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PROPAGATION OF THE MAIN SIGNAL IN A DISPERSIVE LORENTZ MEDIUM

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Evolution of the main signal in a Lorentz dispersive medium is considered. The signal propagating in the medium is excited by a sine-modulated pulse signal, with its envelope described by a hyperbolic tangent function. Both uniform and non-uniform asymptotic representations for the signal are found. It is shown how the uniform representation reduces to the non-uniform one. The results obtained are illustrated with a numerical example.

1. Introduction

Investigations on propagation of pulse signals in dispersive media date back to the beginning of 20th century. The fundamental research in this area is due to SOMMERFELD [1] and BRILLOUIN [2, 3]. Although steady interest in this kind of propagation was observed in the literature since then, a new impetus has been added recently due to new applications of the theory in fiber-optics communication and integrated-optics. Also, the knowledge of pulse propagation in dispersive medium, and of accompanying electromagnetic energy losses in the medium, became of vital importance in radiotherapy. A significant contribution to the research on dispersion phenomena in Lorentz media is due to OUGHSTUN and SHERMAN [4]. Equipped with better computation techniques and advanced asymptotic methods, they extended the analysis to models better reflecting practical applications. In particular, they considered signals with a finite rise time and employed uniform asymptotic expansions in their analysis. (Uniform expansions remain valid as their parameters vary in predetermined intervals while non-uniform expansions break down at some parameters values.)

Here, we also consider the evolution of the main signal excited in a Lorentz dispersive medium by a signal with a finite rise time. However, unlike Oughstun and Sherman work, where the envelope of the initial signal is described by an everywhere smooth function of time which tends to zero as time goes to minus infinity, our exciting signal is switched abruptly at a finite time instant, and vanishes identically for earlier times. In the analysis carried out in this paper we apply the BLEISTEIN and HANDELSMAN [10] theory of uniform asymptotic evaluation of integrals with nearby saddle point and an algebraic singularity. We show, how the uniform representation of the evolution of the main signal reduces to the non-uniform representation, which can otherwise be obtained by residues.

The results obtained here are illustrated with a numerical example.

2. Formulation of the problem

Consider the problem of an electromagnetic plane wave propagation in a homogeneous, isotropic medium, whose dispersive properties are described by the Lorentz model of resonance polarization. The complex index of refraction in the medium is given by [4]

$$n(\omega) = \left(1 - \frac{b^2}{\omega^2 - \omega_0^2 + 2i\delta\omega}\right)^{1/2}. \tag{1}$$

Here, $b^2 = 4\pi Ne^2/m$, where N, e and m represent the number of electrons per unit volume, electron charge and its mass, δ is a damping constant, respectively, and ω_0 is a characteristic frequency.

Let the signal $A_0(t)$ in the plane $z = 0$ be a sine wave of a fixed real frequency ω_c with its envelope described by a real function $u(t)$, identically vanishing for $t < 0$, i.e.

$$A_0(t) = \begin{cases} 0 & t < 0 \\ u(t)\sin(\omega_c t) & t \geq 0. \end{cases} \tag{2}$$

Then arbitrary component of the wave propagating in the direction of increasing z (or of a corresponding Hertz vector) can be represented in the medium by the scalar function [4]

$$A(z,t) = \frac{1}{2\pi}\,\mathrm{Re}\left\{ i \int\limits_{ia-\infty}^{ia+\infty} \tilde{u}(\omega - \omega_c) \exp\left[\frac{z}{c}\phi(\omega,\theta)\right] d\omega \right\}, \tag{3}$$

where $\tilde{u}(\omega)$ is the Laplace transform of $u(t)$. The complex phase function $\phi(\omega,\theta)$ is given by

$$\phi(\omega,\theta) = i\omega[n(\omega) - \theta], \tag{4}$$

where the dimensionless parameter

$$\theta = \frac{ct}{z} \tag{5}$$

describes the space-time point $(z,\, t)$.

