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Research Paper

Acoustic Decaphonic Piano: Calculating Safe Retunings from 12-TET to 10-TET and Beyond

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This paper presents a method for safe retuning of fixed-pitch string instruments to alternative musical scales with fewer degrees than their original design. Our approach uses a systematic monotonic surjective mapping to assign the existing set of strings to a new, smaller set of pitch classes. The primary goal is to preserve the instrument's timbre and structural integrity by keeping string tension changes within safe limits. We demonstrate the method on a grand piano and an upright piano retuned from 12-tone equal temperament (12-TET, 12EDO) to 10-tone equal temperament (10-TET, 10EDO). Presented approach may be generalized for retuning from N- to M-step scales ($N > M$) and to other fixed-pitch string instruments. A grand piano was safely retuned using the proposed method and successfully used in a professional concert.

Keywords: monotonic surjective mapping; decaphonic piano; 10-tone equal temperament; 10TET; 10EDO; alternative instrument tuning; xenharmonic; string tension.

1. Introduction

The interest in alternative scale tunings among musicians arises from their ability to enable more accurate performance in selected harmonic progressions, which historically led to the development of various temperaments in tuning. The microtonal approach, which divides the octave into more than the typical twelve semitones (*tasto spezzato*), has been explored since the 16th century, as seen in Nicola Vicentino's *archicembalo* with divided keys known as *tasto spezzato* (Pilch, Toporowski, 2014). Alternative musical scales can also originate from the characteristic spectrum of a given instrument, as in the case of gamelan music, which employs the *slendro* and *pélog* scales (Sethares, 1998). Some scales are based on repeating intervals larger than the typical octave, such as the Bohlen–Pierce scale, which uses a tritave (3:1 frequency ratio) as its fundamental unit (Mathews *et al.*, 1988), or the *hyperpiano*, which follows a hyperoctave structure with a 4:1 ratio (Hobby, Sethares, 2016).

Alternative tunings are gaining visibility beyond specialist contexts. Popular musicians such as Jacob Collier and Dua Lipa have incorporated microtonal elements in widely streamed songs, reaching millions of listeners (Bandy, 2025; Fraser, 2023). Online communities also play a growing role: YouTube creators including Adam Neely, David Bruce, Georg Vogel regularly explore harmony, tuning, and compositional techniques – often presenting alternative temperaments in an accessible format. Microtonality is also present in mainstream entertainment; for instance, the track *Trees in the Depths of the Earth* from the 1996 video game *Kirby Super Star* uses the microtonal Maqam Rast scale and has reached a broad audience through a franchise that has sold over forty million copies (Martin, 2025). This growing engagement is rooted in a longer musical tradition. Early adopters such as Alois Hába and Julián Carrillo, along with 20th-century figures like Harry Partch, Ben Johnston, and Wendy Carlos, developed extensive bodies of microtonal work. Acoustic instruments such as the Sauter Microtone piano (Thomas, 1996) and the Clavemusicum Omnitonum by Krebs Cembalobau reflect continued interest in physical realizations of microtonal tuning systems.

Another musically interesting approach is to reduce the number of scale steps. In the case of the 10-tone equal temperament, where 'tone equal temperament' is abbreviated as TET, also referred to in the literature as the 10 equal division of the octave (10EDO), the octave is divided into ten equal steps, leading to larger intervals between each step compared to a standard semitone. The theoretical foundations of the 10-TET scale were discussed in (Sethares 1998, pp. 259–270). In particular, three structural features make 10-TET a musically functional example. This includes:

1) it supports neutral intervals, such as the neutral third and neutral sixth. These lie between traditional major and minor forms and allow for the construction of neutral chords – harmonic entities that extend the available vocabulary;
2) 10-TET enables chord cycles built on repeated neutral thirds. This structure forms a circle of thirds, functionally analogous to the circle of fifths in 12-TET. While the steps are different, the pattern supports harmonic progression and modulation in a coherent way;
3) 10-TET admits two types of tritone-based cadences, which can resolve to neutral chords in distinct ways. These structures provide multiple paths for harmonic motion and modulation, compensating for the absence of a major-minor dichotomy.

Several compositions have already been written for 10-TET (Xenharmonic Wiki, n.d.), with some specifically composed for a 10-TET piano by Hunt (2022), Senpai (2023), Sevish (2017), and Hideya (2021). The 10-TET scale also exhibits unique mathematical properties, which we demonstrate in Appendix. Despite its theoretical foundation and existing compositions, to our knowledge, no acoustic 10-TET piano has ever been built, and all performances in this tuning have relied on electronic synthesizers.

Retuning an acoustic piano to an alternative scale presents significant technical challenges. A change in tuning affects string tension and may risk breaking strings, altering the instrument's timbre, or making some strings too slack to vibrate properly.

A clear example of this challenge came when renowned (Stanevičiūtė, Janicka-Słysz, 2022) jazz pianist Leszek Możdżer approached us with a practical request. He wanted to perform on an acoustic piano tuned in 10-TET rather than the standard 12-tone equal temperament (12-TET, 12EDO). His aim was to achieve this new tuning without making major physical modifications – retuning alone should suffice. Although this idea may seem straightforward, direct methods of retuning can lead to extreme pitch deviations in the upper or lower registers, creating problems for both tone quality and instrument safety.

There are multiple reasons why acoustic 10-TET pianos have never been built. To construct an acoustic piano designed for the 10-TET scale, one must overcome all the challenges associated with designing a new standard piano, including significant economic costs and numerous design decisions specific to 10-TET, a largely unexplored field that can only be fully evaluated in a finished instrument. These same obstacles also contribute to the slow evolution of standard piano development.

In this paper, we have chosen the opposite approach: instead of building a new instrument from scratch, we start with an existing piano and introduce the minimum necessary modifications – exclusively through retuning – to achieve the desired effect: an acoustic 10-TET piano. Thus, we focus on a safe and practical method for working within the piano's existing mechanical limits.

We propose a method called monotonic surjective mapping that safely retunes a piano by preserving the original frequency range and maintaining string tensions within acceptable limits. The method can also be extended to other fixed-pitch string instruments, such as harpsichords and harps, and generalized to retune from any N-step scale to an M-step scale where $N > M$.

To validate our approach, we studied two instruments: the Nyström upright piano, which has 85 keys and served as a testing platform, and the Steinway Model B grand piano, which has the typical 88 keys. The presented method was successfully applied and demonstrated at a jazz concert performed by Leszek Możdżer on July 13, 2023 (Tomala 2024), and is used on his albums (Możdżer *et al.* 2024; 2025).

2. Retuning from 12-TET to 10-TET in a standard way demands extending the frequency range of an acoustic instrument

A key consequence of transitioning from 12-TET to 10-TET while utilizing all available strings is ex-

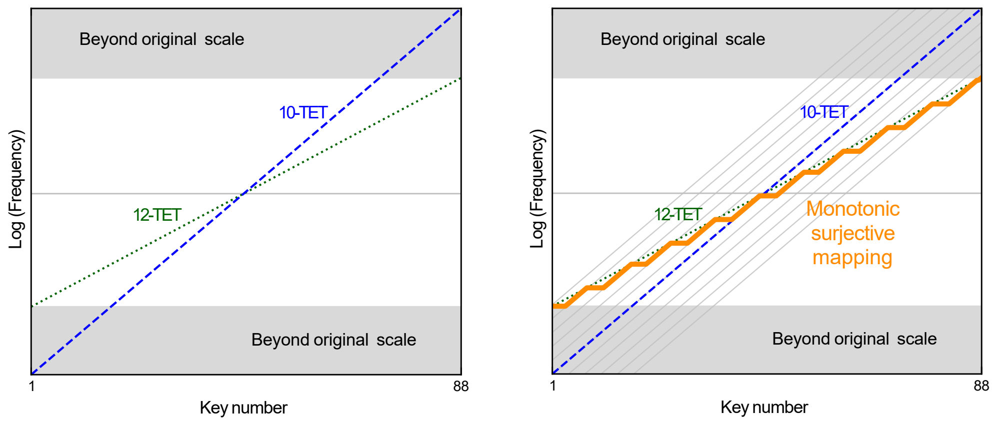


Fig. 1. Idea of monotonic surjective mappings: the problem of extending the original instrument scale when retuning from the 12-TET to the 10-TET scale (left panel); how to solve this problem using monotonic surjective mappings (right panel).

ceeding the instrument's total frequency range span. In 12-TET, the frequency progression follows a well-defined logarithmic slope, where each step corresponds to a fixed frequency ratio of $2^{1/12} \approx 1.0595$. However, in 10-TET, each step is larger, with a ratio of $2^{1/10} \approx 1.072$, meaning that for the same number of keys, the frequency span is stretched. As a result, if an instrument originally designed for 12-TET is simply retuned to 10-TET without additional constraints, its lowest strings may become too loose to function properly, while the highest strings can be subjected to excessive tension, increasing the risk of breakage.

