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FACTORS THAT INFLUENCE THE CALCULATION OF ACOUSTIC SCATTERING BY THE METHOD OF SOURCE SIMULATION

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In this work the source simulation technique was used to calculate the scattering of a plane wave by a cylinder with radial or elliptical transverse section. The basic idea of the source simulation technique is to replace the scattering (or radiating) body with a system of simple sources located within the envelope of the scatterer (or radiator). The extent to which the simulated field reproduces the original one depends on the degree of correspondence between the simulated and the given boundary conditions. Numerical simulations have shown that: 1) the shape of the auxiliary surface, 2) the number of sources, and 3) the way the sources are distributed are the most relevant parameters to ensure an accurate solution of the problem. In the case of the single-layer method, the sources should not be positioned close to the surface or to the center of the body, because the problem becomes ill-conditioned. The auxiliary surface and the scatterer should be as similar as possible in order to minimize the boundary error. With respect to the number of sources (N), there are two opposite effects: 1) if (N) is too small, the sound field is not reproduced accurately; 2) if (N) is too large, the computing time increases and the solution accuracy decreases. The method breaks down when the excitation frequency coincides with the eigenfrequencies — a narrow range of frequencies — of the space formed by the auxiliary surface. As the auxiliary surface is frequently represented by simple surfaces (cylinder, sphere), one can easily calculate the eigenfrequencies and therefore avoid them.

1. Introduction

The mathematical treatment of radiation and acoustic scattering represents a very old and much studied problem by mathematical physics (see [1] and [2]). Radiation and scattering are present in all ondulatory phenomena (elastic waves in rigid bodies, electromagnetic waves, surface waves on the water, etc). The present study, however, deals only with "pure" acoustical waves, that is, acoustical waves in gases or liquids. Another important limitation is that all steps of the solution of the problem are considered linear. Consequently, the Superposition Principle is valid. As radiation and scattering represent classical problems of mathematical physics, there is a vast literature on the subject. The solution methods can be classified into three different groups:

a) Analytical methods, which make possible an exact solution for the problem. However, these methods are limited to a few very simple geometric figures (sphere, cylinder).

The classical works of LORD RAYLEIGH [1], P. MORSE [2] and many other authors present the analytical solution for the radiation and acoustic scattering problems in bodies with spherical and cylindrical shapes.

b) Semi-analytical and semi-empirical methods for the calculation of radiation and acoustic scattering. The utilization of the radiation coefficient, for example, belongs to this category (e.g. GÖSELE [3] and CREMER and HECKL [4]).

c) Numerical methods in which the problem is approached by a set of differential or integral equations. In this case the equations model a linear system, which is in general very large. It is meant here the use of the finite elements method (FEM) or the boundary elements method (BEM). The relatively new method of simulation by elementary sources also belongs to this group.

The method of simulation by elementary sources leads, similarly to FEM and BEM, to the solution of large systems of equations. It comes into question what are the differences between those methods. From the literature [4, 7], one can say:

— In FEM, the number of equations to be solved depends on the volume of the space considered in the solution of the problem. As this number (for problems of radiation and acoustic scattering) increases with the third power of the relationship D/λ (D is a dimension typical of the space of the solution of the problem and λ is the sound wave length), one can expect for high frequencies a system of equations with extremely large matrices.

— In BEM, the number of elements to be calculated depends on the size of the surface of the body considered. The number of equations to be solved increases with the second power of the relationship D/λ, where D is a dimension typical of the body and not of the total solution space of the problem. One can have, in consequence, with BEM a significantly smaller number of equations than with FEM. However, the matrix elements are practically all non-zero numbers, and as a result one has full matrices. The great problem in the use of BEM is that there are no solutions for the problem close to the characteristic frequencies, which correspond to the resonance frequencies of the internal space delimited by the surface of the body. Several methods were developed in order to solve this problem.

