

Research Paper

Series Expanding of the Ultrasound Transmission Coefficient Through a Multilayered Structure

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To calculate the transmission coefficient of ultrasonic waves through a multi-layered medium, a new approach is proposed by expanding it into Debye's series. Using this formalism, the transmission coefficient can be put in the form of resonance terms series. From this point of view, the relative amplitude of the transmitted wave can be considered as an infinite summation of terms taking into account all possible reflections and refractions on each interface. Our model is then used to investigate interaction between the ultrasonic plane wave and the N -plane-layer structure.

Obviously, the resulting infinite summation has to be reduced to a finite one, according to some level of accuracy. The numerical estimation of the transmission coefficient using the exact expression (Eq. (1)) is then compared to the one of our method in the case of two or three plane-layer structure. The effect of the order of the finite summation on the calculated value of the transmission coefficient is, as well, studied. Finally, our proposed method may be used, with the decomposition into Gaussian beams of a pressure field created by a circular source, to draw a 3D image of the pressure field transmitted through a multilayered structure.

Keywords: multilayered structure; Debye's series; resonance formalism; ultrasonic NDT.



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Nomenclature

- Γ_N – transmission coefficient of N -layer structure,
- d_n – thickness of the n -th layer,
- \mathbf{k}_n – propagation wave vector in the n -th layer,
- ρ_n – mass density of the n -th layer,
- c_n – sound speed in the n -th layer,
- α_n – absorption coefficient in the n -th layer,
- t_{n-1} – transmission coefficient of the n -th interface,
- r_n – reflection coefficient of the n -th interface,
- n_i – summation degree in i -layer structure,
- ε – precision needed to define the minimal value of n_i ,
- p_n – acoustic pressure in the n -th layer,
- Z_n – acoustic impedance of the n -th layer.

1. Introduction

The measurement of ultrasound reflection and transmission coefficients, from and through a layered

structure, is of great interest in many nondestructive testing and characterization applications. Biological tissues or rocks are some natural example of the layered structure. Experimental data with a theoretical model are exploited to extract acoustical properties as the attenuation coefficient, density, sound speed and other mechanical properties. Some of these applications are reviewed in (HSU, 2009).

Several techniques can be used to solve the inverse problem. The inverse problem solution can be obtained by minimizing a cost functional formulated as the least square error between the waveform calculated using an equivalent model, and the measured waveform obtained from ultrasonic transmission tests (MESSINEO *et al.*, 2016). A particle swarm optimization (PSO) algorithm based least squares estimation and using the ultrasonic reflection spectrum has been

used (YANG *et al.*, 2019). Multilayer structures are involved in the design of piezoelectric transducers. Implementing two-layer matching structure improves the transmission of the acoustic power into the medium (BAKHTIARI-NEJAD *et al.*, 2020). In the case of several layers, genetic algorithms can help in an optimal selection of the materials used as adaptation layers (GUDRA, BANASIAK, 2020).

Ultrasound propagation through periodic structures is another area of interest of the layered structure. Indeed, periodic structures with a wavelength scale periodicity (POTEL, BELLEVAL, 1993; SHENAND, CAO, 2000; KHALED *et al.*, 2013; MARÉCHAL *et al.*, 2014) are known to exhibit acoustical band gaps, which is of great interest for many applications like wave filtering, guiding, focusing waves, silent blocks, and it can also help improving the efficiency of transducers (MARÉCHAL *et al.*, 2008).

One way to study these structures is to calculate the transmission and reflection coefficients of the structure immersed in a fluid such as air or water. The problem can be resolved numerically. However, analytic solutions are still a great way to understand the physical mechanisms involved. Different approaches and techniques like the plane wave method (PWM) (POTEL, BELLEVAL, 1993; DESCHAMPS, CHENGWEI, 1991) and the transfer matrix method (TMM) (SOLYANIK, 1977; ROKHLIN, WANG, 1992; HASKELL, 1953; FOLDS, LOGGINS, 1977; LOWE, 1995) or the global matrix method (STORHEIM *et al.*, 2015) are often adopted. The iterative method (SCOTT, GORDON, 1977) or the equivalent impedance (MESSINEO *et al.*, 2013) can also be used.

Debye's series decomposition method (MARÉCHAL *et al.*, 2014; GÉRARD, 2022) allows developing the reflection and transmission coefficients into a sum of multiple reflection terms. Typically, some resonance terms are ignored. However, expressing the transmission coefficient into a sum of these terms is very useful for resonance analysis.

