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PIEZOELECTRIC GUIDED WAVE PROPAGATION

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A perfectly conducting strip embedded in a piezoelectric medium is considered. Waves propagating along the strip are investigated with the use of a surface impedance function which is suggested by the known surface impedance function for a perfectly conducting plane embedded in the medium. A dispersion equation for the waves is found and solved in the case of a narrow strip.

1. Introduction

The method of the solution to the problem is explained in Ref. [1], where an approximate surface impedance (a function of slowness) for a conducting strip of a finite width is constructed from the corresponding function for a conducting plane [2]. Although the wave propagation along the conducting strip is three-dimensional, some basic features of the surface impedance are similar to those of the two-dimensional problem of wave propagation along the conducting plane.

For an isotropic medium, the slowness of the wave propagating along a conducting strip is found in Ref. [1]. It can serve as an approximate slowness for an anisotropic medium in special cases (specific medium or specific crystal cuts with respect to the strip, see Sec. 2). In general, however, anisotropic effects in applications are significant and cannot be neglected. In the paper, we take into account contributions to the slowness along the strip due to anisotropy of the medium.

The Fourier transform of the surface impedance (with respect to the slowness across the strip) is a function defined in the plane of the strip. Using this function we can find the electric potential across the strip, provided the electric charge distribution across the strip is given. Since the electric potential should vanish on the strip, we get an equation for the slowness along the strip, in other words, the dispersion equation for the wave that propagates along the strip.

2. The function Z

Let the electro-mechanical field in a homogeneous piezoelectric medium depend on time as $\exp(j\omega t)$. In the rectangular system of coordinates (x, y, z), we consider a perfectly conducting strip of the width $2w$ in the plane $z = 0$, and assume that its axis of symmetry coincides with the x axis (see Fig. 1 in Ref. [1]). The thickness of the strip is infinitesimal, so that its mechanical properties may be neglected.

In the plane $z = 0$, the field is assumed to depend on x and y as $\exp(-jrx - jsy)$, where r and s are components of the wave vector. It will be convenient, occasionally, to call r and s slownesses (and use the concept of slowness curve), although in fact the slownesses are equal to r/ω and s/ω. Since ω is constant, the possible confusion should not be serious.

The relation between the electric potential ϕ and the electric charge (surface) density σ in the plane $z = 0$ is given by the equality

$$\phi = Z(k)\sigma. \tag{1}$$

The function $Z(k)$, where $k = (r^2 + s^2)^{1/2}$, may be called a surface impedance (to be exact, it is Z/ω that has the physical dimension of surface impedance). The approximation of $Z(k)$ proposed in Ref. [1] is

$$Z(k) = \frac{C}{\sqrt{k^2}} - \frac{C\kappa}{\sqrt{k^2 - k_{\mathrm{c}}^2}}, \tag{2}$$

where C and κ are positive constants, and k_{c} is the cutoff slowness of bulk waves. In terms of the variables r and s, Eq. (2) can be rewritten as

$$Z(k) = \frac{C}{\sqrt{r^2 + s^2}} - \frac{C\kappa}{\sqrt{r^2 + s^2 - R^2}}, \tag{3}$$

where $R = k_{\mathrm{c}}$. In the plane (r, s), the second term of Eq. (3) is singular at the circle $r^2 + s^2 = R^2$. In the vicinity of the r axis, where the approximation is meant to be valid (for $r > 0$), a section of the circle should approximate the slowness curve related to the slowest bulk wave. It is seen that the cutoff point $(r_{\mathrm{c}}, s_{\mathrm{c}})$ of the slowness curve lies on the r axis, and is given by the equalities $r_{\mathrm{c}} = R$ and $s_{\mathrm{c}} = 0$.

In general, cutoff points of slowness curves are shifted off the r axis. Therefore, the approximation given by Eq. (3) is valid only when the shift is sufficiently small (specific crystal cuts or weak anisotropy). To take fully into account anisotropy of the piezoelectric medium we should make some of the parameters C, κ, k_{c} depend on the variable s. The simplest way is to leave C and κ constant, and to shift the circle that approximates the slowness curve. Instead of Eq. (3) we have

$$Z(k) = \frac{C}{\sqrt{r^2 + s^2}} - \frac{C\kappa}{\sqrt{f(r, s)}} \tag{4}$$

with

$$f(r,s) = (r-(r_c-R))^2 + (s-s_c)^2 - R^2. \tag{5}$$

The circle $f(r,s)=0$ approximates the slowness curve in the vicinity of the cutoff point (r_c, s_c). The radius R of the circle becomes an additional parameter, not related to r_c. If $r_c = R$ and $s_c = 0$, then$f(r,s) = r^2 + s^2 - R^2$.