— In the method of simulation by elementary sources there is in principle no minimal number of elements to be utilized. However, the number of elements has a crucial role in the accuracy attained by the solution of the problem. BOBROVNITSKII [7] states that the source method is the only one among those here cited that allows for a quantitative evaluation of the the precision of the calculations. This statement is however only valid for the precision with which the velocity of the surface of the body is approached by the elementary sources. A statement about the accuracy of the calculation of intensity and potency has not been shown possible until now. Likewise, in BEM the critical frequencies exist. These frequencies correspond to the resonance frequencies of the internal surface over which the sources are positioned (e.g., monopoles). Choosing an adequate internal surface, as will be demonstrated later in this work, this problem can be easily overcome. Using an open internal surface or a multipole positioned only in one point in the interior of the body, the resonance frequencies do not appear.

2. Description of the scattering problem and the source simulation technique

Considering a harmonic wave with sound pressure amplitude p_e in an infinite and homogeneous space E which encounters in its displacement the body K, internal space is defined by I, the scattered wave by p_s and the surface of the body by S. Over the surface of the body the unitary vector $\mathbf{n}$ is defined:

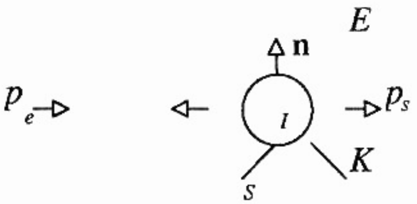

Fig. 1. Geometry of the acoustic scattering problem.

Pressure and velocity of the particle can be determined as the result of the sum of the components p_e and p_s. Respectively,

$$p_t = p_e + p_s \qquad \text{and} \quad v_t = v_e + v_s\,. \tag{2.1}$$

The complex sound pressure p_t has to satisfy the Helmholtz equation:

$$\Delta p_t + k^2 p_t = 0 \tag{2.2}$$

in E, where $k = \omega/c$ is the wave number, ω is the circular frequency, c the speed of sound, and Δ is the Laplace operator. Since sound scattering into the free three-dimensional space is considered, the pressure p_t has to satisfy the Sommerfeld radiation condition:

$$\lim_{r\to\infty}\left[\frac{\partial p_t}{\partial n} + jkp_t\right]\cdot r = 0 \tag{2.3}$$

which can be interpreted as a boundary condition at infinity. Here,

$$r = \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

denotes the distance from x to the origin, where we represent points in space by simple letters like $x = (x_1, x_2, x_3)$. Solutions of the Helmholtz equation in E which also satisfy the radiation equation condition are called radiating wave functions. To get a complete description of the problem, boundary conditions on the surface are needed. For simplicity, we only consider the Neumann boundary value problem where the normal velocity and therefore the gradient of pressure

$$\partial p/\partial n = -j\omega\rho v \tag{2.4}$$

is prescribed [8]. Here ρ is the fluid density and ∂/∂_n is the derivative in the direction of the outward normal.

The principle of the method here presented is based on a treatment of the scattering problem through a system of radiating sources, which should be chosen so that they reproduce as well as possible the sound field generated by the body of Fig. 1. The sources are taken as point sources, and therefore do not represent an obstacle to the sound field. As a consequence the field generated by each one can be summed without taking into consideration interference effects. As the sources are known, i.e., their amplitudes, the sound field can then be easily calculated through the sum of the fields generated by each source individually. The true problem consists then in finding the sources that can best replace the original body. As a consequence, two important questions arise:

a) Which is the type of source to be used and how should they be placed inside the body?

b) Which optimization method should be employed for the results?