FIORITO and ÜBERALL (1979) showed that the acoustic transmission and the reflection coefficient of a fluid layer embedded in another fluid can be written in the form of a sum of resonance terms. The resonance theory of a fluid layer has been extended to include viscous effects (FIORITO *et al.*, 1981). Three layered elastic medium have been investigated (AINSLIE, 1995) using ray path analyses. The solution of the reflection/refraction of a plane wave at a single solid layer has been expanded into Debye's series (DESCHAMPS, CHENGWEI, 1991). Using a matrix notation and a generalized Debye theory, GÉRARD *et al.* (1979; 1980; 1982; 1987) derived an exact solution in an elastic multilayered sphere. The case of submerged cylinders (DEREM, 1982) and plates (CONOIR, 1991; DERIBLE, TINEL, 2011) were studied too. Earlier we exploited this resonance formalism to study the interaction of a bounded ultrasonic beam with an immersed plate

(SOUCRATI *et al.*, 2018). A global transfer matrix has been constructed to study the interaction of harmonic elastic waves with n -layered anisotropic medium (NAYFEH, 1991).

In the present work, firstly, we determined the transmission coefficient (T_N) through N -layered structure using the plane wave theory. The details of the calculus are given in Appendix A, the solution named an exact solution is then given in Eq. (1). This exact solution is broken down into series translating the individual contribution of each resonance. A novel way is applied to write this solution as a product of the Debye series. Then a new resonance model for the transmission (T_N) was developed. The model provides analytical expressions for the characteristics of each resonance. This facilitates the resonance decomposition of the transmitted wave and help understanding resonance phenomena. Further, the model provides a useful tool to solve the inverse problem. In the same manner, our model can be applied to the calculation of the reflection coefficient.

In this paper, we describe firstly the problem of the propagation of ultrasonic waves in multilayered media as well as the theoretical formalism that governs this propagation. Applying the pressure continuity and the particle velocity continuity at the two interfaces of each layer we derived the exact formulation of the transmission coefficient named T_N . Details are given in Appendix A. The result is given in Eq. (1). Then T_N is expanded into a sum of resonance terms like Debye's series. Details about the method used to expand the transmission coefficient into Debye's series are given in Appendix B. The expanded expressions are given in the Eqs. (10), (15), (18), and (22), respectively, for one, two, three, and N -layers. Finally, numerical evaluation of the transmission coefficient T_N given by the exact solution (Eq. (1)) and the expanded formulation are compared. The comparison shows good agreement subject to choosing the right number of resonance terms.

2. Theoretical formalism

2.1. Studied configuration

We consider a layered structure hit by an ultrasonic plane wave in normal incidence (Fig. 1). The structure to be analyzed is composed of N -layers indexed from 1 to N . Each layer is of the thickness d_n . Note

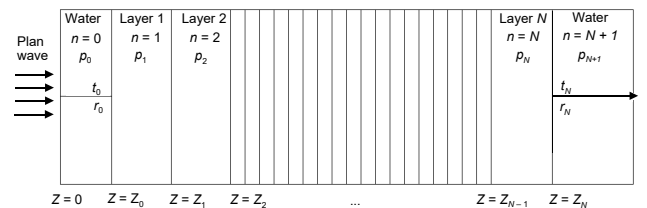


Fig. 1. Geometrical arrangement.

that the propagating vector is \mathbf{k}_n , the mass density is ρ_n , the layer's sound celerity is c_n , the acoustic impedance Z_n , and the attenuation coefficient α_n .

The structure is immersed in water characterized by its density ρ_0 , velocity c_0 , acoustic impedance Z_0 , and wave number k_0 . The surrounding medium is taken to be nonabsorbent, so $\alpha_0 = 0$. The second half medium surrounding the layers is also water and corresponds to the slice indexed $N + 1$. Then $Z_{N+1} = Z_0$.

The N -layered structure is hit, in a normal incidence, by a plane harmonic wave. So, only a longitudinal wave is to be considered. The transverse waves are not considered.

Plexiglas is used as a layer with $\rho = 1200 \text{ kg/m}^3$, $c = 2650 \text{ m/s}$, and $\alpha = 1.13 \text{ dB/[MHz} \cdot \text{cm}]$. For aluminum $\rho = 2800 \text{ kg/m}^3$, $c = 6380 \text{ m/s}$.

2.2. Transmission coefficient

We demonstrate (Appendix A) that the transmission coefficient Γ_N of N -layered structure can be expressed as:

$$\Gamma_N = T_N \frac{\varphi_N}{D_N} \quad (1)$$

with:

$$T_N = \prod_{n=0}^N t_n; \quad t_n = \frac{2Z_{n+1}}{Z_n + Z_{n+1}}, \quad (2)$$

$$\varphi_N = \prod_{n=1}^N X_n; \quad X_n = e^{-i\gamma_n d_n}; \quad \gamma_n = \mathbf{k}_n - i\alpha_n, \quad (3)$$

where T_N is the transmission coefficient through the N -layer structure, while φ_N is the accumulation of the phase induced by the propagation into the different N -layers.