It is here assumed that the envelope of the incident pulse is described by

$$u_\beta(t) = \begin{cases} 0 & t < 0 \\ \tanh \beta t & t \geq 0, \end{cases} \tag{6}$$

where the parameter $\beta > 0$ determines the rate of the pulse growth.
The Laplace transform of $u(t)$ is

$$\tilde{u}_\beta(\omega) = \frac{1}{\beta}\mathcal{B}\left(-\frac{i\omega}{2\beta}\right) - \frac{i}{\omega}, \qquad \mathrm{Im}\,\omega > 0, \tag{7}$$

and the beta function $\mathcal{B}$ is related to the psi function ψ by [9]

$$\mathcal{B}(x) = \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right]. \tag{8}$$

By using (7) in (3) we obtain the formula

$$A(z,t) = \frac{1}{2\pi}\,\mathrm{Re}\left\{ i \int\limits_{ia-\infty}^{ia+\infty} \left[\frac{1}{\beta}\mathcal{B}\left(-\frac{i(\omega-\omega_c)}{2\beta}\right) - \frac{i}{\omega-\omega_c}\right] e^{\frac{z}{c}\phi(\omega,\theta)}\, d\omega \right\}, \tag{9}$$

which describes the dynamics of the signal excited at $z = 0$ by $A_0(t)$, and propagating in the Lorentz dispersive medium in the direction of growing z. The uniqueness of this solution is proved in Sec. 2 of [3].

In this work we study the poles contribution to the asymptotic expansion of $A(z,t)$. We denote this contribution by $A_c(z,t)$ and find both non-uniform and uniform asymptotic expressions for it.

3. Non-uniform asymptotic expression for $A_c(z,t)$

In finding an asymptotic expansion of the integral defined by (9) it is essential to determine the location of its critical points, including saddle points and the poles in the complex ω-plane. The equation governing the location of the saddle points does not seem to be solvable exactly. Instead, different approximate solutions were obtained by Brillouin [3], Kelbert and Sazonov [6], and Oughstun and Sherman [4]. Recently, a new approximation for this location was obtained in [7]. In this work, however, we shall employ a numerical approximation of the saddle point solution obtained with the help of *Mathematica* computer program, and based on interpolation techniques.

As in Oughstun and Sherman study, we deform the original contour of integration to the Olver type contour $P(\theta)$ [8], which passes through the near and distant saddle points. The pole contribution to the asymptotic expansion of (9) occurs if in the process of deformation one or more poles of $\tilde{u}_\beta(\omega)$ are crossed. From the series representation of this function [5]

$$\tilde{u}_\beta(\omega) = \frac{i}{\omega} - 2i\left(\frac{1}{\omega+2i\beta} - \frac{1}{\omega+4i\beta} + \cdots\right) \tag{10}$$

it follows that the integrand in (9) has an infinite set of poles at $\omega = \omega_c - 2im\beta$, $m = 0, 1, 2, \cdots$ in the half-plane $\mathrm{Im}\,\omega \le 0$, which are located along a line, parallel to the ω imaginary axis. If β is big enough, only the real pole $\omega = \omega_c$ is of importance, since the remaining poles are not crossed in the process of contour deformation. If, however, β is small, one or more of the remaining poles can be crossed, and their contributions must then be taken into account.

Let θ_s be the value of θ, at which the deformed contour crosses the pole at $\omega = \omega_c$ in (9), ω_c being real and positive. Then, by the Cauchy theorem,

$$A_c(z,t) = \begin{cases} 0, & \theta < \theta_s, \\ e^{-\frac{z}{c}\omega_c n_i(\omega_c)} \sin\left[\frac{z}{c}\omega_c(n_r(\omega_c) - \theta)\right], & \theta > \theta_s. \end{cases} \tag{11}$$

Here, $n_r(\omega_c)$ and $n_i(\omega_c)$ stand for real and imaginary parts of $n(\omega_c)$, respectively.

Upon introducing the amplitude attenuation coefficient [4]

$$\alpha(\omega_c) = \frac{\omega_c}{c} n_i(\omega_c), \tag{12}$$

and the propagation factor

$$\zeta(\omega_c) = \frac{\omega_c}{c} n_r(\omega_c), \tag{13}$$

$A_c(z,t)$ can be written down as

$$A_c(z,t) = \begin{cases} 0, & \theta < \theta_s, \\ e^{-z\alpha(\omega_c)} \sin\left[\zeta(\omega_c)z - \omega_c t\right], & \theta > \theta_s. \end{cases} \tag{14}$$

It follows then that for real and positive ω_c the pole contribution to the asymptotic expansion of $A(z,t)$ oscillates in time at the frequency ω_c and decreases along its propagation distance z with time independent attenuation coefficient $\alpha(\omega_c)$.

