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GOLAY SEQUENCES – SIDE-LOBE – CANCELING CODES FOR ULTRASONOGRAPHY

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In this paper three different methods and algorithms used to calculate the pairs of Golay sequences of the different length are described. The program for simulating the pairs of the Golay sequences of different length is given. This program (using Matlab$^{\text{TM}}$ routines) is based on the basic properties of complementary series and is simple and computationally effective. Experimental results are presented in the form, which in a clear way illustrates the resolution, depth penetration and contrast dynamics of ultrasonic images obtained using the Golay coded excitation.

1. Introduction

Nowadays we can observe increasing interest in using elongation of the excitation signals rather than transmitting the single brief burst pulses, which are currently used in standard medical ultrasound scanners. There are several reasons for using coded excitation signals in preference to single pulses. First of all, it is an increase of the signal-to-noise ratio (SNR) that plays the main role in ultrasonographic imaging. Secondly, using longer signals and compressing them later on with the help of matched filtering, a short received echoes can be obtained, similar to that obtained using single brief pulses, but with much higher amplitude. It makes possible to explore the ultrasonic echoes with lower peak pressure that, in turn, is very important because it decreases the patient's exposure to potential biological effects. Another, but no less important reason of using coded excitation, is the fact that it allows using higher ultrasonic frequencies improving the axial resolution.

Among the different excitation sequences proposed in ultrasonography, Golay codes evoke more and more interest in comparison with other signals. The reason of that lies in the fact that Golay codes, like no other signals, suppress to zero the amplitude of side-lobes. This type of complementary sequences has been introduced by GOLAY [6, 7]. The pairs of Golay codes belong to a bigger family of signals, which consist of two binary sequences of the same length n, whose auto-correlation functions have the side-lobes equal in magnitude but opposite in sign. The sum of these auto-correlation functions gives a single auto-correlation function with the peak of $2n$ and zero elsewhere.

One of the main problems associated with all the binary codes discussed so far in literature is the high side-lobe level for short code lengths. Although a delta ambiguity function is rather utopian, one solution of this problem can be the utilization of a set of waveforms with complex ambiguity functions, which are in some sense as "different" as possible. By coherently combining the matched-filter responses (the complex ambiguity functions of the set), the restrictions imposed on a single ambiguity function can be bypassed.

In this paper the principle of construction and basic properties of the Golay complementary sequences are described. Also the algorithm and the program of calculation the Golay sequences of different length are given. In particular, the possibility of calculation the pairs of Golay sequences using the cyclic principle is shown.

2. Golay complementary sequences

Golay complementary sequences are pairs of binary codes belonging to a bigger family of signals called complementary pairs, which consists of two codes of the same length n, whose auto-correlation functions have side-lobes equal in magnitude but opposite in sign. Summing them up results in a composite auto-correlation function with a peak of $2n$ and zero side-lobes. Figure 1 illustrates the principle of the side-lobe-canceling for a pair of Golay codes of length 4 bits each.

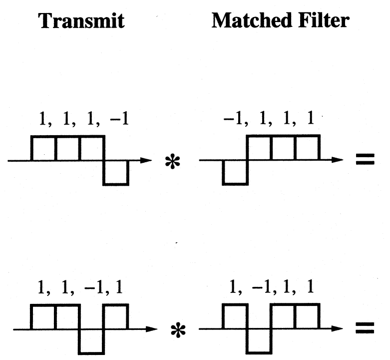

Fig. 1. The principle of the side-lobe-canceling for a pair of codes of length 4 bits each, $*$ – denotes correlation.

3. Synthesis of complementary sequences

There are several essentially different algorithms for generating Golay pairs.

For example, DOKOVIC [5] described the method of calculation the Golay sequences as follows. Let the variables a_i and b_i ($i = 1, 2, \ldots n$) be the elements of two n-long complementary series, which are equal to either "+1"or "−1".

$$\begin{aligned} A &= a_1, a_2, ..., a_n, \\ B &= b_1, b_2, ..., b_n. \end{aligned} \tag{1}$$

The ordered pair $(A; B)$ are Golay sequences of length n if and only if their associated polynomials are:

$$\begin{aligned} A(x) &= a_1 + a_2 x + \ldots + a_n x^{n-1}, \\ B(x) &= b_1 + b_2 x + \ldots + b_n x^{n-1}, \end{aligned} \tag{2}$$

and satisfy the identity

$$A(x)A(x^{-1}) + B(x)B(x^{-1}) = 2n \tag{3}$$

in the Laurent polynomial ring $Z[x, x^{-1}]$.