Mathematically the problem is based on representing the sound field by summing up the contributions of the individuals sources

$$p = \sum_{q=1}^{N_q} \sum_{m=-\infty}^{\infty} A_{q,m} \Phi_{q,m} , \qquad (2.5)$$

where p represents the scattered pressure or the radiated pressure in the field; $A_{q,m}$ is the complex source strength of the q-th source at a point x_q in the field; m is the order of each source and $\Phi_{q,m}$ is the sound field generated by the sources. In Eq. (2.5) $\Phi_{q,m}$ could also be called the source function [8]. Equation (2.5) intrinsically has the condition that each field can be represented by a sum of functions of the type $\Phi_{q,m}$. This is naturally the case, only if all functions $\Phi_{q,m}$ satisfy the wave equation and if they form a complete function system. The first condition is certainly satisfied if $\Phi_{q,m}$, represents for example the field generated by a monopole, dipole or quadrupole. The second condition, i.e., if it is possible to represent any acoustic field as a sum in the form of Eq. (2.5), seems to be as yet unproven with all mathematical rigor [8]. As no difficulty has been noted by other authors (CREMER [9]; HECKL [10], OCHMANN [11]) when miltipole sources were used for the reconstruction of the acoustic field, the same procedure will be used in the present work. In other words, the multipole sources will be used to represent the radiation or scattering problem of the original body.

We have then two distinct situations:

a) one can use a variable order multipole localized in a single point inside the body, that is, in Eq. (2.5) $N_q = 1$ and M is very large,

or

b) one can use only monopole sources positioned in several points inside the body, which renders N_q very large and $M = 1$ in Eq. (2.5).

One can also have a combination of both extreme cases presented in a) and b), that is, to use a multipole (for example, monopole + dipole) positioned in several points. Together with the choice of the type and the positioning of the sources, the choice of the optimization criterium also imposes a fundamental question for the use of the source simulation technique. Basically the idea is to try to approximate the field generated by the sources, determining its source strength (which are ultimately the solution of the problem), to the real field generated by the original body. The error derived from this approximation should be minimized. Several methods can be used to that end, such as the null field method, the collocation method, the Cremer's method [8]. In this work we have used the least squares minimization.

3. Influence parameters

The parameters that influence the performance of the method, that is, the capacity of the method to reproduce boundary conditions are the following: the type and number of sources, their positioning in the interior of the body, the shape of the source surface, and the existence of critical frequencies. For that purpose a surface error will be defined, which will aid in the evaluation of the performance of the above mentioned parameters:

$$E(\%) = \sum_{i=1}^{N_\varphi} \frac{|Z_m - Z_\varphi|^2}{|Z_m|^2}\,, \tag{2.6}$$

where Z_m is the measured impedance and Z_φ is the calculated impedance with the source simulation technique for each angle φ around the geometry of the studied body.

3.1. Type of sources

By definition the sources must be radiating wave functions. It is convenient to work with available analytical functions. Only solutions of the Helmholtz equation in separable coordinate systems can be constructed explicitly [8]. In practice, spherical radiators (for three-dimensional problems) or cylindrical radiators (for two-dimensional problems). In the usual systems of coordinates, the wave functions or sources will be represented by:

— generalized spherical coordinates

$$\phi = P_{m_1}^{m_2} \cos(\vartheta) h_{m_1}^{(2)}(kr) e^{jm_2\varphi};$$

— spherical coordinates independent from φ

$$\phi = P_m \cos(\vartheta) h_m^{(2)}(kr);$$

— cylindrical coordinates

$$\phi = H_m^{(2)}(kr) e^{jm\varphi},$$

where P are the Legendre polynomials; $h_m^{(2)}$ the spherical Hankel function of the second order and $H_m^{(2)}$ the cylindrical Hankel function of the second order.

3.2. Location of the sources

A general assumption of the source simulation technique is that the sources must be located in the interior of the closed surface S. It is also possible to put sources on the boundary itself. But this leads to boundary integral equations and to the corresponding BEM, which are not topics of this paper. For the choice of the source location, essentially two alternatives are possible [8]: 1) only a few source locations are chosen, but at these locations a source with increasing order is used; or 2) a continuous source distribution of simple sources on an inner auxiliary surface is employed. The contrast between both methods is the greatest if a closed auxiliary surface with a layer of monopoles is chosen as one extreme and a infinite series of multipoles at only one source location as the other extreme [8]. The first method is called "the single-layer method" and the second one "the one-point multipole method" [8]. If the geometry of the body is spherical or cylindrical (or not far from those), the use of the one-point multipole method is recommended with the multipole located in the center of the body. This procedure facilitates the convergence of the wave functions and reduces computing time. On the other hand, sources positioned very close to the center when using the single-layer method tend to cause the matrix of linear equations to became more ill-conditioned, leading to an increased surface error (see Fig. 2 and Fig. 3). If the sources are positioned very close to the boundary, the accuracy will deteriorate due to the inadequate integration of the source singularity.