Moreover, t_n correspond to:

- t_0 : transmission from water of the first half medium surrounding the structure, to the first layer (layer 1),
- t_N : transmission from the last layer (layer N) to the water of the second half medium surrounding the structure.

The transmission coefficient Γ_N consists of a fraction of two terms, namely the numerator ($T_N \varphi_N$) that takes into account transmission attenuation and the denominator (D_N) that takes into account multiple reflections at each interface. This latter can be expressed as $D_N = 1 + \Phi_N$, where Φ_N is responsible for reflections/refractions at all the interfaces.

A similar formula has been already given in (STOVAS, ARNSTEN, 2006). Indeed, starting from Eq. (40) in Appendix B, we can deduce:

$$D_N = D_1 + \sum_{m=0}^{N-1} r_{N-m} x_{N-m} D'_{N-m}. \quad (4)$$

So, one can see that D_N can be written in the form of:

$$D_N = 1 + \Phi_N \quad (5)$$

with:

$$\Phi_N = r_0 r_1 x_1 + \sum_{m=0}^{N-1} r_{N-m} x_{N-m} D'_{N-m}. \quad (6)$$

For $N > 1$, D_N is developed in a new manner to allow decomposition into series terms. Details are given in Appendix B. So D_N can be expressed in a simple way as:

$$D_N = D_{N-1} + r_N x_N \tilde{D}_{N-1} x_N = X_n^2, \quad (7)$$

$$r_n = \frac{Z_{n+1} - Z_n}{Z_n + Z_{n+1}},$$

where

$$\tilde{D}_N = r_N D_{N-1} + x_N \tilde{D}_{N-1}, \quad \tilde{D}_1 = r_1 + r_0 x_1, \quad (8)$$

where r_N is the reflection coefficient at the last interface from water to the layer N . It is expressed as:

$$r_N = \frac{Z_{N+1} - Z_N}{Z_N + Z_{N+1}} = \frac{Z_0 - Z_N}{Z_N + Z_0}. \quad (9)$$

Now, the idea is to expand Γ_N into a sum of resonance terms as Debye's series.

For one layer ($N = 1$), we have:

$$\Gamma_1 = T_1 X_1 \sum_{n_1=0}^{\infty} (-r_0 r_1 x_1)^{n_1}. \quad (10)$$

For more than one layer ($N > 1$), we derive an expression of D_N (Appendix B) that allows expanding Γ_N into Debye's series.

Let us start with two layers $N = 2$:

$$\frac{1}{D_2} = \frac{1}{C_1} \sum_{n_2=0}^{\infty} (-\beta_1 r_2 x_2)^{n_2}. \quad (11)$$

We express β_1 as:

$$\beta_1 = \frac{1}{r_1} \left(1 - \frac{t_1 t'_1}{C_1} \right), \quad (12)$$

thus:

$$\frac{1}{D_2} = \frac{1}{C_1} \sum_{n_2=0}^{\infty} \left(-\frac{r_2}{r_1} x_2 \right)^{n_2} \sum_{m=0}^{n_2} \frac{n_2!}{m! (n_2 - m)!} \left(-\frac{t'_1 t_1}{C_1} \right)^m. \quad (13)$$

The term $(1/C_1)^m$ can be expanded in series as:

$$\frac{1}{C_1^m} = \sum_{n_1=0}^{\infty} (-r_0 r_1 x_1)^{n_1} \frac{(m + n_1)!}{m! n_1!}. \quad (14)$$

Replacing Eq. (10) in (9), we get the final form of the transmission coefficient through two layers:

$$\Gamma_2 = T_2 \varphi_2 \sum_{n_2=0}^{\infty} (-r_2 x_2)^{n_2} \sum_{n_1=0}^{\infty} W_{n_2 n_1} (-r'_0 x'_1)^{n_1} \quad (15)$$

with:

$$W_{n_2 n_1} = r_1^{n_1 - n_2} \frac{n_2!}{n_1!} \sum_{m=0}^{n_2} \frac{(m+n_1)!}{m!m!(n_2-m)!} (-t'_1 t'_1)^m. \quad (16)$$