With respect to the variable s, the function $f(r,s)$ is a polynomial of degree 2,

$$f(r,s) = as^2 + 2bs + c, \tag{6}$$

where $a = 1$, $b = -s_c$, and $c = s_c^2 + (r-r_c)^2 + 2R(r-r_c)$. Hence, we have

$$Z(k) = \frac{C}{\sqrt{r^2+s^2}} - \frac{C\kappa}{\sqrt{as^2+2bs+c}}. \tag{7}$$

The function $Z(r,s) = Z(k)$ will be used below.

3. The function G

The Fourier transform of $Z(r,s)$ with respect to s,

$$G(r,y) = \int_{-\infty}^{\infty} Z(r,s)\exp(-jsy)\,ds, \tag{8}$$

can be calculated with the use of formulas given in Ref. [3] (see p. 122, f. (32), and p. 117, f. (5)). We obtain

$$G(r,y) = -2CK_0(r|y|) + 2C\kappa\exp(-js_c y)K_0(B|y|), \tag{9}$$

where K_0 is the modified Bessel function of the second kind (of order zero), and $B = (c-b^2)^{1/2} = ((r-r_c)^2 + 2R(r-r_c))^{1/2}$. The value of B is real for $r > r_c$ and for $r < r_c - 2R$. We assume that $r > r_c$. If $r_c = R$ and $s_c = 0$, then $G(r,y)$ given by Eq. (9) is equal to the corresponding function for the isotropic case (cf. Ref. [1], Eq. (3.4)).

The function $G(r,y)$, which is defined in the plane $z = 0$ and depends on the parameter r, solves the problem of finding the electric potential when the electric charge density is given. The corresponding integral formula is the Fourier transform of Eq. (1) with respect to the variable s, i.e.

$$\phi(y) = \int_{-\infty}^{\infty} G(r, y-y')\sigma(y')\,dy', \tag{10}$$

where $\phi(y)$ and $\sigma(y')$ are amplitudes of the wave propagating along the strip.

For calculation purposes, we represent the exponential function in the second term of Eq. (9) as a series. This gives

$$G(r,y) = -2CK_0(r|y|) + 2C\kappa \sum_{m=0}^{\infty} \frac{(-js_{\mathrm{c}})^m}{m!} y^m K_0(B|y|). \tag{11}$$

4. A narrow strip

If the width $2w$ of the strip is infinitesimal, we may assume that the charge density across the strip is given by the function

$$\sigma(y) = \left(w^2 - y^2\right)^{-1/2} \tag{12}$$

for $|y| < w$, and $\sigma(y) = 0$ for $|y| > w$.

It suffices to calculate the potential $\phi(y)$ for $y = 0$. We have

$$\phi(0) = \int_{-w}^{w} G(r, -y)\sigma(y)\, dy \tag{13}$$

(the prime has been omitted), and then

$$\begin{aligned}\phi(0) = -2C \int_{-w}^{w} \left(w^2 - y^2\right)^{-1/2} K_0(r|y|)\, dy \\ + 2C\kappa \sum_{m=0}^{\infty} \frac{(-js_{\mathrm{c}})^m}{m!} \int_{-w}^{w} y^m \left(w^2 - y^2\right)^{-1/2} K_0(B|y|)\, dy.\end{aligned} \tag{14}$$

In order to calculate the above integrals we use K-transforms defined as

$$g(\rho;\nu) = \int_{0}^{\infty} f(y) K_\nu(\rho y)(\rho y)^{1/2}\, dy \tag{15}$$

(cf. Ref. [4], p. 125). The first term of Eq. (14), and the second term for $m = 0$, can be calculated with the use of the function

$$f(y) = y^{-1/2}\left(w^2 - y^2\right)^{-1/2} \tag{16}$$

for $y < w$, and $f(y) = 0$ for $y > w$. We get

$$\begin{aligned} g(\rho;0) &= \rho^{1/2} \int_{0}^{w} \left(w^2 - y^2\right)^{-1/2} K_0(\rho y)\, dy \\ &= \frac{1}{2}\pi \rho^{1/2} I_0(\rho w/2) K_0(\rho w/2) \end{aligned} \tag{17}$$

(cf. Ref. [4], p. 129, f. (10) for $\nu = 0$), and hence

$$
\begin{aligned}
\phi(0) = &-2\pi C I_0(rw/2)K_0(rw/2) \\
&+ 2\pi C\kappa I_0(Bw/2)K_0(Bw/2) + 2C\kappa \sum_{m=2}^{\infty} \frac{(-js_c)^m}{m!} g_m(B),
\end{aligned} \tag{18}
$$

where

$$
g_m(B) = \int_{-w}^{w} y^m (w^2 - y^2)^{-1/2} K_0(B|y|)\,dy \tag{19}
$$

for even m, and $g_m(B) = 0$ for odd m.