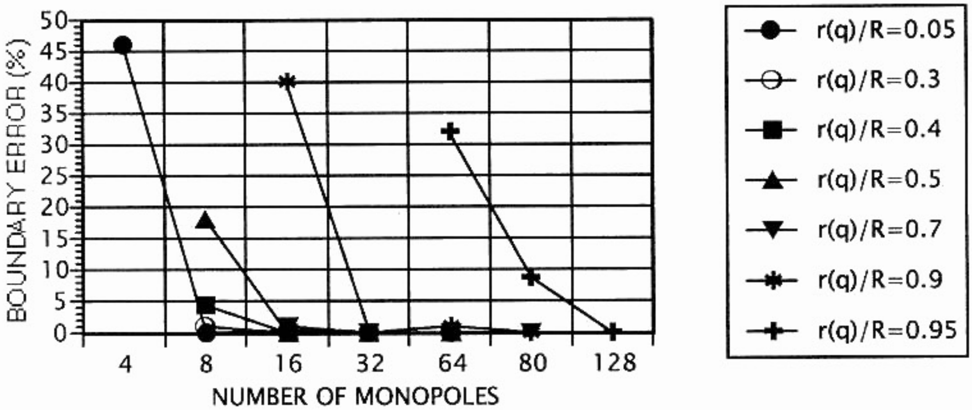

Fig. 2. Error when satisfying the boundary error as a function of the position of the source surface with radius $r(q)$ and a circular cylinder with radius R, $kR = 0.73$.

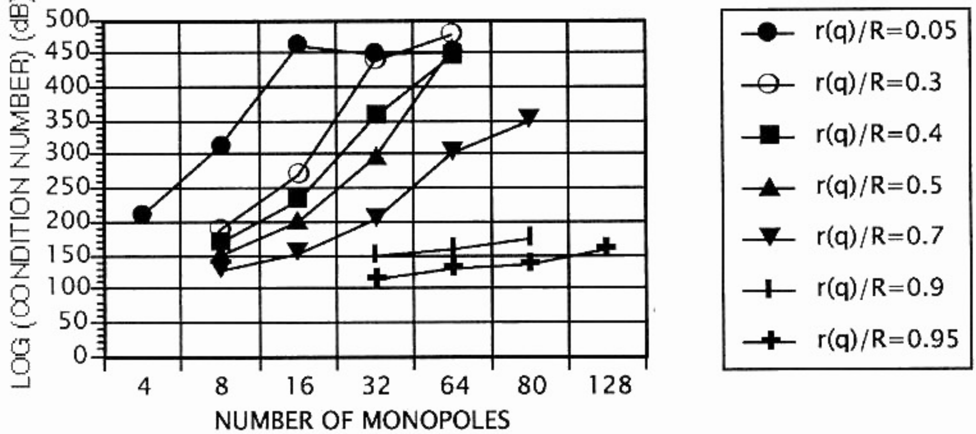

Fig. 3. Condition number in dB as a function of the position of the source surface with radius $r(q)$ and a circular cylinder of radius R, $kR = 0.73$.

There is again substantial increase in computing time due to the increase in the number of sources necessary in order to minimize the surface error. Our findings are not in agreement with TOMILINA [12] and BOBROVNITSKII and TOMILINA [13], who say that the source surface should be close to the body surface in order to improve the accuracy of the problem of reducing the boundary error. This is only correct when kR is very large.

