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INTERACTION OF RAYLEIGH WAVES WITH PERIODIC BAFFLES

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Rayleigh waves in two solids separated by an air gap containing two alternate periodic in-plane baffle systems are considered. A Bragg reflection of the waves occurs when the distance between the baffle planes and between baffles and solids are half wavelength long in air. This may support an idea of contactless application of comb transducers to generate Rayleigh waves. The analysis is carried out using the BIS expansion method to account for the wavefield singularity at the baffle edges.

1. Introduction

There is growing interest in application of Rayleigh waves, long exploited in nondestructive evaluation of materials and in a variety of sensors (see papers presented on recent IEEE Ultrasonics Symposia, for instance). Whatever the application is, there is high interest in contactless generation of these waves in solids.

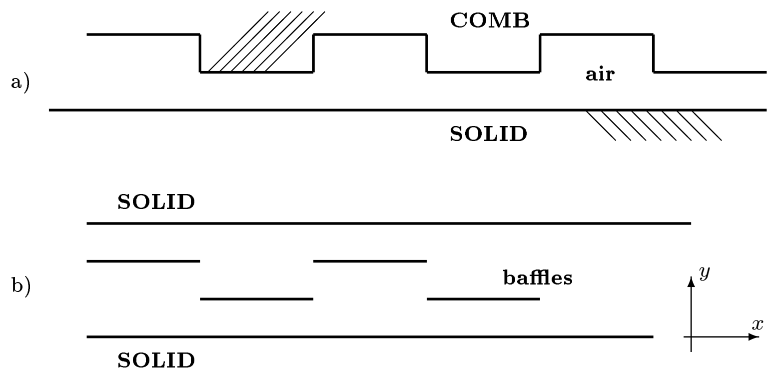

Fig. 1. a) In contrast to the lower solid where the Rayleigh wave propagates on the plane surface, in the upper solid (the comb), the wave would be strongly perturbed by the surface corrugation. To avoid complication of the analysis, in the model b) both solids have plane surfaces allowing Rayleigh waves to propagate. The baffles represent the interaction of the acoustic wave in air with comb surfaces; due to the large difference in acoustic impedances of solids and air, the baffles are considered acoustically stiff. The side surfaces of teeth which height is about half wavelength in air, are neglected in the model.

One of possible solutions widely discussed in recent literature is a micromachined electro-mechanical transducer [1]. Here, we consider a comb transducer [2] in a novel configuration by applying it through thickness-optimized layer of air (a liquid couplant may also work). The study presented in this paper is limited to the analysis of the most fundamental phenomenon – the propagation of Rayleigh waves in the proposed system. To simplify the analysis, the system is modeled here by a doubly periodic baffle systems placed in an air gap separating two elastic halfspaces; the system is symmetric and the Rayleigh waves in solids interact with baffles by means of the wavefield in the gap. Fig. 1. explains the correspondence between the original comb/solid arrangement and the analyzed system. The paper solves the boundary-value problem for the system presented in Fig. 1b) whatever its origin is.

2. Planar characterization of acoustic media

It is convenient to introduce the planar harmonic Green's function for elastic halfspaces and for an air layer of thickness h. Assuming the general harmonic wavefield of the form

$$e^{j\omega t}e^{-jrx} \tag{1}$$

on the media boundaries, where ω is angular frequency and r is the wavenumber, we have after [3]

$$u_2 = GT_{22} \tag{2}$$

for upper solid, and the analogous equation, with replacement of G by $-G$, for the bottom solid. Here, u_2 is the normal particle displacement and T_{22} is the normal traction; only these wavefield components are involved in interaction of the solids with the adjoined air layer.

$$\begin{aligned} G &= \frac{jk_l^2 q_l/\mu}{(k_t^2-r^2)^2+4r^2q_lq_t}, k_l^2=\omega^2\rho/(2\mu+\lambda), k_t^2=\omega^2\rho/\mu, \\ q_m &= \sqrt{k_m^2-r^2} \;\text{ if }\; r<k_m \text{ otherwise } -j\sqrt{r^2-k_m^2},\; m=l,t, \end{aligned} \tag{3}$$

where ρ is the mass density and μ, λ are Lamé constants; k_l, k_t are wavenumbers of longitudinal and shear waves in solids. The choice of the square-root values for $q_{l,t}$ makes the solution satisfying the radiation conditions in depth of the body at $y \to \infty$, assuming the wavefield dependence on y in the form $\exp(-jq_{1,2}y)$. Eqs. (2), (3) sufficiently characterize the elastic halfspaces in the problem under consideration.