The pole contribution (14) represents a discontinuous function of θ, while the integral representation of $A(z,t)$ changes continuously with θ. As pointed in [4], this fact is of little significance if z is finite and the pole is bounded away from the dominant saddle point at $\omega = \omega_s$. Denote $X(\omega,\theta) = \operatorname{Re}\phi(\omega,\theta)$. Then $e^{-(z/c)X(\omega_c,\theta)}$ is negligible in comparison to the saddle point contribution which has the magnitude $e^{-(z/c)X(\omega_s,\theta)}$. Hence, the discontinuous behaviour of $A_c(z,t)$ is then also negligible.

4. Uniform asymptotic expression for $A_c(z,t)$

The situation becomes different if the dominant saddle point approaches the pole at $\omega = \omega_c$ (or any other pole). In this case $X(\omega_c,\theta)$ is comparable with $X(\omega_s,\theta)$ and so are the pole and the branch point contributions to the asymptotic expansion of $A(z,t)$. To obtain a continuous asymptotic representation for $A_c(z,t)$, a uniform approach, as proposed by Bleistein and Handelsman [10] will be here used.

Let us consider the first pole at $\omega = \omega_c$. From (9) and (10) it follows that the function to be asymptotically evaluated is

$$A_c(z,t) = -\frac{1}{2\pi} \operatorname{Re}\left\{ \int\limits_{P(\theta)} \frac{e^{\lambda\phi(\omega,\theta)}}{\omega - \omega_c}\, d\omega \right\}, \tag{15}$$

where $\lambda = z/c$. (One can verify that the contour $P(\theta)$ through a near (distant) saddle point $\omega = \omega_s$ makes the angle $\pi/4$ $(3\pi/4)$ with the real axis.)

In accordance with the Bleistein and Handelsman method we introduce a new variable of integration τ, defined by

$$\phi(\omega,\theta) = -\frac{\tau^2}{2} - \gamma\tau + \rho = \Psi(\tau,\theta). \tag{16}$$

The quantities γ i ρ are chosen so that $\tau = -\gamma$ is the image of the saddle point $\omega = \omega_s$ and $\tau = 0$ is the image of $\omega = \omega_c$. Then,

$$\rho(\theta) = \phi(\omega_c,\theta) \quad \text{and} \quad \gamma(\theta) = \sqrt{2[\phi(\omega_s,\theta) - \phi(\omega_c,\theta)]}. \tag{17}$$

The complex-valued function $\gamma(\theta)$ is defined such that it is a smooth function of θ when its argument varies in the interval $-\pi < \operatorname{Arg}\gamma \leq \pi$.

One finds from (16) that

$$\tau + \gamma = \sqrt{2[\phi(\omega_s, \theta) - \phi(\omega, \theta)]}, \tag{18}$$

and hence for ω near ω_s:

$$\tau + \gamma \approx \sqrt{-\phi_{\omega\omega}(\omega_s, \theta)}\,(\omega - \omega_s)[1 + O(\omega - \omega_s)]. \tag{19}$$

The steepest descent path through the saddle point $\tau = -\gamma$ runs parallel to the real axis. Upon using (16) in (15) the function $A_c(z,t)$ takes the form

$$A_c(z,t) = -\frac{1}{2\pi}\operatorname{Re}\left\{\int\limits_{C(\theta)} \frac{G_0(\tau,\theta)}{\tau} e^{\lambda\Psi(\tau,\theta)}\,d\tau\right\}, \tag{20}$$

where

$$G_0(\tau,\theta) = \frac{\tau}{\omega - \omega_c}\frac{d\omega}{d\tau} \tag{21}$$

and $C(\theta)$ is the image of $P(\theta)$ under (16).

We now expand G_0 in the form

$$G_0(\tau,\theta) = a_0 + a_1\tau + \tau(\tau+\gamma)H_0(\tau,\theta), \tag{22}$$

where $H_0(\tau,\theta)$ is a regular function of τ. Since the last term vanishes at both critical points $\tau = -\gamma$ and $\tau = 0$, the coefficients a_0 and a_1 are given by

$$a_0 = G_0(0,\theta) \quad \text{and} \quad a_1 = \frac{G_0(0,\theta) - G_0(-\gamma,\theta)}{\gamma}. \tag{23}$$

