to each other at an arbitrary time instant

$$t + \left(\frac{\bar{n}}{a+\overline{W}\bar{n}}\right)_{k+1}\bar{r} = t + \left(\frac{\bar{n}}{a+\overline{W}\bar{n}}\right)_k \bar{r}. \tag{2}$$

This expression can be written in the following form

$$\left[\left(\frac{n_x}{a+\overline{W}\bar{n}}\right)_{k+1} - \left(\frac{n_x}{a+\overline{W}\bar{n}}\right)_k\right]x + \left[\left(\frac{n_y}{a+\overline{W}\bar{n}}\right)_{k+1} - \left(\frac{n_y}{a+\overline{W}\bar{n}}\right)_k\right]y +$$

$$+ \left[\left(\frac{n_z}{a+\overline{W}\bar{n}}\right)_{k+1} - \left(\frac{n_z}{a+\overline{W}\bar{n}}\right)_k\right]\Delta z = 0.$$

Assuming $\Delta z \to 0$, we obtain

$$\frac{d}{dz}\left(\frac{n_x}{a+\overline{W}\bar{n}}\right) = 0, \quad \frac{d}{dz}\left(\frac{n_y}{a+\overline{W}\bar{n}}\right) = 0.$$

The expressions

$$\frac{n_x}{a(z)+W_x(z)n_x+W_y(z)n_y+W_z(z)n_z} = C_1$$
$$\frac{n_y}{a(z)+W_x(z)n_x+W_y(z)n_y+W_z(z)n_z} = C_2 \tag{3}$$

describe the generalized SNELL's law for a layered medium.

3. The differential equations of rays

An arbitrary point of the wave front traces in space a curve called the ray. According to (1) the components of the velocity vector at this point are

$$U_x = an_x+W_x, \quad U_y = an_y+W_y, \quad U_z = an_z+W_z, \tag{4}$$

where n_x, n_y, n_z are the directional cosines of the tangent to the ray in a immobile medium.

From Fig. 3 it follows that

$$n_x = \cos\theta, \quad n_y = \cos\theta\tan\vartheta, \quad n_z = \frac{1}{\cos\vartheta}\sqrt{\cos^2\vartheta-\cos^2\theta}.$$

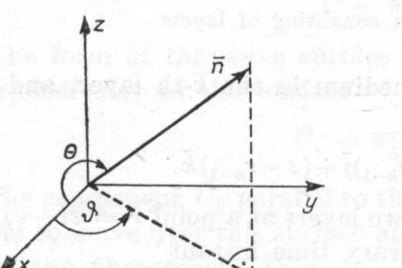

Fig. 3. Determination of vector $\bar{n}$ by means of angles θ, ϑ

By substituting the directional cosines expressed in such a way into (4) and taking into account that

$$U_x = \frac{dx}{dt}, \quad U_y = \frac{dy}{dt}, \quad U_z = \frac{dz}{dt}$$

we obtain the differential equations of the ray

$$\frac{dx}{dz} = \frac{[a(z)\cos\theta+W_x(z)]\cos\vartheta}{a(z)\sqrt{\cos^2\vartheta-\cos^2\theta}+W_z(z)\cos\vartheta},$$
$$\frac{dy}{dz} = \frac{[a(z)\cos\theta\tan\vartheta+W_y(z)]\cos\vartheta}{a(z)\sqrt{\cos^2\vartheta-\cos^2\theta}+W_z(z)\cos\vartheta}. \tag{5}$$

To perform the integration effectively, it is necessary to know the relations $\theta = \theta(z)$ and $\vartheta = \vartheta(z)$ in which x, y are not involved because the medium is layered along the direction of axis z. From (3), we get

$$\operatorname{tg}\vartheta = C_2/C_1,$$

$$\cos\theta = C_1\frac{a[1-(W_xC_1+W_yC_2)]\pm W_z\sqrt{[1-(W_xC_1+W_yC_2)]^2-(a^2-W_z^2)(C_1^2+C_2^2)}}{[1-(W_xC_1+W_yC_2)]^2+W_z^2(C_1^2+C_2^2)}.$$

Instead of C_2, it is convenient to assume $\vartheta = \vartheta_0$ as a constant and to denote $C = C_1$

$$\cos\theta = C\frac{a\cos^2\vartheta_0[1-C(W_x+W_y\tan\vartheta_0)]}{\cos^2\vartheta_0[1-C(W_x+W_y\tan\vartheta_0)]^2+W_z^2C^2}+$$

$$+C\frac{W_z\cos\vartheta_0\sqrt{\cos^2\vartheta_0[1-C(W_x+W_y\tan\vartheta_0)]^2-(a^2-W_z^2)C^2}}{\cos^2\vartheta_0[1-C(W_x+W_y\tan\vartheta_0)]^2+W_z^2C^2}.$$