To derive the analogous planar characterization of an air layer of thickness h, we try the solution to the acoustic potential ϕ

$$\phi = (Ae^{-jsy}+Be^{jsy})e^{-jrx} \tag{4}$$

(the time harmonic dependence is dropped throughout the paper) to the acoustic equations

$$\begin{gathered} v=-\nabla\phi,\; p=\rho_o\partial\phi/\partial t, \\ c^2\nabla^2\phi = \partial^2\phi/\partial t^2, \\ k=\omega/c,\; s=\sqrt{k^2-r^2} \;\text{ if }\; r<k \text{ otherwise } -j\sqrt{r^2-k^2}, \end{gathered} \tag{5}$$

where ρ_o is the air mass density and c is the acoustic wave velocity in air. Elimination of unknown constants A and B by the wavefield components on both sides of the layer: $v^u = v_2$ and $p^u = p$ on the upper side, and $v^b = v_2$ and $p^b = p$ on the bottom side, where p is the acoustic pressure and v_2 is the y-component of particle velocity in the acoustic field, yields

$$\begin{aligned} p^u &= j\omega\rho_o \frac{v^u \cos sh - v^b}{s \sin sh}, \\ p^b &= j\omega\rho_o \frac{v^u - v^b \cos sh}{s \sin sh}. \end{aligned} \tag{6}$$

Next, we introduce the corresponding description of a composed media: the system of an air layer of thickness h on an elastic halfspace. Note that $p = -T_{22}$ at the air/solid contact and $j\omega u_2$ in solid equals v_2 in air as well. For the upper solid, it suffices to substitute

$$\begin{aligned} v^u &= j\omega u_2, \\ p^u &= -T_{22} \end{aligned} \tag{7}$$

in Eqs. (6) to obtain the relation between the wavefield components at the lower side of the air layer: v^b, p^b; for further convenience they are denoted here by v^+ and p^+ (these will be the velocity and pressure at the upper side of the upper periodic baffle system)

$$\begin{aligned} p^+ &= -j\frac{\omega\rho_o}{s}\frac{s\cos sh + \omega^2\rho_o G \sin sh}{s \sin sh - \omega^2 \rho_o G \cos sh} v^+, \\ p^- &= j\frac{\omega\rho_o}{s}\frac{s\cos sh + \omega^2\rho_o G \sin sh}{s \sin sh - \omega^2 \rho_o G \cos sh} v^-, \end{aligned} \tag{8}$$

where the second equation concerning the upper boundary of the air layer placed on the bottom solid, has been derived analogously using $v^b = j\omega u_2$ and $p^b = -T_{22}$ with $u_2 = -GT_{22}$ describing the lower elastic halfspace.

Naturally, the air layer between the lower and upper periodic baffle systems is characterized by the original Eqs. (6) (this explains why the new notations $v^\pm, p^\pm$ were introduced above).

The baffles are considered infinitesimally thin thus the velocity on both their sides are equal: $v^+ = v^u$ and $v^- = v^b$, according to notations used in Eqs. (8), (6). The pressure however, may be different on both sides yielding

$$\begin{aligned} \Delta p^+ &= p^+ - p^u, \\ \Delta p^- &= p^u - p^-, \end{aligned} \tag{9}$$

in the notations of Eqs. (6), (8), (9). When inverted, the above equations result in

$$\begin{gathered} v^+ = -\frac{z\Delta p^+ + \Delta p^-}{(z-1)(z+1)}\frac{s \sin sh}{j\omega\rho_o}, \quad v^- = -\frac{\Delta p^+ + z\Delta p^-}{(z-1)(z+1)}\frac{s \sin sh}{j\omega\rho_o}, \\ z = \frac{s \sin 2sh - \omega^2\rho_o G \cos 2sh}{s \sin sh - \omega^2 \rho_o G \cos sh}, \end{gathered} \tag{10}$$

which are the fundamental equations, in spectral domain, for the considered boundary-value problem. Here, $r \in (-\infty, \infty)$ is the spectral variable and $s = \sqrt{k^2 - r^2}$.