To quantify this effect, consider a standard 12-TET piano, where the fundamental frequency range spans from 27.5 Hz (A_0) to 4186 Hz (C_8). Applying a standard mapping from 12-TET to 10-TET (centered at $C_4(40)$ = 261.63 Hz[1]) shifts the lowest fundamental frequency to 17.53 Hz, far below the playable limit for an acoustic piano, while the highest frequency extends to 7288.3 Hz, well beyond the structural limits of typical piano strings. As the tension of a string is proportional to the square of its frequency, the tension ratio is given by $T/T_{\text{orig}} = (f/f_{\text{orig}})^2$, which results in lowering the tension of the first string to approximately 40 % of its original value and increasing the tension of the highest string to above 300 % of its original tension. This extreme expansion in the frequency range, leading to severe tension changes, is the fundamental reason why such retunings have, until now, only been implemented in electronic synthesizers rather than in acoustic instruments. An illustration of this problem is presented in the left panel of Fig. 1. The left panel explains the problem of extending the original instrument scale when retuning from the 12-TET to the 10-TET scale. The vertical axis represents the frequencies of each note on a logarithmic scale. Both scales appear as straight lines but with different slopes. For the 10-TET scale (blue dashed line), each scale step is larger than for the original 12-TET scale (green dotted line). The gray areas at the bottom and top parts of the plot show the frequency range by which the instrument scale must be extended if a standard mapping is used for retuning from 12-TET to 10-TET. The right plot illustrates how to solve this problem using monotonic surjective mappings. One example of a monotonic surjective mapping is marked with an orange curve. This line connects the frequency of the first key in the original 12-TET tuning with the frequency of the last key in the original tuning by following either lines with the same slope as the alternative 10-TET tuning (examples of them are marked with gray lines) or remaining constant (horizontal).

[1]In this paper, we use a simplified musical notation: $C_4(40)$ is written as C(40), meaning that the 40th key of the analyzed instrument corresponds to a key associated with the note C. This notation also emphasizes that we consider only the strings, which, in the case of a piano, are always connected to the piano action. The action is triggered by a key positioned within a keyboard that follows the fixed Halberstadt layout. In other fixed-pitch string instruments, such as a diatonic harp, the fixed pattern of the diatonic scale is represented by the colors of the strings.

3. Monotonic surjective mappings

The solution to the above problem proposed in this paper aims to possibly preserve the two outermost frequencies of the instrument's original scale while following the alternative 10-TET scale in between. However, this creates a contradiction, as the slope of the alternative scale is steeper than that of the original one. To resolve this, we allow for a monotonic surjective mapping, meaning that some pitches from the alternative scale can be repeated.

This approach results in a large number of possible mappings, many of which do not align naturally with the standard keyboard layout. A logical way to introduce order into these mappings is to preserve the octave interval (12 key distance) on a normal keyboard. This has the advantage that trained pianists already

Table 1. 12-TET and 10-TET scales expressed in cents.

a) 12-TET scale

Key name	C	C#	D	D#	E	F	F#	G	G#	A	A#	B
Interval [¢]	0	100	200	300	400	500	600	700	800	900	1000	1100

b) 10-TET scale

Step	1	2	3	4	5	6	7	8	9	10
Interval [¢]	0	120	240	360	480	600	720	840	960	1080

have the octave distance embedded in their muscle memory, making adaptation to the alternative mapping significantly easier.

This leads to a more formal definition of the assumptions that describe monotonic surjective mappings:

– the alternative scale is mapped onto the original scale surjectively. This ensures that no string (key) is omitted, meaning that every string set is assigned a pitch from the new scale;
– the mapping is monotonic, meaning that each subsequent string has an equal or higher frequency. This allows for repeated sounds in the alternative scale while maintaining the conventional left-to-right increasing pitch layout expected by pianists;
– the octave interval is preserved, ensuring that a pianist playing an octave in the original tuning will still play an octave after retuning.

The right panel of Fig. 1 demonstrates the idea how a monotonic surjective mapping resolves described problem by selectively repeating alternative scale pitches while maintaining the original frequency range.

4. Key definitions and computational framework for safe retuning

A piano consists of a set of strings (for a typical grand piano, like Steinway Model B, with 88 keys, there are 236 strings), each corresponding to a specific key on the keyboard. These keys are arranged in a repeating pattern of black and white keys known as the Halberstadt layout (Mendel, 1949), which has been standardized for Western instruments. The pitch of each string follows a predefined tuning system, traditionally in modern Western culture based on the 12-tone equal temperament (12-TET), where each octave is divided into twelve equal steps. In this convention, the keys are assigned names based on letter notation (C, C#, D, D#, E, F, F#, G, G#, A, A#, B), with C often serving as a convenient reference point.

4.1. Expressing intervals in cents

A useful measure of the frequency ratio (musical interval) is the cent, a logarithmic unit that divides one octave (ratio 2:1) into 1200 cents. If f_1 and f_2 are two frequencies, their difference in cents, Δc, is given by $\Delta c = 1200 \log_2(f_2/f_1)$. A single semitone in 12-TET spans exactly 100 ¢. In contrast, in a 10-TET scale, each step between intervals is larger, measuring 120 ¢ as presented in Tables 1a and 1b.

4.2. Tuning point

To define a tuning system, one particular note is chosen as a reference, here called the tuning point (TP). This is the frequency from which all other pitches in the scale are derived. A common TP is $A_4 =$ 440 Hz, which serves as the modern international tuning standard[2]. In this paper, unless explicitly stated otherwise, we assume the TP to be $A_4 = A(49) =$ 440 Hz. However, this is an arbitrary choice, and tuning can be established from any key.

4.3. Pinning point

The pinning point (PP) we define as a main frequency alignment point between the original and alternative scales. Unlike the TP, which defines the frequency system, the PP is selected based on where the two tuning systems coincide at a particular key (string). At the PP, one note has the same frequency in both scales, ensuring that this pitch remains unchanged during the transition from 12-TET to 10-TET. The choice of PP affects how the mapping between old and new pitches relate to each other.

As illustrated in Fig. 2, the blue curve represents relative frequencies (intervals) in the 12-TET scale within one octave, where one note is chosen as a reference (called here TP) for tuning the entire system. A common example of a TP is the international standard pitch, $A_4 = 440$ Hz. Starting from this note, the frequencies of other steps in the 12-TET scale are calculated. In a 12-TET scale, each step corresponds to 100 ¢, whereas in a 10-TET scale, each step is larger, corresponding to 120 ¢. As a result, the relative frequency curve for the 10-TET scale is steeper (120 ¢/step) compared to the 12-TET scale (100 ¢/step). In the plot, the blue curve represents

[2] Other examples include scientific pitch (Verdi pitch) with C_4 = 254 Hz or French pitch (diapason normal) with $A_4 = 435$ Hz. In some historical instruments tuning was as low as $A_4 = 415$ Hz (Rose, Law, 2001).