By L'Hospital's rule:

$$\lim_{\tau\to 0}\frac{\omega - \omega_c}{\tau} = \lim_{\tau \to 0}\frac{d\omega}{d\tau}, \tag{24}$$

and hence

$$G_0(0,\theta) = 1. \tag{25}$$

Furthermore, from (19)

$$\lim_{\tau\to -\gamma}\frac{d\omega}{d\tau} = -\frac{1}{\sqrt{-\phi_{\omega\omega}(\omega_s,\theta)}}, \tag{26}$$

and thus,

$$G_0(-\gamma,\theta) = -\frac{\gamma}{\omega_s - \omega_c}\frac{1}{\sqrt{-\phi_{\omega\omega}(\omega_s,\theta)}}. \tag{27}$$

In this manner we obtain

$$a_0 = 1 \quad \text{and} \quad a_1 = \frac{1}{\gamma} + \frac{1}{\omega_s - \omega_c}\frac{1}{\sqrt{-\phi_{\omega\omega}(\omega_s,\theta)}}. \tag{28}$$

If now (22) is inserted into (20), and the resulting canonical integrals ([10]) are expressed by special functions, the following result is found

$$A_c(z,t) = \frac{1}{2\pi}\operatorname{Re}\left\{e^{\lambda\rho}\left[W_{-1}(\sqrt{\lambda}\gamma) + \frac{a_1}{\sqrt{\lambda}}W_0(\sqrt{\lambda}\gamma)\right] + R_0(\lambda,\theta)\right\}, \tag{29}$$

where,

$$W_0(z) = \sqrt{2\pi} e^{\frac{z^2}{2}} \quad \text{and} \quad W_{-1}(z) = i \int_{-iz}^{\infty} e^{-\frac{s^2}{2}} ds. \tag{30}$$

The remainder of the expansion, R_0, is given by

$$R_0(\lambda, \theta) = \lambda^{-1} \int_{C(\theta)} G_1(\tau, \theta) e^{\lambda \Psi(\tau,\theta)} \, d\tau, \tag{31}$$

with

$$G_1(\tau, \theta) = \tau \frac{dH_0}{d\tau}. \tag{32}$$

In arriving at (31) we integrated the last term in (22) by parts and neglected the boundary contributions as being asymptotically negligible.

The function W_{-1} can be expressed in terms of the complementary error function $\mathrm{erfc}(z) = 2/(\sqrt{\pi}) \int_z^{\infty} e^{-s^2} ds$. By using (28) and (30) in (29) we arrive at the following uniform asymptotic representation

$$A_c(z,t) \sim \tag{33}$$
$$\mathrm{Re}\left\{ e^{\lambda\rho} \left[\frac{i}{2} \,\mathrm{erfc}\left(i\gamma \sqrt{\frac{\lambda}{2}} \right) - \frac{e^{\frac{\lambda\gamma^2}{2}}}{\sqrt{2\pi\lambda}} \left(\frac{1}{\gamma} + \frac{1}{(\omega_s - \omega_c)\sqrt{-\phi_{\omega\omega}(\omega_s, \theta)}} \right) \right] \right\},$$

of the main signal in the medium, provided only the first pole $\omega = \omega_c$ interacts with the saddle point.

This asymptotic formula is valid for small and large values of $\sqrt{\lambda}\gamma$. In particular, if $\gamma \to 0$, the components of the last parentheses blow up, but their sum remains bounded.

If $\sqrt{\lambda}|\gamma|$ is large, the error function in (33) can be approximated by its asymptotic expansion (comp. [11])

$$\mathrm{erfc}(iy) = \eta(y) - e^{y^2} \left[\frac{i}{\sqrt{\pi} y} + O(y^{-3}) \right], \tag{34}$$

where

$$\eta(y) = \begin{cases} 0, & -\pi < \mathrm{Arg}(y) < 0, \\ 1, & \mathrm{Arg}(y) = -\pi \;\text{ or }\; 0, \\ 2, & 0 < \mathrm{Arg}(y) < \pi. \end{cases} \tag{35}$$

Upon using this expansion in (33), the non-uniform asymptotic representation of the main signal evolution results:

$$A_c(z,t) \sim \mathrm{Re}\left\{ \frac{i e^{\lambda\phi(\omega_c,\theta)}}{2} \eta(\gamma) + \frac{e^{\lambda\phi(\omega_s,\theta)}}{\omega_s \sqrt{-\phi_{\omega\omega}(\omega_s,\theta)}} \right\}. \tag{36}$$

If $\mathrm{Arg}(\gamma) < 0$, which occurs when the pole at $\omega = \omega_c$ is located to the right with respect to the contour $P(\theta)$, the main signal is absent in $A_c(z,t)$, and the only term that appears is that proportional to $(-\phi_{\omega\omega}(\omega_s,\theta))^{-1/2}$. This term can be interpreted as due to the saddle point ω_s.