Let the auto-correlation functions N_A and N_B, corresponding to the sequences A and B respectively, be defined by the following expressions:

$$\begin{aligned} N_A(j) &= \sum_{i \in \mathrm{Z}} a_i a_{i+j}, \\ N_B(j) &= \sum_{i \in \mathrm{Z}} b_i b_{i+j}, \end{aligned} \tag{4}$$

where the set $a_k = 0$ if $k \notin (1, ..., n)$. Now the condition (3) can be substituted by the sum $N_A + N_B$, and

$$N_A(j) + N_B(j) = \begin{cases} 2N, & j = 0, \\ 0, & j \neq 0. \end{cases} \tag{5}$$

The sum of both autocorrelation functions is $2N$ at $j = 0$ and zero otherwise.

BUDISIN [2] in his work described the recursive method for constructing the Golay's sequences that is presented below. Let the variables $a(i)$ and $b(i)$ be the elements ($i = 0, 1, 2, \ldots 2^n - 1$) of two complementary sequences with elements "+1" and "−1" of length 2^n:

$$\begin{aligned} a_0(i) &= \delta(i), \\ b_0(i) &= \delta(i); \end{aligned} \tag{6}$$

$$\begin{aligned} a_n(i) &= a_{n-1}(i) + b_{n-1}(i - 2^{n-1}), \\ b_n(i) &= a_{n-1}(i) - b_{n-1}(i - 2^{n-1}), \end{aligned} \tag{7}$$

where $\delta(i)$ is the Kronecker delta function.

Expression (7) shows that on each step new elements of the sequences are produced by concatenation of elements $a_n(i)$ and $b_n\,(i)$ of length n.

Example:
Let $n = 1$, then i equals to 0 and 1.

$$\begin{aligned} a_1(0) &= a_0(0) + b_0(-1) = 1, \\ b_1(0) &= a_0(0) - b_0(-1) = 1, \\ a_1(1) &= a_0(1) + b_0(0) = 1, \\ b_1(1) &= a_0(1) - b_0(0) = -1. \end{aligned}$$

As a final result, we obtain two complementary sequences of the length 2^n:

$$a_1 = \{1, 1\}; \qquad b_1 = \{1, -1\}.$$

If these operations are performed recursively for $n = 2, 3, 4, \ldots$, the following complementary sequences are obtained:

$$\begin{aligned} a_2 &= \{1,\ 1,\ 1,\ -1\}; \\ b_2 &= \{1,\ 1,\ -1,\ 1\}; \\ a_3 &= \{1,\ 1,\ 1,\ -1,\ 1,\ 1,\ -1,\ 1\}; \\ b_3 &= \{1,\ 1,\ 1,\ -1,\ -1,\ -1,\ 1,\ -1\}; \\ a_4 &= \{1,\ 1,\ 1,\ -1,\ 1,\ 1,\ -1,\ 1,\ 1,\ 1,\ 1,\ -1,\ -1,\ -1,\ 1,\ -1\}; \\ b_4 &= \{1,\ 1,\ 1,\ -1,\ 1,\ 1,\ -1,\ 1,\ -1,\ -1,\ -1,\ 1,\ 1,\ 1,\ -1,\ 1\}. \end{aligned}$$