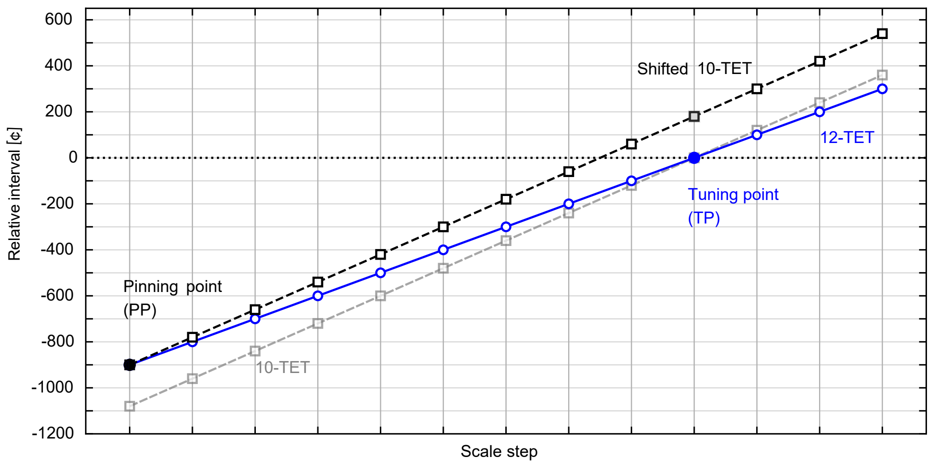


Fig. 2. Difference between the TP and the PP; the blue curve – 12-TET; gray – 10-TET from same TP; black – shifted 10-TET intersecting the 12-TET curve at the PP.

the 12-TET scale, while the gray curve represents the 10-TET scale, both starting from the TP. However, the second curve can be arbitrarily shifted to start from a different step. This shift introduces another important point, referred to here as the PP. The note at this point has the same frequency in both the 12-TET and 10-TET scales, meaning that at this position, the two frequency curves intersect. The black curve represents the 10-TET scale shifted so now it intersects the blue 12-TET curve at the PP. The placement of the PP is independent of the TP, though in some cases, they may coincide. Selecting different PP values results in distinct mappings, influencing how the new scale aligns with the existing instrument layout.

Moreover, choosing a particular PP determines how the instrument interacts with other instruments tuned to 12-TET. A well-chosen PP increases the number of shared (common) notes between the two systems, allowing for better harmonic compatibility. If the PP is poorly chosen, the retuned instrument may lack critical common notes, making ensemble performance with 12-TET instruments more challenging. Thus, selecting an appropriate PP is not only a technical decision but also a musical one, balancing structural feasibility with practical usability.

4.4. Standard mapping

We define a standard mapping as the simplest way to retune from 12-TET to 10-TET. One selects a TP and a PP, then assigns each key to consecutive steps of the new scale, moving outward from the PP. Sethares (1998, pp. 259–270) and Hunt (2021) describe similar direct approaches for the 10-TET scale. Sethares suggests middle C as the starting point, PP = C_4 = C(40) = 261.63 Hz, while Hunt uses PP = C_2 = C(16) = 65.41 Hz. In both cases, they perform on electronic synthesizers rather than acoustic instruments.

4.5. Number of possible monotonic surjective mappings

As a consequence of the assumptions made regarding the monotonic surjective mappings, we retain the octave (12 keys apart) but distribute 10 steps of the alternative scale within it. This requires selecting two keys per octave for repeated sounds, leading to 66 possible key assignments for a given PP. This situation can be generalized to arbitrary scales by considering the number of ways to assign M-steps of the alternative scale to N-keys of the original scale, allowing for repeated steps while preserving order. The number of such mappings is given by the binomial coefficient $C(N, M)$:

$$
\begin{aligned}
C(N, M) &= C(N-1, M-1) + C(N-1, M) \\
&= \frac{N!}{M!(N-M)!},
\end{aligned}
\tag{1}
$$

where $N \geq M > 1$ and '!' denotes the factorial operation. The first term counts all monotonic mappings while excluding the 'cyclicity' of the musical scale, meaning that the last step is not equivalent to the first step. The second term accounts for mappings where the last step is equal to the first step due to the cyclic nature of the scale. While there are only 12 unique steps of the original scale, the physical properties of the instrument introduce additional complexity. Unlike the keyboard, which maintains translational symmetry across octaves – where shifting by an octave results in an equivalent musical structure – the strings do not share this symmetry. Each string has a unique tension. Consequently, there are 88 (number of keys) unique PP, leading to a total of 88 × 66 = 5808 possible mappings that fulfill our assumptions. In this paper, for simplicity, we discuss subset of PP from middle octave (from C(40) to B(51)).

4.6. Operating point

The operating point (OP) of a given string is defined as the ratio, expressed as a percentage, of the string tension to the breaking tension of that string, based on manufacturer data: OP = $T/T_{\text{break}} \times 100\,\%$. For an instrument designed for a specific tuning system, such as 12-TET, the original OP values correspond to an optimal tension that ensures the desired timbre and sound quality.

4.7. Mapping signature

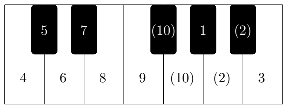
As briefly explained in Fig. 1, a monotonic surjective mapping describes how to assign 10 steps of the 10-TET scale to 12 keys of the 12-TET scale. To unambiguously identify the mapping, we must specify the PP, which indicates which step of the original 12-TET scale corresponds to the 1st step of the alternative 10-TET scale. We also need to identify which steps 'break' the ascending sequence by repeating the previous steps from the alternative scale. For example, for PP = G#(48) and choosing 2nd and 10th steps of alternative scale, the full mapping is fully defined, resulting in the sequence —1, 2, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10— assigning them to the 12-TET-based keyboard. This means that key G# plays the 1st step of the 10-TET scale, A plays the 2nd step, A# also plays the 2nd step, B the 3rd, C the 4th, C# the 5th, D the 6th, D# the 7th, E the 8th, F the 9th, F# the 10th, and G the 10th step. Noticeably, keys A and A# play the same pitch that corresponds to 2nd step of the 10-TET scale. Similarly keys F# and G play now the same sound that is 10th step of the 10-TET scale as presented in Table 2a. To make the notation easier to read, we refer to this mapping as PP = G#(48), MS = '2':(A, A#); '10':(F#, G), where MS is mapping signature. For convenience, we also assign each mapping a numerical label (e.g., mapping no. 18). This label is arbitrary but helps us quickly refer to different mappings in software or in larger plots such as presented in Fig. 9 without spelling out the entire step sequence each time.

Table 3 demonstrates a special repeated-key scenario, where the same new step is assigned multiple times in a row – including the possibility of three repetitions – while still respecting monotonic surjective criteria. Finally, Table 4 shows how cyclicity can come

Table 2. Typical mapping cases.

a) Mapping no. 18: PP = G#, MS = '2':(A, A#); '10':(F#, G)

Original key	C	C#	D	D#	E	F	**F#**	**G**	G#	**A**	**A#**	B
Alternative	4	5	6	7	8	9	**10**	**10**	1	**2**	**2**	3



b) Mapping no. 31: PP = G, MS = '4':(A#, B); '8':(D#, E)

Original key	C	C#	D	**D#**	**E**	F	F#	G	G#	A	**A#**	**B**
Alternative	5	6	7	**8**	**8**	9	10	1	2	3	**4**	**4**

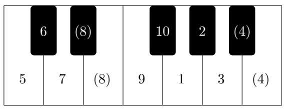


Table 3. Special double-key case, mapping no. 10: PP = C, MS = '2':(C#, D, D#).