3. Boundary conditions

The analogous equations in spatial domain, although never explicitly written in this paper (the same notations are used for variables in spatial and spectral domains), help us in formulation of the boundary conditions on the planes of the baffle systems:

$$\begin{gathered} v^+ = 0, x \in \mathcal{D}_1;\ \Delta p^+ = 0, x \in \mathcal{D}_2, \\ v^- = 0, x \in \mathcal{D}_2;\ \Delta p^- = 0, x \in \mathcal{D}_1, \\ \mathcal{D}_1 : x \in -\Lambda/4, \Lambda/4 + n\Lambda, \\ \mathcal{D}_2 : x \in -\Lambda/4, \Lambda/4 + (n+1/2)\Lambda, \end{gathered} \tag{11}$$

where Λ is the period of baffles which width is assumed $\Lambda/2$, n is integer number.

For the sake of the BIS expansion method used below to solve the problem, we introduce the x-derivative of Δp. In spectral domain and new notation applied here for convenience, $\dot{p}^\pm = -jr\Delta p^\pm/(\omega\rho_o)$ and Eqs. (10) transform into

$$v^+ = \frac{s}{r}\frac{(z\dot{p}^+ + \dot{p}^-)\sin sh}{(z-1)(z+1)},\ v^- = \frac{s}{r}\frac{(\dot{p}^+ + z\dot{p}^-)\sin sh}{(z-1)(z+1)}, \tag{12}$$

which relations have certain asymptotic property that will be exploited later:

$$v^+ = -S_r\dot{p}^+/2,\ v^- = -S_r\dot{p}^-/2, \tag{13}$$

at $r \to \pm\infty$ (note that $G \to 0$ and $z \to \exp|r|h$). The function S_ν is defined (for real ν) by $S_\nu = 1$ for $\nu \geq 0$ and -1 otherwise.

Using $\dot{p}^\pm$ in boundary conditions (11) instead of Δp does not suffice because the condition $\dot{p} \sim \partial_x\Delta p = 0$ would admit a solution $\Delta p = \text{const}$. To set this const to zero, the boundary conditions formulated for $\dot{p}^\pm$ must be appended by a discrete condition at arbitrary point of the corresponding domain. Explicitly

$$\begin{aligned} \dot{p}^+ &= 0, x \in \mathcal{D}_2, \text{ and } \Delta p^+ = 0, x = n\Lambda + \Lambda/2, \\ \dot{p}^- &= 0, x \in \mathcal{D}_1, \text{ and } \Delta p^- = 0, x = n\Lambda, \end{aligned} \tag{14}$$

must be applied instead of the boundary conditions (11) concerning the acoustic pressure.

4. The BIS expansion

In periodic systems like the one under consideration, the acoustic wavefield is represented by the Bloch expansion

$$\dot{p}^\pm = \sum_{-\infty}^{\infty} p_n^\pm e^{-jr_n x},\ r_n = \xi + nK,\ \xi \in (-K/2, K/2), \tag{15}$$

for acoustic pressure, and similarly for normal velocity in the baffle planes (with expansion coefficients $v_n^\pm$). The domain of ξ is constrained to the first Brillouin zone, as usual, however due to the system symmetry, it suffices to consider $\xi > 0$. According to this expansion, r should be substituted by $\xi + nK$ in Eqs. (12).

The subsequent expansion is required by the BIS expansion method [3], where $P_\nu = P_\nu(0)$ is Legendre function:

$$\begin{aligned} v_n^+ &= \sum_m \alpha_m S_{n-m} P_{n-m}, \; p_n^+ = \sum_m \alpha'_m P_{n-m}, \\ v_n^- &= \sum_m \beta_m P_{n-m}, \; p_n^- = \sum_m \beta'_m S_{n-m} P_{n-m}, \end{aligned} \tag{16}$$

to satisfy the boundary conditions (11), (14) (without discrete constraints for $\Delta p^\pm$ which will be considered later). Indeed,

$$\begin{aligned} &\sum_n P_{n-m} e^{-jnKx} = 0 \text{ in } \mathcal{D}_2, \text{ and} \\ &\sum_n S_{n-m} P_{n-m} e^{-jnKx} = 0 \text{ in } \mathcal{D}_1 \end{aligned}$$

as required. The size of the finite vectors α, α', β, β' is a matter of the expansion accuracy discussed below.