Let us rewrite the two first terms of Γ_2 with respect to n_2 , in order to give a physical interpretation of each of them:

$$\Gamma_2 = T_2 \varphi_2 \left\{ \sum_{n_1=0}^{\infty} (-r_0 r_1 x_1)^{n_1} + \sum_{n_1=0}^{\infty} \left[1 - (n_1+1)(-r_2 x_2) \frac{t_1 t'_1}{r_1} \right] (-r_0 r_1 x_1)^{n_1} + \dots \right\}. \quad (17)$$

The first summation in Eq. (13) counts for the resonance into the first layer and transmitted through the second (Fig. 2a). The second summation term is obtained for $n_2 = 1$ and composed of two components: the first sum is the same as for $n_2 = 0$, the second sum counts for the reflection from the interface between layer 2 and the ambient medium followed by resonances into layer 1 (Fig. 2b). The coefficient (n_1+1) in the second sum is due to the Fabry-Pérot like effect, which means that the response is composed of many echoes that arrive at the same time as shown in Fig. 2.

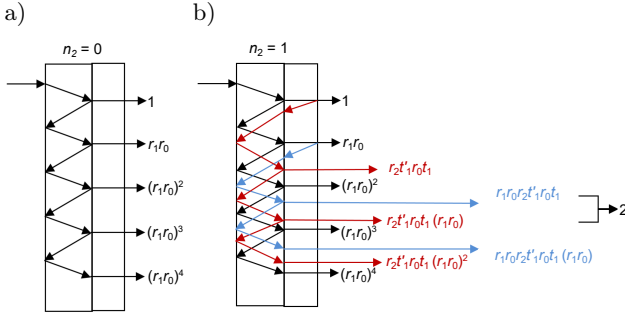


Fig. 2. Schematic of (a) the first layer resonance and transmission ($n_2 = 0$) and (b) the multiple reflections from the second layer and ambient medium interface ($n_2 = 1$).

In the same manner we derive the expression for three adjacent layers:

$$\Gamma_3 = T_3 \varphi_3 \sum_{n_2=0}^{\infty} (-x_2)^{n_2} \sum_{n_1=0}^{\infty} W_{n_2 n_1} (-r'_0 x'_1)^{n_1} \cdot \sum_{n_3=0}^{\infty} W_{n_2 n_3} (-r'_3 x'_3)^{n_3}, \quad (18)$$

where

$$W_{n_2 n_3} = r_2^{n_3 - n_2} \frac{n_2!}{n_3!} \sum_{m=0}^{n_2} \frac{(m+n_3)!}{m!m!(n_2-m)!} (-t'_2 t'_2)^m. \quad (19)$$

In the case of four layers, we have:

$$\Gamma_4 = T_4 \varphi_4 \sum_{n_2=0}^{\infty} (-x_2)^{n_2} \sum_{n_1=0}^{\infty} W_{n_2 n_1} (-r'_0 x'_1)^{n_1} \cdot \sum_{n_3=0}^{\infty} W_{n_2 n_3} (-x'_3)^{n_3} \sum_{n_4=0}^{\infty} W_{n_3 n_4} (-r'_4 x'_4)^{n_4} \quad (20)$$

with:

$$W_{n_3 n_4} = r_3^{n_4 - n_3} \frac{n_3!}{n_4!} \sum_{m=0}^{n_3} \frac{(m+n_4)!}{m!m!(n_3-m)!} (-t'_3 t'_3)^m. \quad (21)$$

In the general case of N -layers with $N > 3$:

$$\Gamma_N = T_N \varphi_N \sum_{n_2=0}^{\infty} (-x_2)^{n_2} \sum_{n_1=0}^{\infty} W_{n_2 n_1} (-r'_0 x'_1)^{n_1} \cdot \prod_{j=3}^{N-1} \left[\sum_{n_j=0}^{\infty} W_{n_{j-1} n_j} (-x'_j)^{n_j} \right] \cdot \sum_{n_N=0}^{\infty} W_{n_{N-1} n_N} (-r'_N x'_N)^{n_N}, \quad (22)$$

where

$$W_{n_j n_{j+1}} = r_j^{n_{j+1} - n_j} \frac{n_j!}{n_{j+1}!} \sum_{m=0}^{n_j} \frac{(m+n_{j+1})!}{m!m!(n_j-m)!} (-t'_j t'_j)^m. \quad (23)$$

Γ_N can be interpreted as a sum of waves of the amplitude W_{ij} reflected at each interface.

We derived here a more generalized formula than those based on the amplitude of echoes derived in the reflection mode (CHERN, NIELSEN, 1989) and through the transmission mode (CHERN, NIELSEN, 1990).

3. Simulation results

In this section, we show the validity of the proposed model by simulating the transmission coefficient of different configurations of multilayered structures. The cases of one, two, and three layers are studied for different numbers of resonances.