Original key	C	**C#**	**D**	**D#**	E	F	F#	G	G#	A	A#	B
Alternative	1	**2**	**2**	**2**	3	4	5	6	7	8	9	10

Table 4. Cyclicity cases.

a) Mapping no. 55, MS = '1':(B, C, C#)

Original key	**C**	**C#**	D	D#	E	F	F#	G	G#	A	A#	**B**
Alternative	**1**	**1**	2	3	4	5	6	7	8	9	10	**1**

b) Steinway, TP = A(49) 440 Hz, PP = C(40), mapping no. 65, MS = '1':(A#, B, C)

Key	40	41	42	43	44	45	46	47	48	49	50	51
Original key	**C**	C#	D	D#	E	F	F#	G	G#	A	**A#**	**B**
Alternative	**1**	2	3	4	5	6	7	8	9	10	**1**	**1**
f_{orig} [Hz]	**261.63**	277.18	293.66	311.13	329.63	349.23	369.99	392.00	415.30	440.00	466.16	493.88
f_{alt} [Hz]	**261.63**	280.40	300.53	322.10	345.22	369.99	396.55	425.01	455.52	488.21	523.25	523.25

c) Steinway, TP = A(49) 440 Hz, PP = F#(46), mapping no. 34, MS = '5':(A#, B, C)

Key	40	41	42	43	44	45	46	47	48	49	50	51
Original key	**C**	C#	D	D#	E	F	**F#**	G	G#	A	**A#**	**B**
Alternative	**5**	6	7	8	9	10	1	2	3	4	**5**	**5**
f_{orig} [Hz]	261.63	277.18	293.66	311.13	329.63	349.23	**369.99**	392.00	415.30	440.00	466.16	493.88
f_{alt} [Hz]	244.11	261.63	280.40	300.53	322.10	345.22	**369.99**	396.55	425.01	455.52	488.21	488.21

into play, allowing repeated or tripled steps to extend beyond a single octave. These examples confirm that as long as pitch assignments remain non-decreasing (or remain constant across a short span), the mapping fulfills the monotonic surjective definition, even in more complex cases involving octave equivalence. The mappings presented in Tables 4b and 4c are particularly interesting because, at first glance, the mapping MS = '1':(A#, B, C) with PP = C(40) and MS = '5':(A#, B, C) with PP = F#(46) leads to the same assignment of new steps from the 10-TET scale to the notes on the 12-TET keyboard – the three keys A#, B, and C play identical pitch. However, the choice of PP results in different frequencies being assigned to the strings for those mappings (compare last rows in Tables 4b and 4c). As we will see later, this leads to significantly different total tension and OPs.

4.8. Calculations

Figure 3 illustrates the sequence of calculations needed to evaluate each mapping. The 'constants' box provides instrument-specific data, such as the reference frequency at the TP and each string's physical properties. The 'variables' box defines the PP and MS, which change with each tested mapping. The 'original tuning' and 'alternative tuning' boxes list values calculated for each string (for example, tension and OP) and single scalar values for the entire instrument (for instance, total tension). Finally, the 'differences' box compiles parameter differences or ratios (e.g., ΔT, Δf, etc.) that allow us to compare the original and alternative tunings under chosen criteria.

Figure 4 shows the input data used for the calculations for the Steinway Model B grand piano and the Nyström upright piano (both instruments originally were designed for TP = A(49) = 440 Hz). Those parameters correspond to the 'constants' box in Fig. 3. The values for Nyström (e.g., string lengths, diameters) were measured directly, and for the Steinway are taken from MATTHIAS (1990). Notable differences include the longer speaking lengths (the lengths of vibrating part of the strings) for the grand piano (compared to the upright) and distinct transitions between

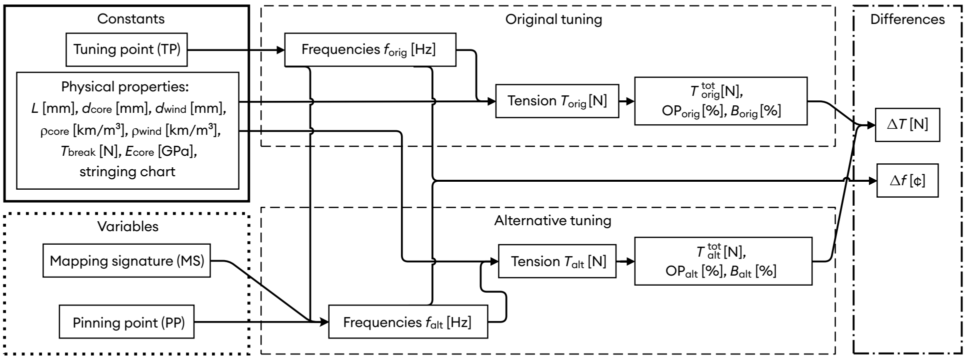


Fig. 3. Flowchart of calculations for mapping an instrument's tuning from an original scale with N-steps (e.g., 12-TET) to an alternative scale with M-steps.

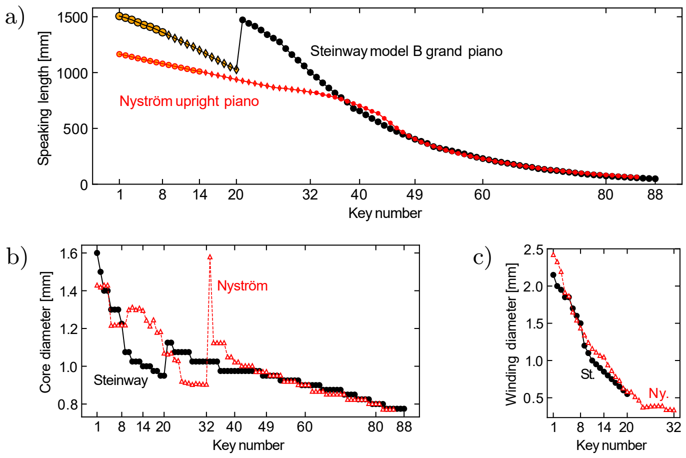


Fig. 4. Stringing scale parameters of the studied instruments: a) speaking length of the strings for each key. Each curve consists of three segments, distinguished by different symbols. The leftmost segment represents the overspun (wound) strings with a single string per key. The middle segment corresponds to the section where there are two overspun strings (bichord) per key. The final part represents the section with three strings per key (trichord) without winding; b) string core wire diameter for each key; c) copper winding wire diameter for the overspun strings.

unichord, bichord-wound, and trichord string sections. Additional windings are used in shorter bass strings to compensate for the reduced length (Rose, Law, 2001). Other parameters like maximum tension, density properties are taken from datasheets provided by manufacturer of piano rounded steel wire (Stahl- und Drahtwerk Röslau GmbH, n.d.).

4.9. String tension

For an idealized perfectly flexible, plain (unwound) string of density ρ and diameter d, vibrating at frequency f over a speaking length L, the tension T is often approximated by $T = 4\,\mu\, f^2\, L^2$, where $\mu = \pi\,\rho\, d^2/4$ is the linear mass density (kg/m). This expression assumes negligible bending stiffness. In case of wounded (overspun) strings the tension is determined by the core wire as the winding is made from soft (compared to core made out of steel) copper wire (Bucur, 2016).

4.10. Overspun (wound) strings

For copper-wound bass strings, the tension formula (Louchet, 2021) generalizes to

$$T = \frac{\pi\, f^2\, L^2}{10^{12}} \left[\rho_{\text{core}}\,(d_{\text{core}})^2 + \left(\frac{\pi\,\rho_{\text{wind}}}{4}\right)(\Omega^2 - d_{\text{core}}^2)\right],$$

where d_{core} and d_{wind} are the core and winding diameters, $\Omega = d_{\text{core}} + 2\, d_{\text{wind}}$ is the outer diameter, and ρ_{core}, ρ_{wind} are material densities (e.g., 7750 kg/m^3 for steel, 8920 kg/m^3 for copper). The factor 10^{12} accounts for unit conversions from millimeters to meters.