3.3. Number of sources

The number of sources is influenced by several parameters, but mainly by the geometry of the body and the type of the source. In the determination of the number of sources, with respect to the single-layer method considering a circular cylinder, an expression was intended which would give the smallest number of monopoles necessary to satisfy the boundary conditions. Additionally, the position $r(q)/R$ (see Figs. 2 and 3) of the source surface should also comply with the assumption of a minimal number of monopoles.

The simulation leads us to the following:

$$N = \beta \cdot kR, \tag{2.7}$$

where β is an unknown factor, and R is the cylinder radius. This expression establishes a relationship between the wave number, the size of the body, and the number of monopoles. With the simulation the following values have been found for:

a) for $0.73 \leq kR \leq 1.46$ and $r(q)/R = 0.4 \rightarrow \beta = 16$,
b) for $1.83 \leq kR \leq 3.66$ and $r(q)/R = 0.5 \rightarrow \beta = 8$,
c) for $4.58 \leq kR \leq 7.33$ and $r(q)/R = 0.6 \rightarrow \beta = 6$,
d) for $9.16 \leq kR \leq 23.10$ and $0.7 \leq r(q)/R \leq 0.8 \rightarrow \beta = 4$.

3.4. Shape of the source surface

One aspect rarely considered in the utilization of the source simulation technique is the shape of the source, that is, the shape of the auxiliary surface over which the sources are positioned (see ZANNIN [5] and see Subsec. 3.2). The object of study here is a circular cylinder, and for the source surface the following shapes have been used: a) cylindrical, b) elliptical, c) non-elliptical (see Fig. 4). Surface error could be minimized and boundary condition satisfied in all tested cases. The number of monopole sources needed grew in direct proportion to the deviation of the source surface from a circular cylindrical shape (case "a"). This can be observed in Table 1 below:

Table 1. Influence of the shape of the source surface in minimizing boundary error.

Shape	Error (%)	Sources — N
(a)	0.01	8
(b)	0.0005	40
(c)	0.5	20

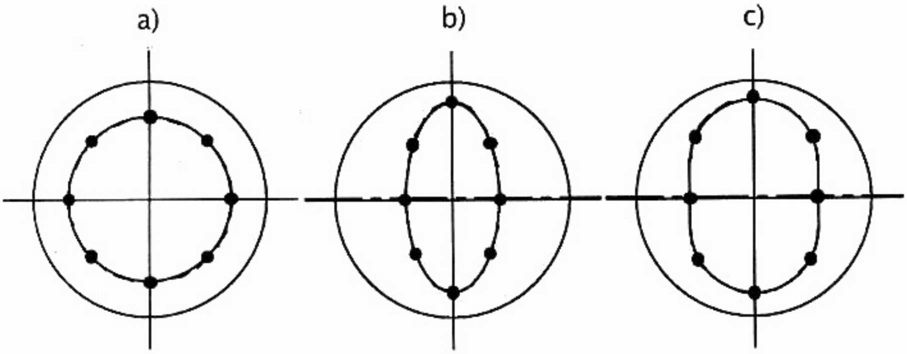

Fig. 4. Shape of the source surface.

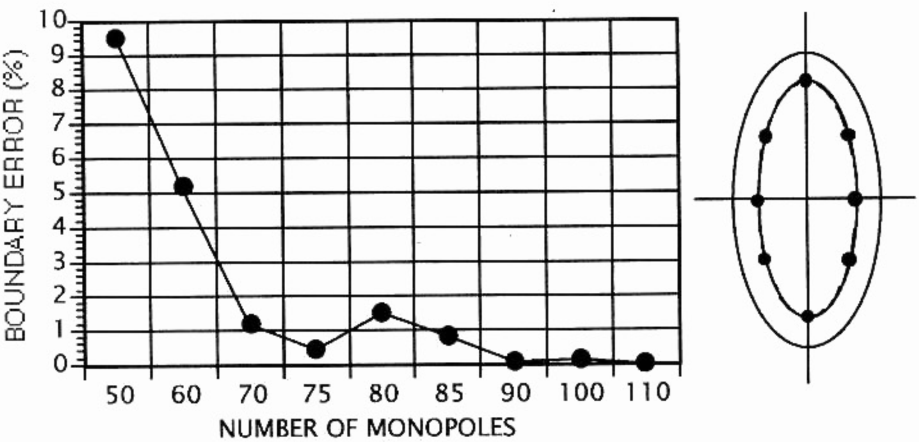

Fig. 5. Elliptical cylinder with elliptical source surface.