If, $\mathrm{Arg}(\gamma)>0$, which occurs after the contour crosses over the pole, then in addition to the term proportional to $(-\phi_{\omega\omega}(\omega_s,\theta))^{-1/2}$, a new term appears

$$\mathrm{Re}\left\{ie^{\lambda\phi(\omega_c,\theta)}\right\} = e^{-\frac{z}{c}\omega_c n_i(\omega_c)}\sin\left[\frac{z}{c}\omega_c[n_r(\omega_c)-\theta]\right]. \tag{37}$$

It represents the main signal and its form fully agrees with (11).

In this manner we have obtained both uniform and non-uniform asymptotic representations for the evolution of the main signal in the medium, described by (33) and (36), respectively. While the uniform representation applies for small and large values of $\sqrt{\lambda}\gamma$, the non-uniform representation is valid only for sufficiently large $\sqrt{\lambda}|\gamma|$.

One remark should now be made. The Bleistein-Handelsman theory assumes that the saddle point is of the first order, i.e. $\phi_{\omega\omega}(\omega_s,\theta)$ is never zero. In the present context this assumption is satisfied everywhere except for the special value of $\theta=\theta_1$, where two coalescing *near* simple saddle points merge on the ω-imaginary axis to form a saddle point of the second order. Hence $\phi_{\omega\omega}=0$ at $\theta=\theta_1$, and consequently both asymptotic representations of $A_c(z,t)$, as given by (33) and (36), are there invalid. Therefore, strictly speaking, if the carrier frequency ω_c lies below the anomalous dispersion region, (33) is uniform for $\theta>\theta_1$.

5. Numerical example

A numerical example given below illustrates the results obtained in the previous section. It is assumed that the Lorentz medium is described by Brillouin's choice of medium parameters

$$b=\sqrt{20.0}\times 10^{16}s^{-1}, \qquad \omega_0=4.0\times 10^{16}s^{-1}, \qquad \delta=0.28\times 10^{16}s^{-1}, \tag{38}$$

and additionally, $\lambda=3.0\times 10^{-15}$, and $\omega_c=2.0\times 10^{16}s^{-1}$. The latter choice implies that in this example the saddle point in question is the near one.

Let us first suppose that the parameter β in (6) is large enough, say of the order of 10^{17} or more, to ensure that the second pole $\omega_{c2}=\omega_c-2i\beta$ is sufficiently distant from the contour P, and, in particular, it is not crossed in the process of the contour deformation. Then only the real pole at $\omega=\omega_c$ is of interest. Under this assumption the real and imaginary parts of $\gamma(\theta)$, as given by (16), are shown in Fig. 1. In order to determine numerical values of the function $\omega_s(\theta)$ an interpolation technique provided by the *Mathematica* computer program has been employed. The evolution of the main signal, as predicted by the uniform asymptotic representation (33), is depicted in Fig. 2. The anomaly in the plot at $\theta=\theta_1\approx 1.5$, results from vanishing $\phi_{\omega\omega}(\omega_s,\theta)$ at $\theta=\theta_1$ and, as discussed in the previous section, the result obtained from (33) breaks down there. Fig. 3 shows the corresponding plot obtained from (33), in which the term proportional to $1/\sqrt{-\phi_{\omega\omega}}$ has been dropped.

Assume now that β is sufficiently small, such that the second pole at $\omega_{c2}=\omega_c-2i\beta$ may appear close to, or be crossed by the deformed contour P. To fix our attention let

$\beta = 5.0 \times 10^{14} s$. In this case the asymptotic expression for $A_c(z,t)$, as given by (33), must be augmented by an expression similar in form, but multiplied by -2 and with ω_c replaced by ω_{c2} (comp. (10)).

The corresponding plot is shown in Fig. 4. It is seen that the growth of the main signal is slower then in the case of large β.