4.11. Inharmonicity for finite stiffness

Real musical strings have finite stiffness, so their partial frequencies deviate from integer multiples of the fundamental. If the fundamental frequency is f_0, then the n-th partial can be approximated by $f_n = n f_0 \sqrt{1 + Bn^2}$, where B is the inharmonicity coefficient. According to Fletcher (1964), for a solid steel string with diameter d (in cm), speaking length L (in cm), and fundamental frequency f_0 (in Hz), the inharmonicity coefficient is given by $B \approx 3.95 \times 10^{10}(d^2/(L^4 f_0^2))$. For copper-wound steel strings, where d_{core} is the core diameter and d_{total} is the total diameter (including winding), Fletcher (1964) provides $B \approx 4.6 \times 10^{10}(d_{\text{core}}^4/(d_{\text{total}}^2 L^4 f_0^2))$. In all cases, a sufficiently small B is important to preserve the instrument's characteristic timbre (Louchet, 2021).

5. Results and discussion

The method described in the previous section allows us to evaluate various monotonic surjective mappings by computing key parameters such as string tension, OPs, and inharmonicity coefficients. Before comparing different mappings in detail, we first examine these parameters for selected cases to illustrate how individual mappings affect the instrument.

Figure 5 presents the OPs of each string, expressed as a percentage of the breaking force. This provides a reference for further comparisons by showing how tension varies across the keyboard. The figure includes results for the original 12-TET tuning (empty markers), the standard mappings (thick curves), and two selected surjective mappings (full markers connected by thin curves). The plot reveals that the original OPs are significantly different between the two instruments which is expected as all string parameters are different. The Steinway grand piano exhibits a smoother distribution of tension, with a maximum OP around 50 %, while the Nyström upright piano reaches above 70 %. This difference shows that the optimal mapping may not be the same for both instruments. In particular, since the Nyström upright piano is already closer to the breaking point near the 31st key, one would prioritize mappings that minimize additional tension.

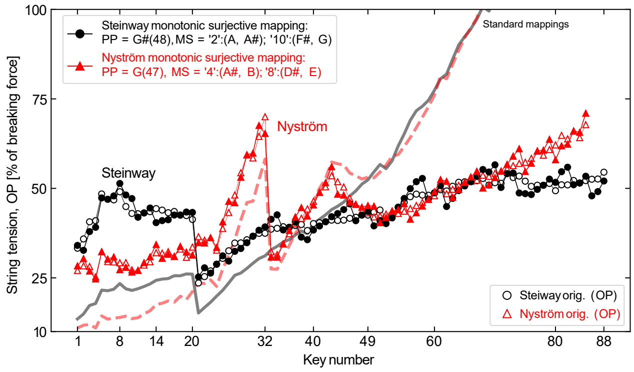


Fig. 5. String tension as a percentage of the breaking force (OPs). The empty markers represent the tension levels for the Steinway (empty black dots) and the Nyström upright piano (empty red triangles), expressed as a percentage of the breaking force for each string. These points define the original OPs (OP_{orig}) of the strings. The Nyström upright piano reaches 70 % of the breaking force around the 31st key. The thick red and black curves represent the tension values when a standard mapping is used to retune from 12-TET to 10-TET. The breaking force is exceeded for both instruments near the 67th key. The filled markers connected by lines represent the calculated (OP_{alt}) values for selected monotonic surjective mappings, which closely align with the original values.

We also observe that the standard mapping leads to significant imbalances in OPs, as predicted earlier. However, Fig. 5 now quantifies this effect across all strings, confirming that standard mapping (for PP = C(40)) introduces excessive tension in the upper range – maximum allowed tension is exceeded around 67th key which leads to string breaking. Unlike the standard mapping, the OP values obtained for the selected monotonic surjective mappings closely match the original ones for both instruments.

Similarly, Fig. 6 presents the inharmonicity coefficients B calculated for the original tuning (markers). In addition, we include values for the selected monotonic surjective mappings (thick black and red curves). The dotted lines represent the inharmonicity coefficients for the standard mapping. The results show an order-of-magnitude difference between the original and standard mappings, highlighting the necessity of careful mapping to preserve the instrument's intended timbre.

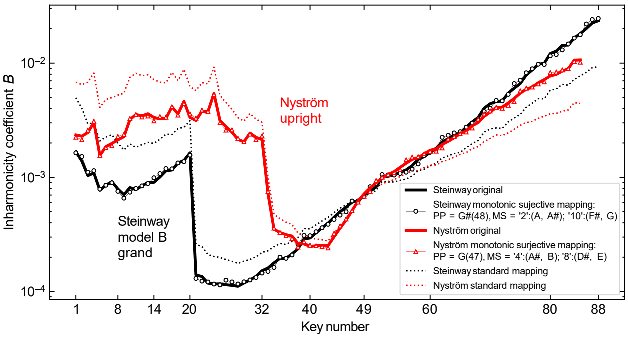


Fig. 6. Inharmonicity. The thick solid lines represent the inharmonicity coefficients B calculated for the original OP. The markers show the calculated inharmonicity coefficients for the optimal mappings: PP = G(47), MS = '4':(A#, B); '8':(D#, E) for Nyström and PP = G#(48), MS = '2':(A, A#); '10':(F#, G) for Steinway. The dotted lines represent the inharmonicity coefficients B calculated for the standard mapping. The difference between the original and standard mapping reaches an order of magnitude.

The details of presented before two selected monotonic surjective mappings, PP = G(47), MS = '4':(A#, B); '8':(D#, E) and PP = G#(48), MS = '2':(A, A#); '10':(F#, G), are explained in Fig. 7. In this figure, unlike in Figs. 1 and 2, the slope of the original 12-TET scale has been subtracted. As a result, the original tuning appears as a horizontal line, and the plot shows deviations from that tuning. This plot serves as a practical reference for piano tuners performing the transition from 12-TET to 10-TET, as it directly indicates how many cents each key must be retuned from the original 12-TET tuning. The inset presents zoomed main plot for one octave around PPs G(47) in case of Nyström and G#(48) for Steinway. For this points the frequency difference is by definition zero. The jumps between 43rd, 44th and 50th, 51st keys for Nyström and between 46th, 47th and 49th, 50th key for Steinway are fingerprints of the presented signatures.

To find the optimal mapping, one must establish suitable criteria for ranking mappings from the most to the least optimal. As mentioned earlier, multiple parameters have conflicting requirements. For the pinblock, where tuning pins are placed, lower total tension is preferable. Similarly, individual pins benefit from reduced tension. From a timbre perspective, a lower

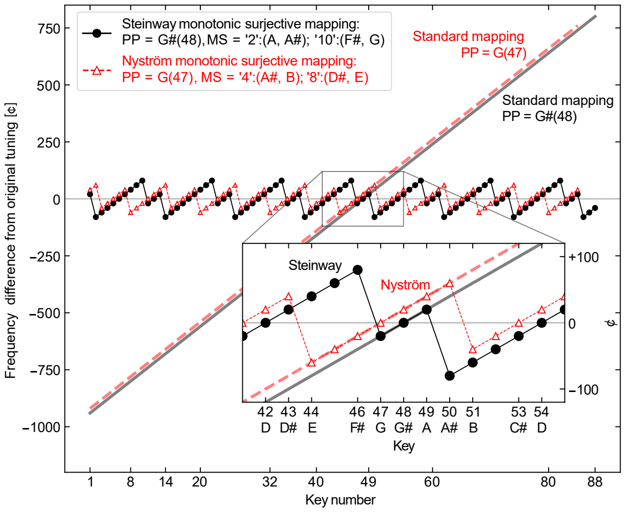


Fig. 7. PP and MS. Detuning from oryginal 12-TET scale expressed in cents for all keys of keyboard. The gray solid line at 0 ¢ represents oryginal tuning in 12-TET scale. The black and red markers show selected monotonic surjective mappings. The rising diagonal thick black and red dashed curves are result of standard mappings from the same PPs as those for optimal mapping. The slope of of diagonal lines is a difference between 12-TETs slope (100 ¢ per key) and 10-TETs slope (120 ¢ per key) resulting in 20 ¢ per step.

inharmonicity coefficient is desirable; however, the inharmonicity coefficient is inversely proportional to tension. Moreover, reducing inharmonicity below its original value is not always the goal, as preserving the original timbre may be a higher priority. This consideration is reflected in the maximum change of the OP criterion. Furthermore, starting from different PP results in varying numbers of shared frequencies (notes) with the original scale, which may be important when performing with other instruments.