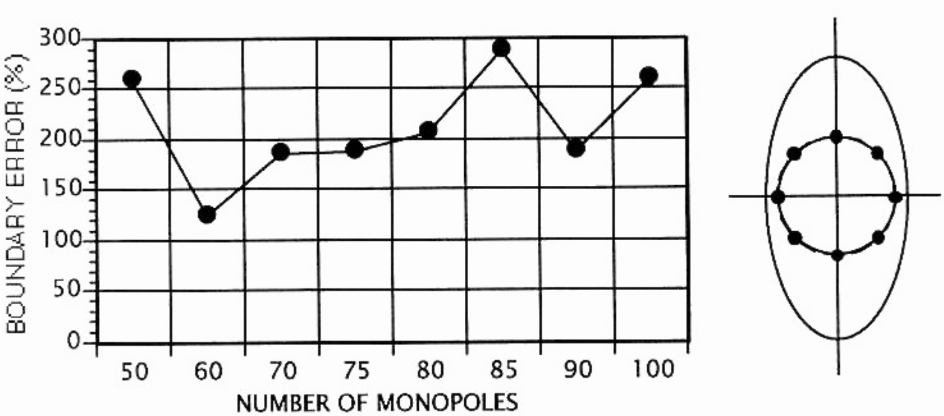

Fig. 6. Elliptical cylinder with cylindrical source surface.

A cylinder with elliptical transverse section was also used as a second object of study. The results obtained were very consistent as long as the source surface was identical with the external surface (see Fig. 5). For the case of a cylindrical source surface located within the elliptical body, the boundary condition could not be satisfied (see Fig. 6). The elliptical body required a significantly larger number of monopole sources in order to get the boundary error minimized.

3.5. Critical frequencies

There are frequencies, sometimes called fictitious eigenfrequencies, at which, or close to which, the solution of the Helmohltz integral equation is non-unique. It is well known that the boundary element method breaks down at these frequencies [14, 15].

JEANS and MATHEWS [16] have demonstrated that in the use of the source simulation technique the critical frequencies correspond to the eigenfrequencies of the internal space formed by the closed source surface, when over this surface the boundary condition of Dirichlet is considered. The eigenfrequencies of a circular space with the Dirichlet condition over its surface are represented by the roots of the Bessel function (see [17]):

$$J_n(kr) = 0, \quad n = 0, 1, 2, 3, \ldots, \tag{2.8}$$

where r is the radius of the source surface.

Nevertheless, other authors report that they have not observed the presence of inner eingenfrequencies when utilizing the source simulation technique, in the calculation of both the acoustic radiation and of the acoustic scattering. Part of the conclusions of these studies are cited in what follows: "*The SUP solution does not appear to be affected by spurious internal resonances which have plagued the integral formulations in the past [18]*"; "*Numerical experiments suggest that this nonuniqueness problem does not exist in the MFS... The reciprocal of the condition number at the eigenfrequencies and* $ka =$ 4.4934094 *is not worse than at the other frequencies and it does not seem to deteriorate at any of the frequencies. Our findings agree with those of Koopman et al., who reported the absence of fictitious eigenfrequency difficulty with this method for acoustic radiation problems [19]*"; "*The results for* $R = 1$ *and* $k = 2.40483$ *which is the smallest zero of the Dirichlet eigenvalue for the circle, illustrate that there is no unique problem at the critical wave number [20]*".