The exact expression of the transmission coefficient Γ_N through an N -layer structure is given by Eq. (1). The series expansion of the resonance term is given by Eqs. (10), (17), and (18), respectively for 1, 2, or 3 layers. In all these expressions we have infinite sums which are truncated during the simulation. The simulation results are then compared to the calculations of Γ_N using Eq. (1) to determine firstly the degree of truncation and secondly to validate our model.

3.1. One-layer case

Figure 3 gives the transmission coefficient versus frequency, for one layer made of 4 mm of plexiglass. The exact solution (red curve) is compared to our model (blue curve). The results in Fig. 3a are given for $n_1 = 1$ which means that the series in Eq. (10) is truncated to the first term. However in Fig. 3b the series is truncated to the 5 first terms. So the maximum number of resonance taken into account should be determined according to the acoustical parameters of the layer. The same results for aluminum plate are given in Fig. 4.

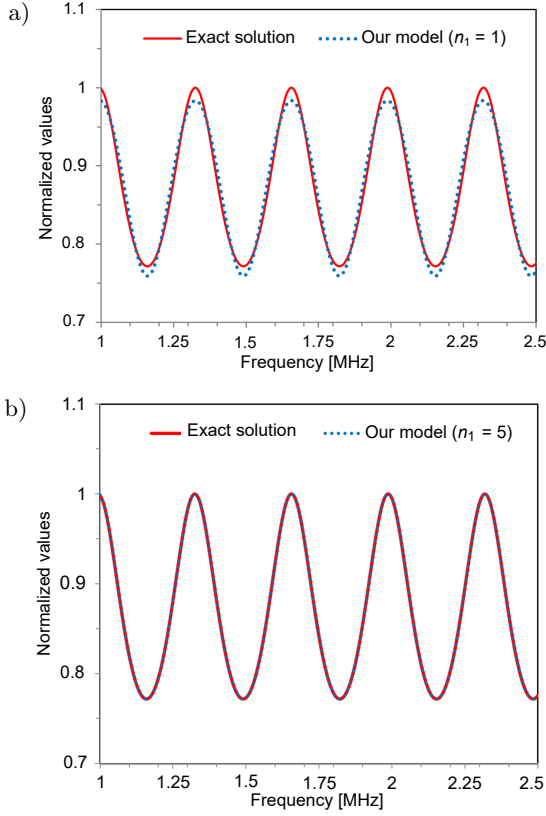


Fig. 3. $|T_1|$, versus frequency, for 4 mm plexiglas plate: exact solution (red), our model for $n_1 = 1$ (blue in a), and $n_1 = 5$ (blue in b).

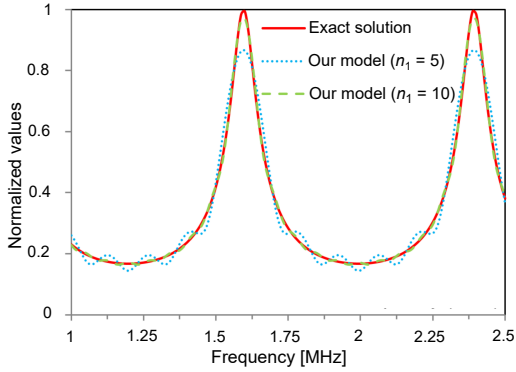


Fig. 4. $|T_1|$, versus frequency, for 4 mm aluminum plate: exact solution (red), our model for $n_1 = 5$ (blue), and $n_1 = 10$ (green).

As expected, Fig. 3 and Fig. 4 show the well-known resonance frequency due to different modes of propagation in the layer.

Discrepancies are noticed, especially for the minima and maxima of the curves, if the number of resonances taken into account is insufficient (blue curve Figs. 3a and 4). To match the exact solution one need to define a precision ε . Then the minimal value of n_i ($i = 1, 2, \dots, N$) should be determined according to this accuracy ε .

The minimal value of n_1 is determined according to the expected accuracy using this expression:

$$|r_0 r_1|^{n_1} < 10^{-\varepsilon} \quad \text{then} \quad n_1 \geq \frac{-\varepsilon}{\log |r_0 r_1|}. \quad (24)$$

If we take $\varepsilon = 3$ for example, one should take the summation of the 20 terms for aluminum and only 4 terms for plexiglas.

3.2. Two layers

We studied a structure made of two layers aluminum/polyethylene (Al/PE) immersed in water. Each layer has 4 mm thickness. Transmission coefficient variation according to the frequency is presented in Fig. 5 for the Al/PE structure. This structure has been studied theoretically and experimentally using the reflection response (LENOIR, MARÉCHAL, 2009).