Ultimately, the criteria for selecting an optimal mapping are subjective. The only truly objective requirement is that each string's tension must remain below its ultimate breaking point. A nearly objective criterion is ensuring tuning stability, but the literature does not define a single reference value for this. Tuning stability depends on factors such as the instrument's materials, environmental conditions (temperature, humidity), and playing intensity. A commonly used guideline is to limit frequency changes for each string to no more than 100 ¢.

Table 5 presents the results of calculations for standard mappings with PPs C(40) and A(49). These mappings were evaluated based on their impact on total string tension and OPs, which measure the string tension as a percentage of the breaking force. The alternative tension values (T_{alt}) and the changes in tension (ΔT) are listed alongside the maximum and minimum OPs for each case.

The results clearly demonstrate that standard mappings, regardless of the chosen PP, introduce severe

Table 5. Summary of tension values for standard mappings with PPs from C(40) and A(49). The original tension is 161.684 kN for Nyström and 168.735 kN for Steinway. The original maximum OP^{max}_{orig} is 70 % (Nyström) and 54.5 % (Steinway). Similarly the original minimum OP^{min}_{orig} is 25.3 % for Nyström and 23.6 % for Steinway.

Instr.	PP	T_{alt} [kN]	ΔT [kN]	OP^{max}_{alt} [%]	OP^{min}_{alt} [%]
Nyström	A(49)	180.955	+19.271	155.7	8.9
Nyström	C(40)	222.781	+61.097	191.7	11.0
Steinway	A(49)	186.174	+17.438	134.3	11.0
Steinway	C(40)	229.207	+60.471	165.3	13.5

structural and acoustic issues. In every case, the maximum OP^{max}_{alt} significantly exceeds the instrument's limit (is over 100 %). Simultaneously, the minimum OP^{min}_{alt} drops drastically from original values of 25.3 % for Nyström and 23.6 % for Steinway, indicating that some strings become too loose to function properly, compromising pitch stability and timbre. The uneven distribution of tension, combined with the extreme departure from the instrument's original inharmonicity characteristics, results in an unbalanced tonal spectrum, rendering the instrument practically unplayable.

On the other hand, Table 6 summarizes the total string tension values for selected monotonic surjective mappings. The mappings listed here represent a subset of all possible monotonic surjective mappings, chosen based on specific PPs and mapping structures previously presented in Tables 2, 3, and 4 as examples of MSs. As shown, some of the presented mappings exhibit a total tension difference between the alternative and original tuning that is close to zero. It is worth noting that even in some cases total tension difference is relatively large (e.g., 21.444 kN) the maximum OP – OP^{max}_{alt} is well below breaking value and minimum OP value OP^{min}_{alt} is close to the original one.

While Table 6 provides a numerical overview of specific mappings, manually comparing each case is inefficient given the large number of possible mappings. A more effective approach is to visualize all computed mappings and sort them based on different criteria to identify the most favorable ones.

Figure 8 presents an extensive analysis of 792 possible monotonic surjective mappings, covering 12 different PP from PP = C(40) to PP = B(51). The plots show absolute total tension change $|\Delta T^{tot}|$, signed total tension difference ΔT^{tot}, maximum and mean

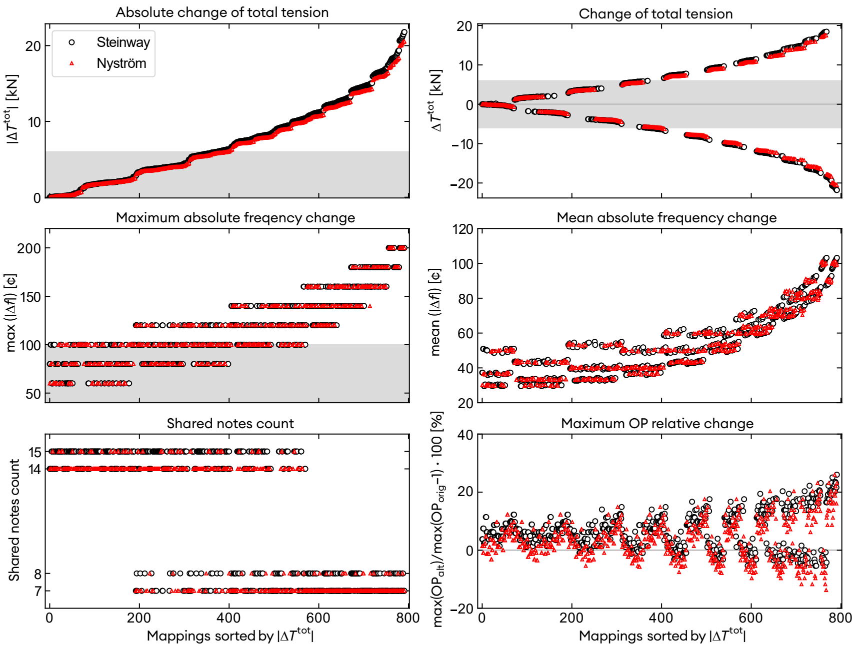


Fig. 8. Criteria for selecting an optimal mapping.

Table 6. Summary of tension values for selected monotonic surjective mappings. The original tension is 161.684 kN for Nyström and 168.735 kN for Steinway. The original maximum OP^{max}_{orig} is 70 % (Nyström) and 54.5 % (Steinway). Similarly the original minimum OP^{min}_{orig} is 25.3 % for Nyström and 23.6 % for Steinway.

Instr.	PP	MS	T_{alt} [kN]	ΔT [kN]	OP^{max}_{alt} [%]	OP^{min}_{alt} [%]
Nyström	G(47)	no. 31: '4':(A#, B); '8':(D#, E)	161.684	−0.000	71.0	24.7
Steinway	G#(48)	no. 18: '2':(A, A#); '10':(F#, G)	168.730	−0.006	56.6	25.3
Steinway	C(40)	no. 34: '1':(A#, B, C)	165.560	−3.175	58.6	23.0
Steinway	C(40)	no. 55: '1':(B, C, C#)	169.724	+0.988	59.1	23.0
Steinway	F#(46)	no. 65: '1':(A#, B, C)	190.179	+21.444	67.3	26.5

absolute frequency change, the number of shared notes with the original tuning, and the relative change in the maximum OP. By sorting mappings based on $|\Delta T^{\mathrm{tot}}|$ (top-left panel), it becomes evident that mappings optimized for one criterion are not necessarily optimal for others. Additionally, mappings for different instruments (Steinway and Nyström) do not overlap, suggesting that a universally optimal mapping does not exist across different instruments.

A key finding of this analysis is that all 792 examined monotonic surjective mappings are structurally safe. This means that none of the mappings exceed the maximum breaking tension for any string. Moreover, as shown in Fig. 8, particularly in the panel displaying the maximum absolute frequency change, no string undergoes a retuning greater than 200 ¢, ensuring that all modifications remain within a reasonable tuning range. The plot in Fig. 8 presents 792 possible monotonic surjective mappings for Steinway (black circles) and Nyström (red triangles), calculated across 12 PP ranging from PP = C(40) to PP = B(51), resulting in a total of $12 \times 66 = 792$ mappings. Each panel corresponds to a different ranking criterion: absolute total tension change $|\Delta T^{\mathrm{tot}}|$, total tension difference ΔT^{tot}, maximum absolute frequency change, mean absolute frequency change, number of shared notes between the original and mapped tuning, and the relative change in the maximum OP. The mappings are sorted by $|\Delta T^{\mathrm{tot}}|$, as shown in the top-left panel. The results demonstrate that mappings optimized for one criterion (±100 ¢ or ±6000 kN, shown as gray regions) are not necessarily optimal for others.