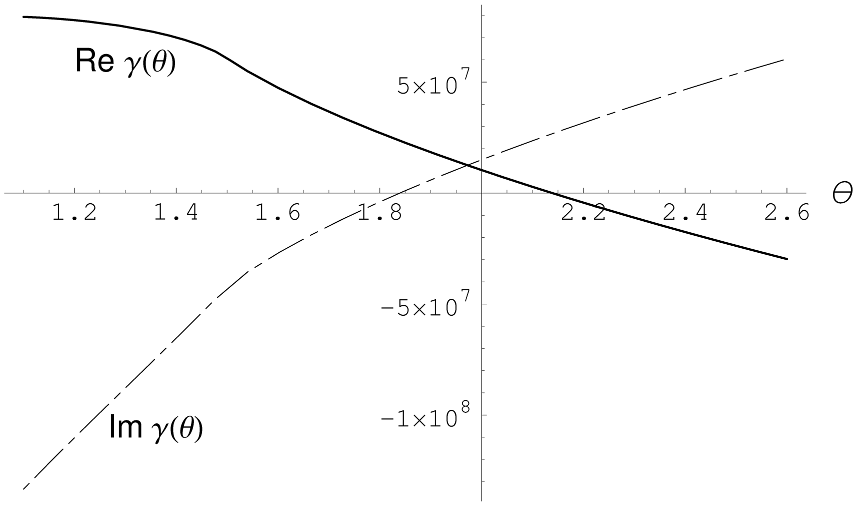

Fig. 1. Real and imaginary parts of the function $\gamma(\theta)$. Here, $\omega_c = 2.0 \times 10^{16}$ s^{-1} and the medium is described by Brillouin's choice of parameters.

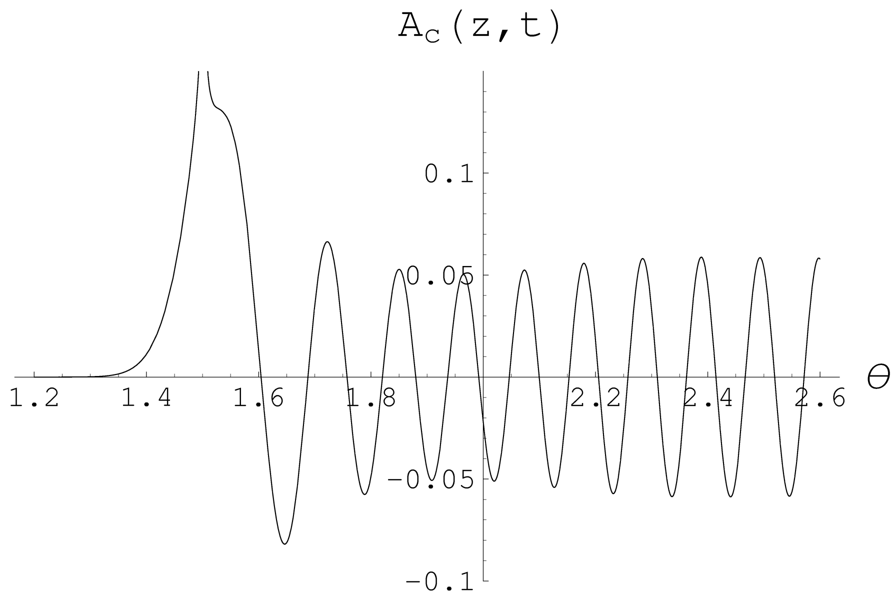

Fig. 2. Representation of the main signal evolution in the medium, when the simple saddle point interacts with the real pole at $\omega = \omega_c$ only. Here, $\omega_c = 2.0 \times 10^{16}$ s^{-1} and $\lambda = 3.0 \times 10^{-15}$ s.