The similar method of generating the complementary code pairs, differing only in the applied mathematical formalism, has been described by MENDIETA *et al.* [8]. This method can be applied to sequences of length n to obtain another code pair of length $2n$:

$$\left[\frac{A}{B}\right] \to \left[\frac{A \oplus B}{A \oplus \overline{B}}\right], \tag{8}$$

where $\overline{B}$ is the inverse of B and $\oplus$ indicates concatenation of functions. This procedure may be iterated in the following way:

$$\begin{aligned} \left[\frac{A}{B}\right] \to \left[\frac{A \oplus B}{A \oplus \overline{B}}\right] &\to \left[\frac{(A \oplus B) \oplus (A \oplus \overline{B})}{(A \oplus B) \oplus (\overline{A \oplus \overline{B}})}\right] \\ &\to \left[\frac{((A \oplus B) \oplus (A \oplus \overline{B})) \oplus ((A \oplus B) \oplus (\overline{A \oplus \overline{B}}))}{((A \oplus B) \oplus (A \oplus \overline{B})) \oplus (\overline{(A \oplus B) \oplus (\overline{A \oplus \overline{B}})})}\right]. \end{aligned}$$

For example, starting with the one element Golay pair, the Golay codes of length 2, 4, etc. are derived as follows:

$$\left[\frac{1}{1}\right] \to \left[\frac{1\ \ 1}{1\ \ -1}\right] \to \left[\frac{1\ \ 1\ \ 1\ \ -1}{1\ \ 1\ \ -1\ \ 1}\right] \to \left[\frac{1\ \ 1\ \ 1\ \ -1\ \ 1\ \ 1\ \ -1\ \ 1}{1\ \ 1\ \ -1\ \ -1\ \ -1\ \ -1\ \ 1\ \ -1}\right] \to \text{etc.}$$

4. Calculating Golay sequences

Golay sequences are known for length 2^n as well as for some other lengths (for example 10, 20, 26 and 40). There are several papers, which consider the Golay sequences. CHAPMAN [3] in his article deals the Golay codes of length 24, URBANKE [12] has been calculated the existence Golay sequences of lengths up to 32. In the latest article, BORWEIN [1] considered the Golay pairs for length up to 100. There are some other articles on this theme but most of authors have been calculated the possible length n for which the Golay sequences exist.

In this paper three different methods of determination of the Golay sequences and providing the numerical outputs are described. In ultrasonography short sequences are used more often than long one. Mainly it is concerned with the time duration of the burst signal and calculation time of the received results. The block diagram of the algorithm for calculating the existence of the Golay sequences of different lengths is shown in Fig. 2. At the heart of this method the basic property of complementary series (5) is laid.

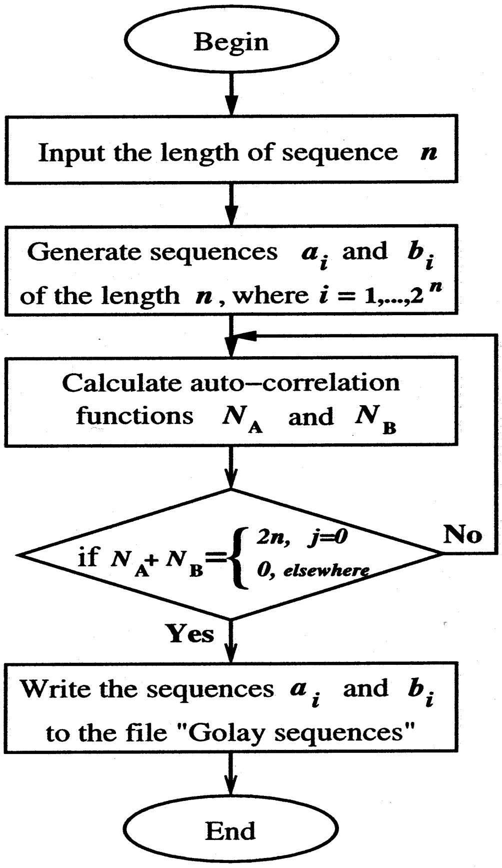

Fig. 2. The algorithm for calculating the Golay sequences of different lengths.

The length of the sequences n is an input element. In the next n-long sequences a_i and b_i with elements "+1" and "-1" are generated. After that, their auto-correlative functions N_A and N_B are calculated. At the end, the condition (5) is verified and if it is true, the pair of sequences are written to the file 'Golay sequences'. Thus, all possible sequences are obtained, which can be called Golay pairs. In Table 1 the number of Golay sequences of different lengths up to 20, which are widely used in ultrasonography, are shown.

Table 1. The number of Golay sequences of different length.