In this work we tried to identify the presence or absence of the eigenfrequencies. Figure 7 shows the error when satisfying the boundary conditions close to and at the first frequency of resonance of a circle: $kr = 2.4048255577$. It is fairly obvious that there is a huge error at this frequency and that the problem formulation breaks down. In Fig. 8 we have the logarithm of the condition number.

One important conclusion that can be drawn from Figs. 7 and 8 is that resonance belongs to a very narrow range of frequencies. For source surfaces like a cylinder or a sphere resonance can be easily calculated and therefore avoided. This is one of the advantages of the source simulation technique with respect to the boundary element method.

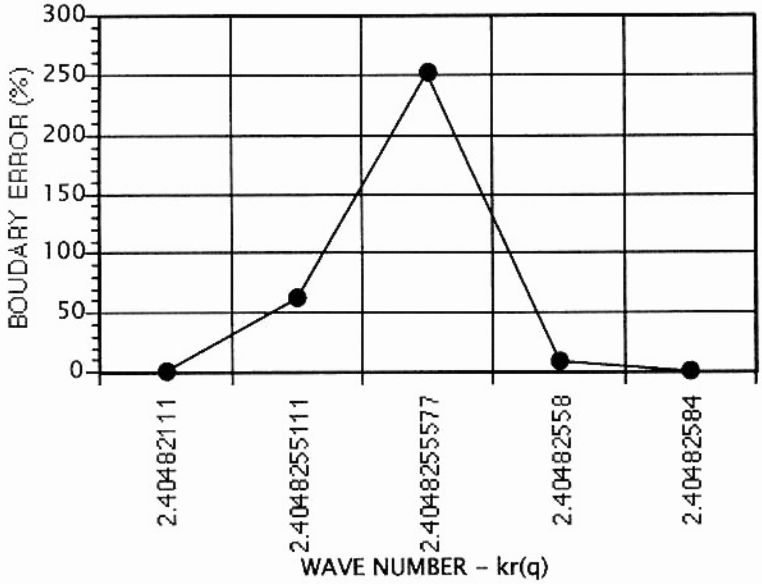

Fig. 7. Boundary error close to and at the resonance frequency of a circle — $kr(q) = 2.4048255577$.

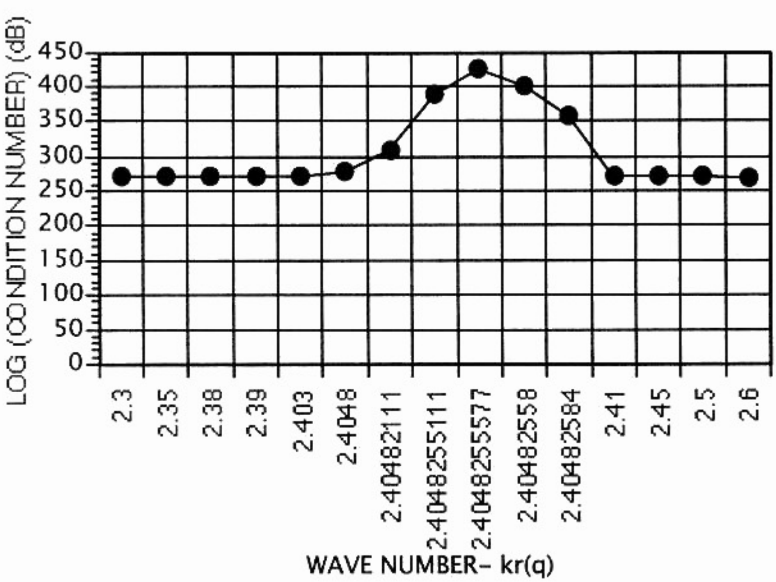

Fig. 8. Logarithm of the condition number at the proximities of the first resonance frequency of a circle — $kr(q) = 2.4048255577$.