We propose a useful tool for selecting an optimal monotonic surjective mapping – a plot that visualizes mapping-related scalar values using a color scale, with PP on the vertical axis and MSs on the horizontal axis. Black dots indicate mappings that fall below an arbitrarily chosen threshold, while the top 20 mappings are ranked and labeled with red numbers.

This type of chart is particularly useful in several ways. First, it helps identify which keys are doubled in a given mapping. To interpret the chart, one starts by selecting a specific field in the 2D map, then reads the corresponding PP on the left and the MS at the bottom. For example, in Fig. 9, which presents total tension differences for Steinway, the best mapping according to this criterion is marked with a red '1'. This corresponds to PP = G# on the left and mapping number 18, which yields the alternative scale step sequence —1, 2, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10—. This means that, starting from the chosen PP key, the sequence follows the structure presented in Table 11a, with repeated key pairs **A, A#** and **F#, G**. This mapping can be expressed as PP = G#(48), MS = '2':(A, A#); '10':(F#, G). Similarly for Nyström with this method we identify mapping PP = G(47), MS = '4':(A#, B); '8':(D#, E) as optimal from the point of view of total tension.

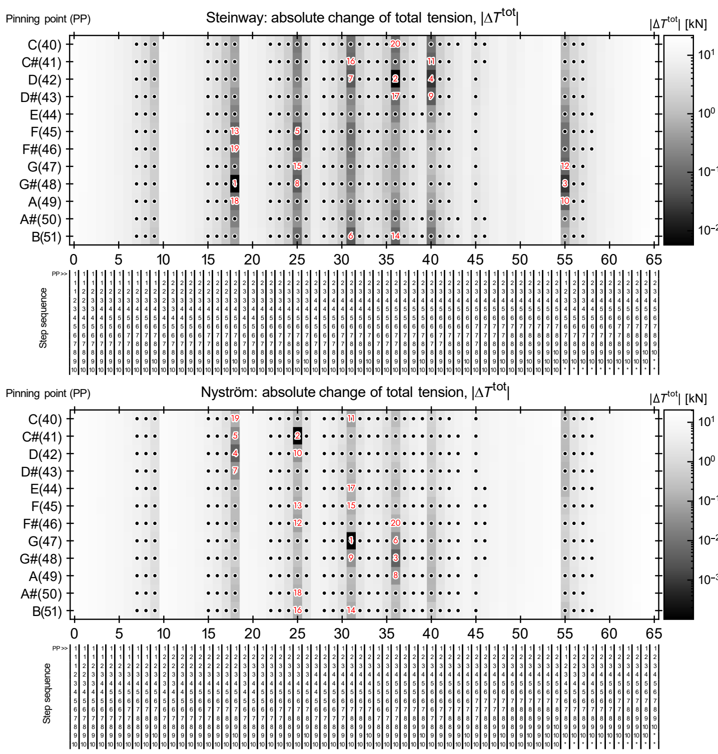


Fig. 9. Absolute total tension change for: a) Steinway; b) Nyström. The color scale represents the absolute total tension change $|\Delta T^{\mathrm{tot}}|$ in kN, using a logarithmic color scale. The black dots indicate mappings where $|\Delta T^{\mathrm{tot}}|$ is below the threshold of 6000 kN. The top 20 mappings are ranked and labeled with red numbers.

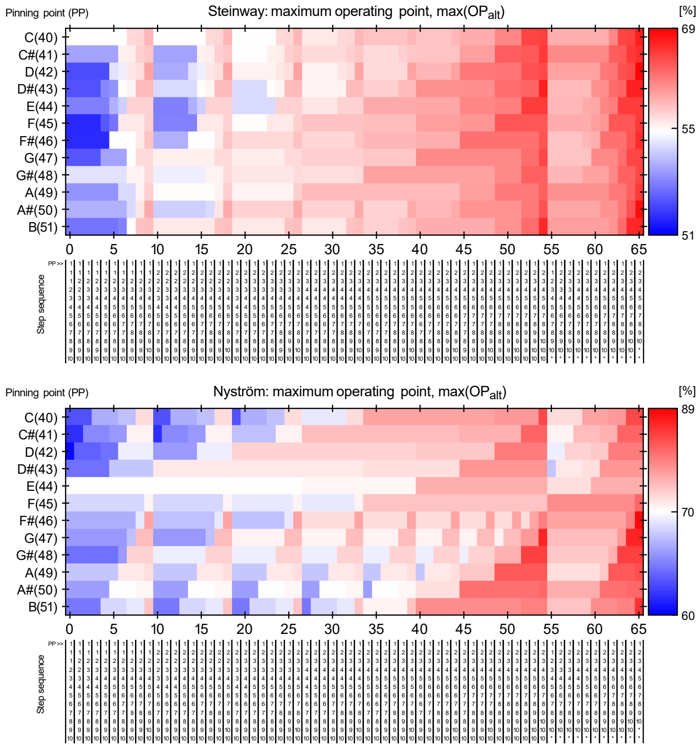


Fig. 10. Maximum operating point for: a) Steinway; b) Nyström. The color scale represents the maximum operating point – $\mathrm{OP}_{\mathrm{alt}}^{\max}$, where white color is assigned to maximum value of original operating point – $\mathrm{OP}_{\mathrm{orig}}^{\max}$.

While all analyzed mappings are structurally safe and do not compromise the instrument, we can further refine our selection by identifying the most optimal ones based on additional criteria. Since total tension remains within acceptable limits and does not threaten the integrity of the pinblock (Fig. 9), we can instead prioritize mappings that minimize changes in the maximum or minimum OPs, as these influence inharmonicity (Fig. 10). Here, we present selected examples, but the final choice of mapping should always be based on a similar analysis tailored to each individual instrument. For this reason, we do not provide 2D visualizations for all possible parameters.

6. Generalization for other fixed-pitch string instruments

The methodology for identifying monotonic surjective mappings, as demonstrated with pianos, can be extended to other fixed-pitch string instruments, such as harps, celestas, cimbaloms, harpsichords, clavichords, lyres, dulcimers, kanteles, spinets, psalteries, virginals, and zithers. In these applications, the traditional octave (2:1 frequency ratio) can be replaced by alternative interval structures, such as the tritave (3:1 ratio) found in the Bohlen–Pierce scale (Mathews *et al.*, 1988) or the hyperoctave (4:1 ratio) used in experimental tuning systems, including some hyperpiano concepts (Hobby, Sethares, 2016). The interval being divided, sometimes called an equave, can be split into unequal steps. Therefore, the approach presented in this paper provides a systematic method for determining safe retuning mappings from an arbitrary N-step scale to a smaller M-step scale, provided that $M < N$.

An example of a non-octave-based system is the set of tuning scales proposed by Wendy Carlos, known as the alpha, beta, and gamma scales. These systems divide the perfect fifth, rather than the octave, into 9, 11, and 20 equal parts, respectively. While our method is not directly applicable to retuning a 12-TET instrument to one of these scales, because the basic periodic unit is not the octave, it becomes fully applicable when retuning between Carlos-type scales. For instance, an instrument tuned in gamma ($N = 20$) can be retuned safely to beta ($M = 11$) or alpha ($M = 9$), and beta can be retuned to alpha. These transformations preserve the periodicity of the fifth and meet all assumptions of the monotonic surjective mapping framework. In this way, retuning between scales with different step counts becomes feasible even when the equave is not the octave but another interval, such as the perfect fifth in Carlos-type tunings.