Lenght of the sequences, n	2	4	8	10	16	20
Number of Golay pairs	2	8	48	32	384	272

The program that allows calculating Golay's sequences of the different lengths is presented in the Appendix.

First of all, the program calculates the number of possible sequences after introducing a desired length n of the sequences. Next the set of combinations, which are equal to 2^n, are being generated. Then the conversion of the set all combination of the decimal integers to the binary strings is done with the help of the command "dec2bin". The next step is changing the strings to the sequences with single elements "+1" and "-1". In the following step the auto-correlation functions are calculated. Finally, the condition (5) is verified, and if it is true, the pair of sequences A and B are written to the file "Golay sequences".

5. Correlation principle

The structural scheme for obtaining the impulse response for the applied complementary sequences is shown in Fig. 3.

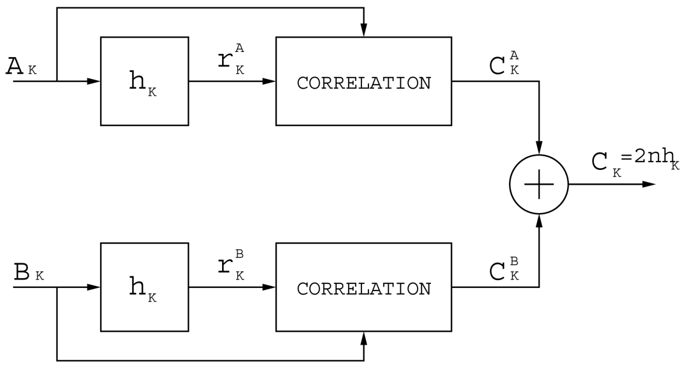

Fig. 3. Impulse response for complementary sequences.

In order to obtain the impulse response, as shown in Fig. 3, the system is first of all initialised with the complementary codes A_n and B_n, resulting in outputs:

$$\begin{aligned} r_k^A &= A_k \times h_k, \\ r_k^B &= B_k \times h_k, \end{aligned} \tag{9}$$

where h_k is impulse response of the transducer.

Next, both the outputs r_k^A and r_k^B with the respective codes are correlated and the individual results are obtained:

$$\begin{aligned} C_k^A &= A_k \otimes r_k^A, \\ C_k^B &= B_k \otimes r_k^B, \end{aligned} \tag{10}$$

and next both the results are added:

$$C_k = C_k^A + C_k^B = 2nh_k. \tag{11}$$

The final output is $2n$ times larger than the response to a single impulse; however, the noise is increased by a factor of $\sqrt{2n}$ ($\sqrt{n}$ for each correlation and $\sqrt{2}$ for the addition) [4]. Therefore, an improvement of the SNR in $\sqrt{2n}$ is obtained in comparison with the single period burst transmission. More realistically, transmitting two sequences per observation time, the SNR improvement factor is actually $\sqrt{n}$.

6. Experimental results

Two images of a tissue phantom RMI 415GX with attenuation of 0.7 dB/[MHz×cm] are shown in Fig. 4. The phantom consists of several nylon wires 0.374 mm in diameter positioned every 1 cm axially. Additional wires are placed at a 30 degree angle at the top of the phantom. Also some wires are placed at the depth of 3 cm with decreasing distances down from 3 mm to 0.5 mm.

The two-cycle pulse of the frequency 3.5 MHz and the pair of the Golay codes of the lengths 16 bits at the same frequency were used. The peak pressure levels of the excitation signals at the transducer were set as low as possible to visually detect the echoes received using a burst transmission slightly larger than the noise level. The same peak pressure has been used for coded transmission. The scanned area of the phantom is marked by the rectangle (Fig. 4c). The resulting images are shown in Fig. 4a and 4b. For quantitative comparisons, the RF-lines are also shown.

The SNR gain is evident when moving from burst to coded transmission. When applying the conventional pulses, the penetration hardly reaches to 4 cm. The scan distance obtained using Golay sequences extends up to 6 cm (the lowest visible white dot at the image (Fig. 4b)). The respective RF-echo lines shown below of each image confirm the outstanding quality of the received signal, when the Golay coded transmission was used.