By inserting (6) into (5), we obtain

$$\frac{dx}{dz} = f_1\{a(z),\, W_i(z),\, C,\, \vartheta_0\},\qquad \frac{dy}{dz} = f_2\{)vz),\, W_i(z),\, C,\, \vartheta_0\}. \tag{7}$$

These equations are integrable. The constants C and ϑ_0 depend on the direction of the ray θ, ϑ_0 in the neighbourhood of the source (Fig. 3).

If the source is located at the point (x_0, y_0, z_0), then we obtain C from (3) as a function of the angles θ_0 and ϑ_0 and quantities a, $\overline{W}$ taken for $z = z_0$

$$C = \frac{\cos\theta_0\cos\vartheta_0}{\cos\vartheta_0[a(z_0)+W_x(z_0)\cos\theta_0]+W_y(z_0)\cos\theta_0\sin\vartheta_0+W_z(z_0)\sqrt{\cos^2\vartheta_0-\cos^2\theta_0}}. \tag{8}$$

4. The explicit form of the equations of rays

Integration of (7) with given functions $\overline{W}(z)$, $a(z)$ and under the assumptions that the coordinates of the source are x_0, y_0, z_0 provides the following equations for the rays:

$$x = x_0+\int_{z_0}^{z} f_1\{a(z),\, W_i(z),\, a(z_0),\, W_i(z_0),\, \vartheta_0,\, \theta_0\}\,dz,$$

$$y = y_0+\int_{z_0}^{z} f_2\{a(z),\, W_i(z),\, a(z_0),\, W_i(z_0),\, \vartheta_0,\, \theta_0\}\,dz. \tag{9}$$

Example. Formulae (9) are usually so complex that the integration can be accomplished only in a numerical way. In some cases, it is possible to perform this integration directly and to obtain the equations of rays of the form $x = F_1(z)$, $y = F_2(z)$.

If we assume, for instance, that the medium is at rest, that is $\overline{W}(z) = 0$ and the sound velocity is a function of the form $a(z) = a_0(1-\beta z)$, then from (6) we have

$$\cos\theta = Ca_0(1-\beta z),$$

and the set of equation (7) simplifies to the form

$$\frac{dx}{dz} = \frac{Ca_0(1-\beta z)\cos\vartheta_0}{\sqrt{\cos^2\vartheta_0 - C^2a_0^2(-\beta z)^2}},$$
$$\frac{dy}{dz} = \frac{Ca_0(1-\beta z)\sin\vartheta_0}{\sqrt{\cos^2\vartheta_0 - C^2a_0^2(1-\beta z)^2}}. \tag{7'}$$

From (8), we obtain

$$C = \frac{\cos\theta_0}{a_0(1-\beta z_0)}$$

and after integration of (7) we can write

$$x = x_0 + \frac{\cos\vartheta_0(1-\beta z_0)}{\cos\theta_0\beta}\left[\sqrt{\cos^2\vartheta_0 - \cos^2\theta_0} - \sqrt{\cos^2\vartheta_0 - \frac{\cos^2\theta_0}{(1-\beta z_0)^2}(1-\beta z)^2}\right],$$
$$y = y_0 + \frac{\sin\vartheta_0(1-\beta z_0)}{\cos\theta_0\beta}\left[\sqrt{\cos^2\vartheta_0 - \cos^2\theta_0} - \sqrt{\cos^2\vartheta_0 - \frac{\cos^2\theta_0}{(1-\beta z_0)^2}(1-\beta z)^2}\right]. \tag{9'}$$

It can be shown that the set of equations (9′) describes a circle that lies in a plane perpendicular to the plane (x, y) and passes through the point (x_0, y_0, z_0) [11].

5. Conclusions

The typical layered medium is the atmosphere. It can be assumed that the velocity of motion of this medium ($\overline{W}(z)$ — velocity of the wind) and the sound velocity $a(z)$ are functions of the altitude.

Under such assumptions, the formulae (9) are equations of acoustic rays in the atmosphere when the wind blows, while (9′) is the set of he same equations when the atmosphere is at rest.

Equations (9) can be utilized to investigate e.g. the phenomenon of refraction under a variety of atmospheric conditions or to the determine the location of sonic booms on the ground.

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