Substituting high order Bloch component of the series (15), (16) into Eqs. (13) where the limit $r \to \infty$ corresponds to $n \to \pm\infty$ at $K > \xi > 0$, and m assumed finite, one obtains

$$\begin{aligned} S_{n-m} \alpha_m P_{n-m} &= S_n \alpha'_m P_{n-m}/2, \\ \beta_m P_{n-m} &= S_n \beta'_m S_{n-m} P_{n-m}/2, \end{aligned} \tag{17}$$

where the summation symbol over m is dropped. We quess that

$$\alpha'_m = 2\alpha_m, \; \beta'_m = 2\beta_m \tag{18}$$

must hold to satisfy an infinite number of equations for $n < -N$ and $n > N$ assuming that $-N \le m \le N+1$. N must be sufficiently large to assure that Eqs. (12), taken at $r = \xi \pm NK$, yields Eqs. (13) with sufficient accuracy. In computations presented in next section, N is chosen about $3k/k_t$.

In this study, we further simplify the considered system by assuming certain symmetry of the wavefield:

$$\begin{aligned} \Delta p = \Delta p^+(x) &= \Delta p^-(x - \Lambda/2), \\ v = v^+(x) &= v^-(x - \Lambda/2). \end{aligned} \tag{19}$$

These assumptions agree with the original comb/solid arrangement: the incident wave in the comb causes the comb surfaces to vibrate almost in phase, neglecting small phase shift resulting from the wave propagation along the tooth height that usually equals only a small fraction of the wavelength in comb.

Applying (19) in (12) and using the identity $P_n(-x) = S_n(-1)^n P_n(x)$, one obtains equations (summation symbol over m dropped)

$$\begin{aligned} \alpha_m S_{n-m} P_{n-m} &= \frac{s_n}{r_n} \frac{z_n \alpha'_m P_{n-m} + \beta'_m S_{n-m} P_{n-m}}{(1+z_n)(1-z_n)} \sin s_n h, \\ \beta_m P_{n-m} &= \frac{s_n}{r_n} \frac{\alpha'_m P_{n-m} + z_n \beta'_m S_{n-m} P_{n-m}}{(1+z_n)(1-z_n)} \sin s_n h. \end{aligned} \tag{20}$$

Combining them with different signs shows that the solution satisfying the assumptions (19) is obtained when

$$\beta_m = (-1)^m \alpha_m, \; \beta'_m = (-1)^m \alpha'_m, \tag{21}$$

what substituted into the first equation yields

$$\left[\frac{s_n}{r_n} \frac{(z_n + (-1)^n) \sin s_n h}{(1-z_n)(1+z_n)} \alpha'_m - \alpha_m S_{n-m} \right] P_{n-m} = 0.$$

Finally using (18), one gets

$$\alpha_m \left[2 \frac{s_n}{r_n} \frac{\sin s_n h}{z_n + (-1)^n} + S_{n-m} \right] P_{n-m} = 0, \tag{22}$$

that must be satisfied for any n, but the equations are nontrivial only for $-N \le n \le N$ (for $0 < \xi < K$, $-N \le m \le N+1$ and N chosen properly according to the above discussion). Here, z_n is z of argument $r = \xi + nK$; similarly s_n.

These equations must be appended by the discrete conditions, Eqs. (14). Integrating $\dot{p}^{\pm} \sim \Delta p_{,x}$ over x and using certain identities for Legendre functions [3], one obtains

$$\Delta p^+|_{x=\Lambda/2} = 2\omega\rho_o \sum_{m,n} \alpha_m \frac{e^{-j(\xi+nK)\Lambda/2}}{-j(\xi+nK)} P_{n-m} = \\ j\Lambda \frac{\omega\rho_o}{\sin \pi\xi/K} e^{-j\xi\Lambda/2} \sum_m \alpha_m P_{-m-\xi/K}(0).$$

Similarly for Δp^- at $x = 0$. Finally, Eqs. (14) yield

$$\sum_m \alpha_m P_{-m-\xi/K} = 0, \tag{23}$$

and $\sum (-1)^m \beta_m P_{-m-\xi/K} = 0$ which, according to Eqs. (21), are identical. Equations (22) and (23) together make a complete system of $2N+2$ equations for $2N+2$ unknowns.

5. The dispersive curves

We seek a nontrivial solution to the homogeneous system of Eqs. (22, 23), its condition of existence is known to be

$$\mathrm{Det}(\xi) = 0, \qquad \xi = r_c, \tag{24}$$

where r_c is small under the assumed conditions that $\Lambda \approx \lambda_R = 2\pi/k_R$; k_R is the wavenumber of Rayleigh wave on a stress-free surface of solids.

In the considered systems, Rayleigh waves in both solids are coupled by an air layer. Moreover the system of periodic baffles couples Rayleigh waves propagating in opposite directions. These are the reason that the wavenumber k_c we seek, differs slightly from k_R,

$$k_c = \pm r_c + K \approx k_R, \tag{25}$$

depending on K as will be shown later below. Naturally, due to the system symmetry, $-k_c$ is the wavenumber of the wave propagating left.