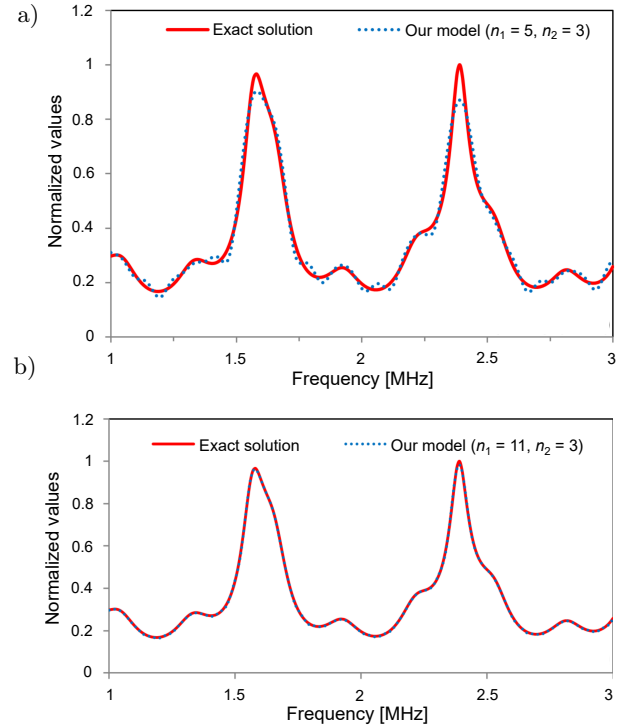


Fig. 5. Variation of $|T_2|$ versus frequency: exact solution (red), our model with $n_2 = 3$ (blue), $n_1 = 5$ in (a), and $n_1 = 11$ in (b).

The minimal value of n_1 is determined using Eq. (24). Taking $\varepsilon = 10^{-2}$ we deduce $n_1 = 11$. For n_2 we should use according to Eq. (7):

$$|\beta_1 r_2|^{n_2} < 10^{-\varepsilon} \quad \text{then} \quad n_2 \geq \frac{-\varepsilon}{\log |\beta_1 r_2|}. \quad (25)$$

If we take $\varepsilon = 2$ it gives $n_1 = 11$ and $n_2 = 5$. Using these minimal values, the approximated solution fits well with the exact one as it is seen in Fig. 5b.

3.3. Three layers

Let us now study the structure made of two plexiglas plates of 5 mm separated by 1 mm of water (Fig. 6).

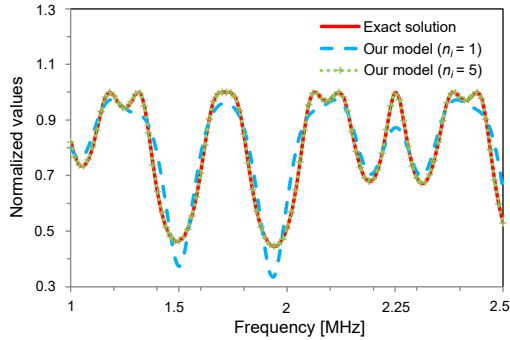


Fig. 6. $|\Gamma_3|$ versus frequency for plexiglas/water/plexiglas of 5/1/5 mm structure: exact solution (red), our model (blue $n_1 = n_2 = n_3 = 1$ and green $n_1 = n_2 = n_3 = 5$).

We can notice from simulation results, that our model based on Debye's series gives very good results for the calculation of the ultrasound transmission coefficient in a multilayered structure. Especially if the number of resonance terms used in calculation is well chosen.

4. Conclusion

In this work, we present a new analytical method for calculating the acoustic frequency response of a multilayered structure. The transmission coefficient calculated using this new method is put into a sum of resonance terms. This way, we can consider the relative amplitude of each wave as a summation of several terms taking into account all possible reflections/refractions.

The expanded solution is in good agreement with the exact solution, subject to take a suitable number of resonances. We have also proposed a method helping defining this resonances' number. Our method gives a more generalized formula than other methods based on reflection or transmission modes. Our work can also help extracting geometrical and acoustical parameters of each layer. The considered layers can be either liquid or solid, since the incidence is normal to avoid the shear waves that are not taken into account by the formula.

Instead of a plane wave, our model can be associated with superposition of a bounded beam to find the 3D diffracted field by a multilayered structure. In the same manner we derived an analytical expression of a 3D ultrasonic field transmitted through a single plane layer (SOUCRATI *et al.*, 2018).