To illustrate the applicability of the method in a more traditional octave-based context, we turn to the example of a folk diatonic harp (not to be confused with a harmonica). This instrument traditionally follows a heptatonic (seven-note) scale (original scale) consisting of the notes C, D, E, F, G, A, and B, meaning $N = 7$. If the goal is to adapt the instrument to a pentatonic (five-note) scale (alternative scale) consisting of C, D, E, G, and A ($M = 5$), then the number of possible monotonic surjective mappings for a given PP can be determined using Eq. (1), yielding 21 potential mappings. Similar to the fixed keyboard pattern of the piano, diatonic harps also incorporate a structured system to indicate pitch relationships. In this case, strings are color-coded, with C strings typically colored red and F strings colored blue or black. This provides a visual reference for the player and reinforces the structural role of the octave. This suggests that the assumption of octave preservation remains valid in this case. Consequently, when applying monotonic surjective mappings to the diatonic harp, maintaining octave equivalence remains a natural and practical choice. The remainder of the analysis would proceed similarly to that of the piano, requiring an evaluation of breaking tensions and other parameters, such as speaking length, to ensure both structural integrity and tonal stability in the alternative tuning.

7. Conclusions – Final selection of mapping

The proposed method of monotonic surjective mappings provides a systematic way to retune a 12-TET instrument to the alternative 10-TET scale while preserving the original frequency range and ensuring that no string exceeds its breaking tension. Our computational analysis confirms that every mapping generated by this method is structurally safe. Nonetheless, various factors – such as absolute frequency changes, total tension differences, and the number of shared notes between the original and alternative tunings – must be taken into account when selecting the optimal mapping.

The final choice of mapping ultimately depends on personal preference and performance context. One important consequence of the monotonic surjective mapping approach is that some neighboring keys on the piano keyboard may be assigned the same pitch. This feature allows for a rapid, percussive repetitions of the same sound during performance, which may influence an artist's preference for a particular mapping. While many mappings can be optimized to minimize tension differences or frequency deviations, the selection remains subjective. For instance, in a somewhat humorous twist, the professional pianist Leszek Możdżer chose not to adopt the mapping with the smallest overall tension change. Instead, he preferred a different signature that maintained symmetry in the keyboard layout. His decision was to repeat the keys F and F# as well as A and A#, while sharing the note A between the original and alternative scales. In his case, his selection corresponds to mapping no. 7 with PP = A(49), where the MS is given by: MS = '1':(A, A#); '8':(F, F#).

In conclusion, the monotonic surjective mapping approach offers a viable solution for retuning acoustic instruments to alternative scales. Although the method ensures structural safety, the optimal mapping must be chosen by balancing technical criteria with the musician's individual artistic taste. Moreover, the concept can be generalized beyond pianos to other fixed-pitch string instruments, such as harps, harpsichords, and dulcimers, thereby broadening its potential applications in diverse musical contexts.

Appendix – Why 10-TET is a unique choice

In the main text, we have focused on dividing the octave into equal steps (TET, EDO). However, there is also broad interest in other scales (temperaments), seeking to resolve the small comma mismatch that arises when combining integer frequency ratios (Pilch, Toporowski, 2014; Rasch, 1984).

The Pythagorean tuning, for example, cycles through perfect fifths (frequency ratio 3:2) while treating octave-equivalent tones (2:1) as musically identical. This produces pure, consonant fifths but also leads to the famous Pythagorean comma, which quantifies the mismatch between twelve stacked just fifths and seven octaves.

The Pythagorean comma is typically defined as the cumulative discrepancy between twelve just fifths and seven octaves:

$$\frac{(3/2)^{12}}{2^7} = \frac{531441}{524288} \approx 1.01364,$$

which corresponds to approximately 23.46 cents. This classical definition is dimensionless and expresses the ratio between the two paths through pitch space.

In this work, however, we use a normalized (per-octave) version of the comma:

$$c_{(3,12,7)} = \left|(3/2)^{12/7} - 2\right| \approx 0.00388,$$

where 3 is the frequency ratio of a perfect fifth, 12 is the number of such intervals stacked, and 7 is the number of octaves they are expected to span. By raising the full stack to the power 1/7, we calculate the average interval needed per octave. The result is then compared to the exact doubling of frequency (ratio 2) expected for one octave. This way, the value $c_{(3,12,7)}$ captures how much the average fifth-based step deviates from the ideal octave size, making it easier to generalize and compare different (t, s) scale configurations.

Although small, this discrepancy confirms that the cycle does not close perfectly, and the frequency does not return exactly to the starting note.

General Pythagorean t-step scales

A more general version of Pythagorean tuning picks an odd harmonic integer denoted n_k and an integer

m_0 that 'folds' the n_k-based interval into the base octave, ensuring the resulting frequency ratio remains within $[1,2]$:

$$h_1 = \frac{n_k}{2^{m_0}}, \quad \text{with} \quad 1 < h_1 < 2.$$

Repeatedly multiplying (or dividing) by h_1, and dividing (or multiplying) by 2 whenever the resulting frequency lies outside $[1,2]$, generates t steps within each octave. If this process nearly reconstructs the octave after t steps across s octaves, the mismatch – called the comma – can be written as:

$$c = \left|(h_1)^{t/s} - 2\right| < \epsilon,$$

where t is the number of steps per octave, s is the number of octaves required to close the cycle, and ϵ is a small upper bound for acceptable mistuning. If the comma is smaller than the Pythagorean $c_{(3,12,7)}$, then the triple (n_k, t, s) is said to define a good comma. This indicates that the generated scale closely approximates octave closure while using a consistent generator interval.

We restrict attention to manageable scales by requiring $t \cdot s < 500$, which limits the total number of notes (i.e., the number of steps per octave times the number of octaves needed to complete the cycle), and limit $n_k \le 21$ to avoid using excessively high harmonics.

Under these constraints, one finds that the smallest comma c – i.e., the best result among all possibilities within these limits – arises for $(n_k, t, s) = (13, 10, 7)$, giving a generating interval $h_1 = 13/8$, often called a 'neutral sixth'. Since here $t = 10$, this results in a 10-step Pythagorean scale whose comma

$$c_{(13,10,7)} \approx 0.00087$$

is more than four times smaller than that of the usual 12-tone Pythagorean system $(3, 12, 7)$, with comma $c_{(3,12,7)}$.

Angles on a musical circle

An insightful way to handle such cyclic issues is to place each frequency ratio on a circle of angles. Specifically, if h_ℓ is the frequency multiplier for the ℓ-th step (where $\ell = 0, 1, \ldots, t-1$), we define the Pythagorean angle by

$$\varphi_\ell = 2\pi \log_2(h_\ell).$$

Because multiplying a frequency by 2 shifts its angle by 2π (one full turn), the angle φ_ℓ neatly captures where h_ℓ lies 'modulo octaves'. In that sense, going once around the circle corresponds to going up by one full octave in frequency.

Comparing with equal temperament

Another way to assess how well a Pythagorean scale approximates its equal-tempered counterpart is the tempered index:

$$\delta_{(n_k,t,s)} = \frac{1}{t-1} \sum_{\ell=0}^{t-1} \left|\Phi_\ell - \varphi_\ell\right|,$$

where $\varphi_\ell = 2\pi \log_2(h_\ell)$ and $\Phi_\ell = 2\pi \frac{s}{t} \ell$ are the equally tempered angles for t notes in each of s octaves. Numerically,

$$\delta_{(3,12,7)} \approx 6.40 \text{ ¢/step}$$

and

$$\delta_{(13,10,7)} \approx 1.46 \text{ ¢/step},$$

where radians per step were recalculated to cents per step.

Hence, from a purely mathematical viewpoint, the 10-TET scale is, at the same time, almost perfectly the 10-step Pythagorean scale built using the 13:8 ratio interval. Moreover, this 10-step Pythagorean scale has the smallest comma c among all general Pythagorean scales calculated with reasonable assumptions.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper. However, part of the research described in this article is the subject of a patent application

Authors' contribution

Aleksander Bogucki developed the method, prepared figures, and wrote the manuscript. Andrzej Włodarczyk conducted string measurements and provided technical validation. Paweł Nurowski initiated and coordinated the project, developed the mathematical framework for 10-TET scale. All authors reviewed and approved the final manuscript.

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