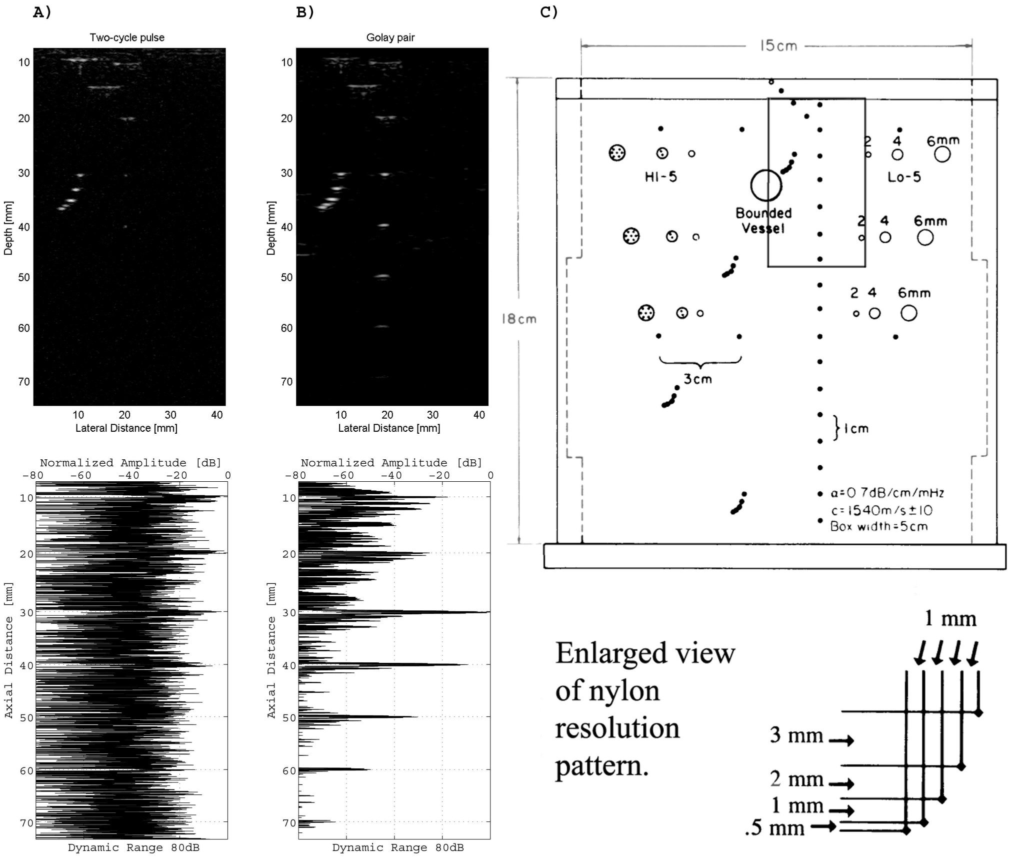

Fig. 4. Ultrasonic images of the tissue phantom RMI 415GX obtained using conventional two cycle sine burst transmission (A) and 16 bit Golay coded transmission (B). Below each image the central RF-lines recorded using the respective transmission are plotted. (C) Schematic diagram of the tissue phantom under examination. The rectangle marks the scanned area.

These two images clearly demonstrate that abdominal ultrasound imaging can benefit from Golay sequences yielding a higher SNR and therefore deeper penetration, while maintaining both axial and lateral resolution. The range resolution that can be achieved, using Golay sequences, is always higher than for a conventional system. However, confirming our hypothesis, the cancellation of the side-lobes is not perfect due to attenuation, and residual shadows are still visible right behind the wires. The main disadvantage of Golay pairs is that they require two transmitting events for every line that decreases the frame rate by half.

As was previously noted, the SNR depends on the length of the applied Golay sequences. In order to compare quantitatively the SNR gain, two sequences 8 and 16 bits were applied and the two central RF echo lines recorded from the tissue phantom were compared (Fig. 5).