One question always present with respect to the source simulation technique is whether the ill-conditioning of the problem (see Figs. 2 and 3) is due to the eigenfrequencies of the inner closed source surface. In Fig. 9 the boundary error can be seen, calculated for the first resonance of a circle — ($kr = 2.4048255577$) — as a function of the number of sources. It can be observed that the error is extremely large (see Fig. 7), though remaining constant despite a substantial increase in the number of sources: 20 to 150 monopoles. In ill-conditioned problems the trend is toward an increase in the error and in the condition number (see Figs. 2 and 3 and [21]) when the number of sources increases [22]. Therefore, one can conclude that the ill-conditioning of the problem is a characteristic of the source simulation technique and is not caused by the eigenfrequencies of the internal space formed by the source surface.

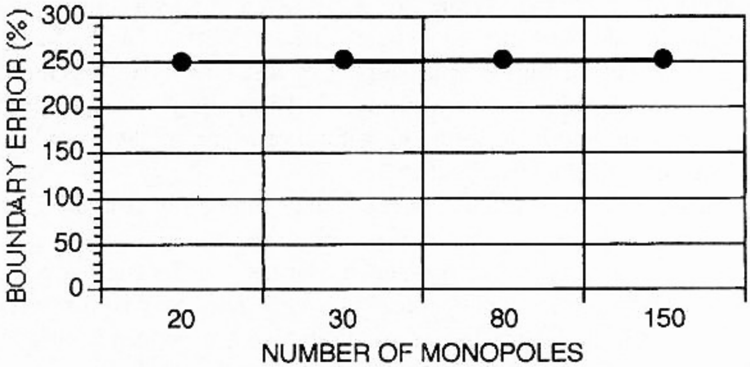

Fig. 9. Influence of the number of monopoles on the boundary error for $kr = 2.4048255577$.

4. Concluding remarks

The quality of the results obtained by the source simulation technique depends on the relationship between some parameters. The most relevant of them are: the shape of the inner source surface, the location of the sources at the source surface, and the number of sources. If one of these parameters is inadequately chosen, this will negatively influence the development of the whole numerical calculation.

In the case of the single-layer method, sources should not be positioned very close to the center of the body, as in this case the condition number of the matrix grows rapidly meaning that it is becoming ill-conditioned. If the wave number is small, one can position the sources close to the center of the body. The advantage of doing so is in the use of a small number of sources in order to minimize the boundary error, which is also translated into less computing time. On the other hand, as the wave number grows, the sources are located closer to the surface. However, the positioning of the sources should obey a relationship between the smallest dimension of the source surface and the largest dimension of the body under study. For the case of a cylinder of radius R and a source surface of radius $r(q)$, the relationship is given by $r(q)/R = 0.9$. Above this value the method becomes very unstable due to the occurence of singularities.

With respect to the shape of the source surface, numerical results have shown that the shape of the studied body and the shape of the source surface should be as similar as possible to each other. Proceeding this way it is possible to minimize the boundary error and to satisfy the boundary conditions, which was not possible when the body considered was an elliptical cylinder and the source surface a circular cylinder.

With respect to the number of sources used there are two opposing effects. If the number of sources is too small the acoustical field cannot be reproduced with precision. If the number of sources is too large, both computing time and computational errors end up increasing. Numerical experiments led us to the conclusion that the ill-conditioning of the problem is not caused by the eigenfrequencies of the source surface, but is a characteristic of the method itself. The method breaks down when the excitation frequency coincides with the eigenfrequencies of the inner space formed by the source surface. The numerical experiments have shown that the eigenfrequencies belong to a very narrow band. In this way, for non-complex surfaces such as a sphere or a circle, they can be easily calculated and avoided. This is one advantage with respect to BEM, as in the last method there is not the possibility of choosing a source surface. In BEM the sources are positioned over the surface of the body and as a result the help of other methods is needed in order to overcome the problem of the eigenfrequencies.

The great disadvantage in the use of the source simulation technique is in the fact that rules for the positioning of the source surface are not known *a priori*. The positioning of the source surface and in consequence of the sources themselves, is based on the experience of the programmer. Further research is necessary to investigate how the method performs with complex surfaces. In that case the main question is about the shape of the source surface.

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