Appendix A

For, $n = 1$ to N , we express the ultrasonic pressure p_n inside each layer as ($z_0 < z < z_N$):

$$p_n = U_n e^{-i\gamma_n(z-z_{n-1})} + V_n e^{i\gamma_n(z-z_{n-1})}. \quad (26)$$

The pressure wave in the first half of embedded medium (water $n = 0$) is then ($0 < z < z_0$):

$$p_0 = U_0 e^{-i\gamma_0 z} + V_0 e^{i\gamma_0 z}. \quad (27)$$

The transmitted wave in water after the layered structure ($z > z_N$) is written as:

$$p_{Nt} = U_t e^{-i\gamma_0(z-z_N)}. \quad (28)$$

We put η_n as the ration of the impedance of the slice before to that of the next slice:

$$X_n = e^{-i\gamma_n d_n}, \quad \eta_n = \frac{Z_n}{Z_{n+1}}. \quad (29)$$

So from the water to the first layer we have:

$$X_0 = e^{-i\gamma_0 d_0}, \quad \eta_0 = \frac{Z_0}{Z_1}, \quad (30)$$

and from the last layer to the water we write:

$$X_N = e^{-i\gamma_N d_N}, \quad \eta_N = \frac{Z_N}{Z_{N+1}} = \frac{Z_N}{Z_0}. \quad (31)$$

The origin of the propagation axis is placed at the interface between layers n and $n - 1$. By applying the pressure continuity (INGARD, MORSE, 1968) and particle velocity continuity equations at each interface and solving them, we get the ratio of the sound pressure level of the transmitted wave (U_t) to the incident wave (U_0). This corresponds to the transmission coefficient (Γ_N). In the same way we can deduce the reflection coefficient.

For one layer, we have:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \frac{1}{\eta_0} \begin{bmatrix} \eta_0 & \eta_0 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} U_1 \\ V_1 \end{pmatrix}, \quad (32)$$

$$\frac{1}{X_1} \begin{bmatrix} x_1 & 1 \\ x_1 & -1 \end{bmatrix} \begin{pmatrix} U_1 \\ V_1 \end{pmatrix} = U_{1t} \begin{pmatrix} 1 \\ \eta_1 \end{pmatrix}. \quad (33)$$

Solving this system, we get the expression of Γ_1 as:

$$\Gamma_1 = \frac{U_{1t}}{U_0} = \frac{t_0 t_1 X_1}{1 + r_0 r_1 x_1}. \quad (34)$$

For a structure of two layers $N = 2$, we have to add this equations system for $n = 1$:

$$\frac{1}{X_n} \begin{bmatrix} x_n & 1 \\ x_n & -1 \end{bmatrix} \begin{pmatrix} U_n \\ V_n \end{pmatrix} = \frac{1}{\eta_n} \begin{bmatrix} \eta_n & \eta_n \\ 1 & -1 \end{bmatrix} \begin{pmatrix} U_{n+1} \\ V_{n+1} \end{pmatrix}, \quad (35)$$

$$\frac{1}{X_N} \begin{bmatrix} x_N & 1 \\ x_N & -1 \end{bmatrix} \begin{pmatrix} U_N \\ V_N \end{pmatrix} = U_{Nt} \begin{pmatrix} 1 \\ \eta_N \end{pmatrix}. \quad (36)$$

Solving this system, we get Γ_2 :

$$\Gamma_2 = \frac{U_{2t}}{U_0} = \frac{t_0 t_1 t_2 X_1 X_2}{1 + r_0 r_1 x_1 + r_1 r_2 x_2 + r_0 r_2 x_1 x_2}. \quad (37)$$

For N layers, we can get the expression of the coefficient of transmission using the same technique:

$$\Gamma_N = \frac{U_N t}{U_0} = \frac{T_N \varphi_N}{D_N}, \quad (38)$$

$$T_N = t_0 t_1 \dots t_N, \quad \varphi_N = X_0 X_1 \dots X_N.$$

We notice that D_N is expressed as a sum of combination of all the possible product of x_n .

For 3 layers for example, we have $x_1, x_2, x_3, x_1 x_2, x_1 x_3, x_2 x_3$, and $x_1 x_2 x_3$. So D_3 is expressed as:

$$D_3 = 1 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_{12} x_1 x_2 + c_{13} x_1 x_3 + c_{23} x_2 x_3 + c_{123} x_1 x_2 x_3, \quad (39)$$

where the coefficients c_i are defined as the product of the reflection coefficient r_n at all the interfaces in which the considered layers are not adjacent.

If we take only one layer from N , we have $c_n = r_{n-1} r_n$. So, for example $c_1 = r_0 r_1$.

If we take m layers from N , we have $\frac{N!(N-m)!}{m!}$ possibilities and we should consider the reflection coefficients at the interfaces where the layers considered are not adjacent.