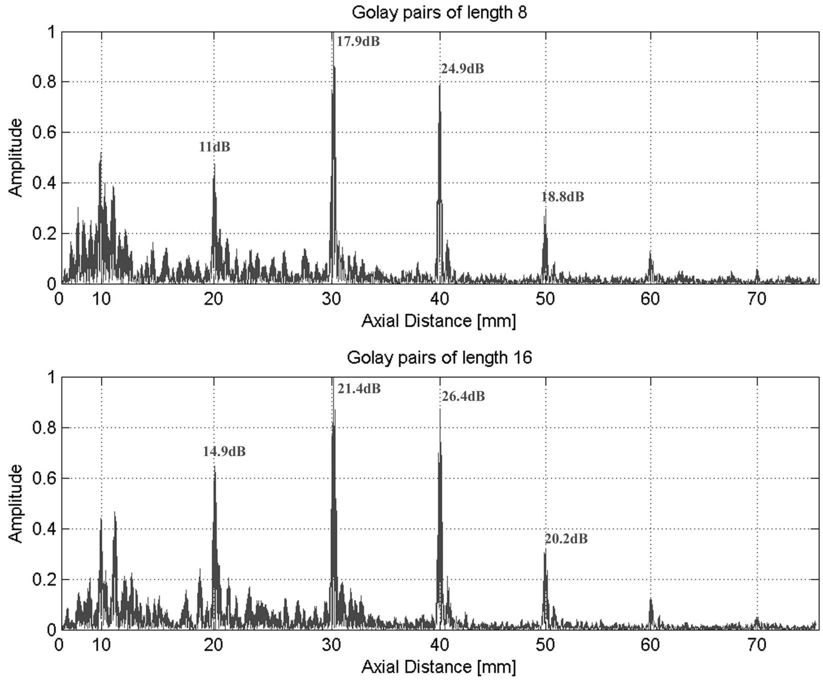

Fig. 5. The central RF-lines of the tissue phantom RMI 415GX recorded using Golay sequences of the length 8 bits (top) and the 16 bits (bottom).

7. Conclusions

Three methods of calculating the pairs of Golay sequences of different lengths were described. Transmission of long coded sequences and compression of the received echoes by means of the matched filtering allow to obtain an axial resolution better than that obtained using conventional short burst pulses but with considerably higher amplitude. Using Golay sequences allow to improve the SNR that plays an important role in the quality of the ultrasonic imaging. For example, the SNR is equal to 24.1 dB when Golay sequences were used and 8.4 dB for two-cycle burst excitation at the depth of 30 mm, improving the contrast and resolution by almost 16 dB. It helps to explore the signals with lower transmitted amplitude that in turn is very important since it decreases the patients' exposure to the ultrasound. Another important reason of using Golay sequences is the fact that they allow a considerably deep penetration using higher frequencies and improving the imaging axial resolution. The algorithm and program for the calculation of Golay sequences has been shown. The program is simple and fast enough to process the data during the laboratory experiments. For example, the exhaustive search for the Golay sequences of the length 16 bit does not exceed 1 minute (PC computer with ATHLON 1.2 GHz processor).

Appendix

The program for the calculation of the Golay sequences of different length is given below.

```
function G=Golay_sequences(n)
% Input length of the sequences
n=length;
% Calculate number sequences
ns=(2^n);
% Calculate all possible sequences
Sequences=dec2bin(0:ns-1);
% Inverse the stream '0' on the element '1'
for k=1:ns
    for p=1:n
        if Sequences(k,p)=='0'
            A(k,p)=1;
        else
            A(k,p)=-1;
        end
    end
end
% Calculate the auto-correlation functions
for p=1:ns
    for i=1:n-1
        B(p,i)=0;
        for j=1:i
            B(p,i)=B(p,i)+A(p,i-j+1)*A(p,n-j+1);
        end
    end
end
% Verify condition (5) and if it is true, the sequences write to the file 'Golay sequences'
for m=1:ns
    for k=m+1:ns
        if B(m,:)+B(k,:)==0;
            fid=fopen('Golay_sequences.txt','a');
            fprintf(fid,'% d',A(m,:));
            fprintf(fid,' ');
            fprintf(fid,'% d',A(k,:));
            fprintf(fid,'\n');
            status=fclose(fid);
        end
    end
end
clear
```

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