If we neglect the series in Eq. (18), which contains higher order terms, and stipulate that $\phi(0) = 0$ on the strip, then we arrive at an equation with respect to the variable r,

$$
-I_0(rw/2)K_0(rw/2) + \kappa I_0(Bw/2)K_0(Bw/2) = 0. \tag{20}
$$

This is the dispersion equation for the wave along the strip. It can be easily solved for $w \to 0$. Indeed, applying asymptotic expressions of the modified Bessel functions $I_m(z)$ and $K_m(z)$ for $z \to 0$,

$$
I_m(z) \sim \frac{1}{m!}(z/2)^m, \quad K_m(z) \sim (-1)^{m+1} I_m(z) \ln(z/2) \tag{21}
$$

(cf. Ref. [5], p. 5, f. (12), and p. 9, f. (37)), we get

$$
\ln(rw/4) - \kappa \ln(Bw/4) = 0 \tag{22}
$$

or

$$
B^2 = (w/4)^{2(K-1)} r^{2K}, \tag{23}
$$

where $K = \kappa^{-1}$. Assuming that $r \approx r_c$ on the right-hand side of Eq. (23), and substituting B, we find the quadratic equation

$$
r^2 + 2(R - r_c)r + r_c^2 - 2Rr_c - X^2 = 0 \tag{24}
$$

for r, where $X = (r_c w/4)^K (w/4)^{-1}$. The solution for $r > r_c$ is

$$
r = r_c - R + (R^2 + X^2)^{1/2}. \tag{25}
$$

(The other solution, for $r < r_c - 2R$, has the minus sign before the square-root.)

It is interesting to note that if $R = r_c$ then the slowness r is equal (approximately, for w sufficiently small) to that of the isotropic case, even though s_c may be different from zero. And conversely, if $R \neq r_c$ (which is possible only in the anisotropic case) then the slowness r is different from that of the isotropic case, even though s_c may be equal to zero, i.e. the cutoff point (r_c, s_c) may lie on the r axis.

Since X is much less than R ($X \to 0$ for $w \to 0$), the square-root in Eq. (25) can be expanded. This gives an even simpler approximate formula,

$$
r = r_c + \frac{1}{2} R^{-1} X^2, \tag{26}
$$

for the slowness of the wave along the strip. It is seen that the value of $r - r_{\mathrm{c}}$ is directly proportional to the curvature R^{-1} of the slowness curve at the cutoff point.

The higher order contributions to the electric potential $\phi(0)$ in Eq. (18) depend on the parameter s_{c}. This dependence can be taken into account (for a specific value of w) by using an approximate value of the integral given by Eq. (19). For w sufficiently small, the expression under the integral sign is less than zero within the integration limits, and tends to minus infinity as $y \to 0$ because $K_0(B|y|)$ has a logaritmic singularity for $y = 0$. Therefore, we can assume some mean value of the function y^m within the integration limits, say w^m, and integrate the rest of the expression. In this way, we obtain

$$g_m(B) \approx w^m \int\limits_{-w}^{w} \left(w^2 - y^2\right)^{-1/2} K_0(B|y|)\, dy \tag{27}$$

for even m. It should be noted that the form of the approximate $g_m(B)$ as a function of the variable w may differ from that of the exact integral.

If we confine ourselves to the first term of the series ($m = 2$) then, instead of Eq. (20), we have

$$-I_0(rw/2)K_0(rw/2) + (1 - s_{\mathrm{c}}^2 w^2/2)\kappa I_0(Bw/2)K_0(Bw/2) = 0. \tag{28}$$

In other words, the only difference is that in the equation, given by Eq. (20), κ is replaced by

$$\kappa' = (1 - s_{\mathrm{c}}^2 w^2/2)\kappa < \kappa, \tag{29}$$

and in the solution, given by Eq. (25) or Eq. (26), K is replaced by

$$K' = K/(1 - s_{\mathrm{c}}^2 w^2/2) > K. \tag{30}$$

As we can see, the more the cutoff point $(r_{\mathrm{c}}, s_{\mathrm{c}})$ in the plane (r, s) is shifted off the r axis, the greater the value of K', and consequently the less the value of $r - r_{\mathrm{c}}$ (for $r_{\mathrm{c}} w < 1$). The higher order contributions for $m = 4, 6, 8 \ldots$ give the same effect.

5. Conclusion

The anisotropy of the piezoelectric medium influences the propagation of the wave guided along the (narrow) strip in two respects. First, the slowness of the guided wave depends on the curvature of the slowness curve (related to the slowest bulk wave) at the cutoff point. The slowness may be greater or less than that in the isotropic case, depending on whether the curvature is greater or less than $1/r_{\mathrm{c}}$. Second, the slowness of the guided wave depends on the shift of the cutoff point off the r axis, given by s_{c}. If the shift is greater, the slowness is less, and vice versa. The two parameters, i.e. the curvature and the shift, are independent of each other.

The narrow-strip waveguide can find application in new piezoelectric surface wave devices which will complement traditional SAW devices.

References

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