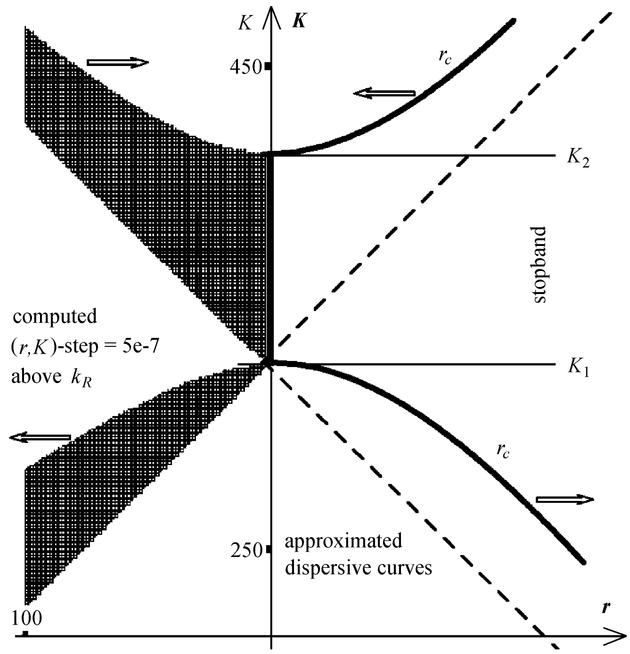

Fig. 2. Two branches of dispersive curves (solid lines, on right) on both sides of stopband domain (K_1, K_2) that approximate the boundary between domains of different sign of computed values of Det, Eq. (24) (on left); $h = (\pi + 0.018)/k$. Dash lines concern the case not discussed here: vanishing average Δp instead on v on baffles (case of $1/\text{Det} = 0$). Arrows mark the direction of Rayleigh wave propagation associated with the given branch.

In the formulated problem, we assumed fixed ω, thus k_R, k_l, k_t are all fixed, while all other parameters, like air gap thickness and baffle period Λ, are variables that can be considered relative quantities to λ_R. Varrying K with ω fixed is thus equivalent to variation of $1/\omega$ with K fixed, neglecting air gap. Small positive variation δK corresponds to small negative variation $-\delta\omega$ (neglecting air gap variation; it is another parameter that will be optimized to obtain the strongest interaction of the wave with the baffle system). This considerations shows that the dependence of k_c on K, or $r_c(K)$, describes in fact, the dispersive curve k_c on ω.

The numerical example presented in Fig. 2 for aluminium solids (at $\omega = 10^6\mathrm{s}^{-1}$: $k_l = 0.1581$, $k_t = 0.3162$, $k_R = 0.3391$, all in mm^{-1}, and c=335ms^{-1}, $\rho = 2700$, $\rho_o = 1.3$ kgm^{-3}) clearly shows that the discussed dispersive relation has an approximation

$$r_c = \sqrt{(K - K_1)(K - K_2)}. \tag{26}$$

There is a stopband, a typical feature of dispersive curves in periodic systems, centered at $K_c = (K_1 + K_2)/2$ where r_c has no longer real value like outside it for $|K - K_c| > \kappa = (K_2 - K_1)/2$, neglecting bulk waves. It was applied $\mathrm{Im}\{G\}$ in the computed examples to neglect bulk waves (although this does not change the condition of Rayleigh wave propagation in the system). This simplification results in real values of computed Det that differs negligible from Re{Det} obtained with with full G accounted for. This also proves that the damping of the wave due to the bulk radiation in the system is very small in the assumed conditions (small $\mathrm{Im}\{r_c\}$).

The relative stopband width κ/K_c is in this example of an order of 1.5e-4. At the stopband center, the imaginary solution to r_c represent a decaying standing wave. It decays on rather long distance of about a thousand wavelength meaning that the interaction of Rayleigh waves with baffles is not very strong, as expected. However, as compared to ordinary combs discussed in [2], it is significant, and thus promising for applications presented in the Introduction.

6. Conclusions

The analysis shows that the interaction between the Rayleigh waves and baffles residing in an air gap is strong enough to manifest itself on dispersive curves by a corresponding stopband. Although not very strong, it is still interesting for applications of contactless generation of Rayleigh waves in solids and deserves deeper theoretical and experimental investigations.

References

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