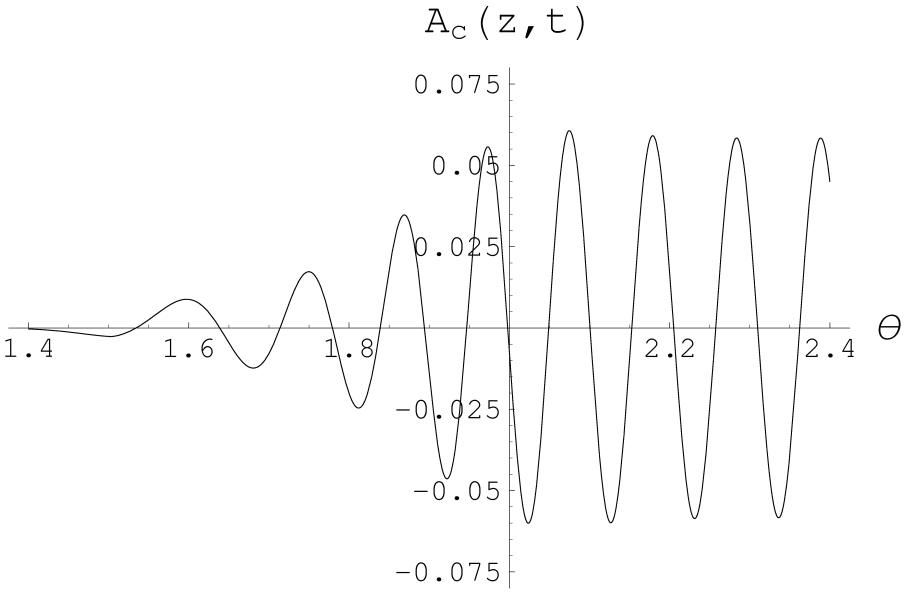

Fig. 3. Representation of the main signal evolution in the medium, with the term proportional to $1/\sqrt{-\phi_{\omega\omega}(\omega(\theta),\theta))}$ dropped from Eq. (33). Here, $\omega_c = 2.0 \times 10^{16}$ s^{-1} and $\lambda = 3.0 \times 10^{-15}$ s.

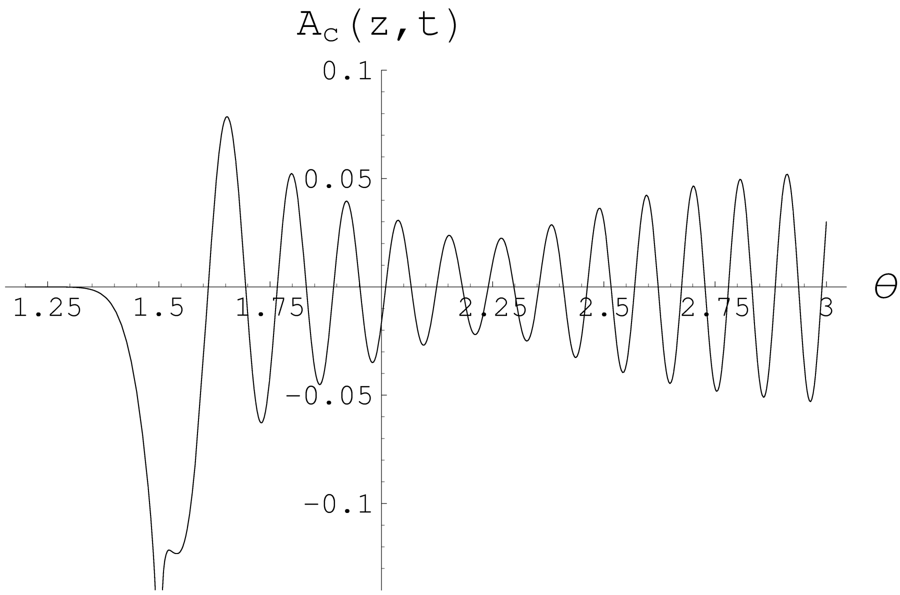

Fig. 4. Representation of the main signal evolution in the medium, when the simple saddle point interacts with two poles at $\omega = \omega_c$ and $\omega = \omega_c - 2i\beta$. Here, $\omega_c = 2.0 \times 10^{16}$ s^{-1}, $\lambda = 3.0 \times 10^{-15}$ s and $\beta = 5.0 \times 10^{14}$ s^{-1}.

6. Conclusions

In this paper the problem of electromagnetic signal propagation in a dispersive Lorentz medium is considered. It is assumed that the exciting signal is turned on at a finite time instant. The signal rapidly oscillates and its envelope is described by a hyperbolic tangent function. While propagating in the medium, the signal splits into three components: Sommerfeld and Brillouin precursors, and the main signal. In this work we find non-uniform and uniform asymptotic representations for the main signal evolution. The former representation is readily obtainable by residues. The latter representation is constructed with the help of Bleistein-Handelsman method of uniform asymptotic evaluation of integrals with nearby simple saddle point and an algebraic singularity. We show how the uniform representation, expressed in terms of complementary error integral, reduces to the non-uniform representation. The results here obtained are illustrated with a numerical example. This paper is a complement to our earlier works on Sommerfeld and Brillouin precursors ([12], [13]).

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