If we consider two layers from three we have three possibilities $x_1 x_2, x_1 x_3$, and $x_2 x_3$ (see Fig. 7).

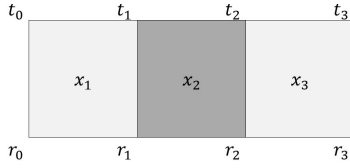


Fig. 7. Theoretical scene we consider three different layers.

For layer 1/layer 2 case, we have $c_{12} = r_0 r_2$. There are only two free interfaces. For layer 2/layer 3 case, we have $c_{23} = r_1 r_3$. There are, again, only two free interfaces. However, for the layer 1/layer 3 case, we got $c_{13} = r_0 r_1 r_2 r_3$. There are four free interfaces.

The expression of every D_N can be deduced using this technique.

Appendix B

We are interested to determine the transmission coefficient Γ_N of the layered structure. The idea is to express the denominator D_N as a sum of D_{N-n} where n goes from 1 to N . We noticed that D_N can be put into the form:

$$D_N = D_{N-1} + r_N x_N \tilde{D}_{N-1}, \quad (40)$$

where \tilde{D}_{N-1} can be written as:

$$\tilde{D}_{N-1} = r_{N-1} D_{N-2} + x_{N-1} \tilde{D}_{N-2}, \quad \tilde{D}_1 = r_1 + r_0 x_1, \quad (41)$$

which yields:

$$D_N = D_{N-1} + r_N r_{N-1} x_N (D_{N-2} + x_{N-1} D'_{N-2}), \quad (42)$$

we put:

$$C_n = 1 + r_n r_{n-1} x_n, \quad (43)$$

$$C_1 = D_1 = 1 + r_1 r_0 x_1 E_n = r_{n-1} + r_n x_n.$$

For two layers:

$$D_2 = D_1 + r_2 x_2 \tilde{D}_1 = C_1 (1 + \beta_1 r_2 x_2) = C_1 C_{21} \quad (44)$$

with:

$$C_{21} = 1 + \beta_1 r_2 x_2, \quad \beta_1 = \frac{\tilde{D}_1}{C_1} = \frac{1}{r_1} \left(1 - \frac{t_1 t'_1}{C_1} \right). \quad (45)$$

For three layers we arrange D_3 in the form:

$$\begin{aligned} D_3 &= D_2 + r_3 x_3 \tilde{D}_2 \\ &= D_1 + r_2 x_2 \tilde{D}_1 + r_3 x_3 (r_2 D_1 + x_2 \tilde{D}_1) \\ &= C_1 C_3 + x_2 \tilde{D}_1 E_3 = C_1 C_3 C_{31}, \end{aligned} \quad (46)$$

where

$$C_{23} = 1 + \beta_1 \beta_3 x_2, \quad \beta_3 = \frac{E_3}{C_3} = \frac{1}{r_2} \left(1 - \frac{t_2 t'_2}{C_3} \right). \quad (47)$$

For four layers:

$$\begin{aligned} D_4 &= D_3 + r_4 x_4 \tilde{D}_3 \\ &= C_4 (D_2 + \beta_4 x_3 \tilde{D}_2) \\ &= C_1 C_4 (C_{34} + \beta_1 x_2 (r_2 + \beta_4 x_3)), \end{aligned} \quad (48)$$

$$D_4 = C_1 C_4 C_{24} C_{341} \quad (49)$$

with:

$$\begin{cases} C_{34} = 1 + \beta_4 r_2 x_3, \\ C_{24} = 1 + \beta_1 \beta_3 x_2, \\ \beta_4 = \frac{E_4}{C_4} = \frac{1}{r_3} \left(1 - \frac{t_3 t'_3}{C_4} \right), \\ \beta_{34} = \frac{r_2 + \beta_4 x_3}{C_{34}}. \end{cases} \quad (50)$$

Generalizing for N layers D_N is put in the form:

$$D_{N \geq 4} = C_N C_1 \prod_{m=2}^{N-1} C_{mN}, \quad (51)$$

$$\begin{cases} C_n = 1 + r_n r_{n-1} x_n, \\ C_1 = 1 + r_0 r_1 x_1, \\ C_{2N} = 1 + \beta_1 \beta_{3N} x_2, \\ C_{mN} = 1 + \beta_{m+1N} r_{m-1} x_m, \\ \beta_n = \frac{E_n}{C_n} = \frac{1}{r_{n-1}} \left(1 - \frac{t_{n-1} t'_{n-1}}{C_n} \right), \\ \beta_{mN} = \frac{r_{m-1} + \beta_{m+1N} x_m}{C_{mN}}. \end{cases} \quad (52)$$

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