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Weakly Nonlinear Dynamics of Short Acoustic Waves in Exponentially Stratified Gas

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The types of linear motion over an ideal gas affected by gravity are specified approximately in the case of large characteristic wave number of perturbation k: $k \gg 1/H$, where H is the scale of density and pressure decrease of the background gas, the so-called height of the uniform gas. The corresponding approximate operators projecting the overall vector of perturbations into specific types are derived, along with equations governing sound in a weakly nonlinear flow. The validity of approximate formulae are verified for the concrete examples of initial waveforms. The numerical analysis reveals a good agreement of these approximate expressions with the exact ones obtained previously by the author. The analysis applies to the weakly nonlinear flow as well, with the small Mach numbers ($M \ll 1$). The links inside modes are redetermined by including terms of order M^2 and M^2/kH.

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1. Introduction

The nonlinear dynamics of fluids affected by external forces is, in general, a very complex problem. The forces make the background of waves propagation non-uniform, with at least density and pressure depending on height [1], what essentially complicates the definition of linear motions (motions of infinitely small amplitude) taking place in the non-uniform media. The mathematical difficulties hamper significantly the studies of nonlinear dynamics of such media.

The difficulty appears also in other than acoustic types of motion. Determination of types of wave motion itself bases on the liner dispersion relation [2, 3]. The number of roots of dispersion relation, or branches of possible types of motion (modes), equals the number of governing equations. In one dimension, there are three types of motion: two acoustic branches and, if attenuation is neglected, the stationary one with zero frequency. In the flows going out of one dimension, the buoyancy waves appear. A possibility of modes distinguishing and prediction of their dynamics analytically, in addition to advance in the theory, is of importance in meteorology and atmosphere dynamics applications [4]. It may be resolved by means of linear operators uniquely separating the modes in the linear flow [5, 6]. Further analysis should take into account the nonlinearity of governing equations.

The one-dimensional motions in the isothermal atmosphere affected by constant gravity force are considered. Starting from the precise determination of modes, the consideration is limited by the perturbations rapidly changing in the space (as compared to the characteristic scale of atmosphere H): $kH \gg 1$, where k is a characteristic wavenumber of perturbation. That takes place in the most important cases and permits to simplify consideration essentially, with insignificant loss of accuracy. The links inside modes, including integro-differential operator with a kernel being a sum of special functions, tend in this limit to simple integrals. The nonlinear governing equation of sound is written, including terms with accuracy of order M^2/kH in Sec. 4.2. The validity of expansion of operators in series is examined in Sec. 4.3.1. Some examples of approximate subdivision of initial waveform into specific types of motion, valid also in the weakly nonlinear flow, are considered in Sec. 4.3.2. The Sec. 4.3.3 illustrates the nonlinear dynamics of a single pulse.

2. Conservation equations and dispersion relations in one dimension

Studies of nonlinear dynamics should start from the general equations of fluid dynamics. They are nonlinear and determine dynamics of all possible types of motion which may take place in a fluid. Mathematically, general solution of conservation equations is unavailable, except for some well-known types of flow under strongly simplifying conditions [3]. The original method, proposed by the author, is to start from determination of the types of motion in the linear flow by links of excess density, pressure and velocity, specific for every mode. Basing on these links, the system of conservation equations splits into dynamic equation for every mode, accounting for interaction between modes. Inclusion of body forces like gravity complicates mathematical part of the analysis, making links and splitting operators integro-differential.

The governing fluid equations in absence of attenuation manifest conservation of momentum, energy and mass [2, 3] are:

$$
\begin{gathered}
\rho\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial z}+F,\\
\rho\left(\frac{\partial E}{\partial t}+v\frac{\partial E}{\partial z}\right)+p\frac{\partial v}{\partial z}=0, \qquad (1)\\
\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial z}=0,
\end{gathered}
$$

where ρ, p, E, v denote fluid density, pressure, internal energy and vertical particle velocity, $F = -g\rho$ is a projection of gravity force per unit mass on axis OZ, z equals the distance from the Earth surface and t denotes time. The unperturbed pressure and density are functions of vertical coordinate: $\rho_0 = \rho_{00}\exp(-z/H)$, $p_0 = p_{00}\exp(-z/H) = \rho_{00}gH\exp(-z/H)$, where $\rho_{00} = \rho_0(0)$, $p_{00} = p_0(0)$ denote the background density and pressure on the Earth's surface, correspondingly. Physical meaning of H is the following: the background density (or pressure) at the levels z and $z + H$, differs $\exp(1)$ times. The thermodynamic relation for an ideal gas completes the system (1):

$$E = \frac{p}{\rho(\gamma - 1)} \tag{2}$$

with $\gamma = C_p/C_v$ being the specific heats ratio. Instead of perturbations of density ρ', pressure p' and velocity v, we introduce the following quantities:

$$R = \rho' \cdot \exp(z/2H), \qquad P = p' \cdot \exp(z/2H), \qquad V = v \cdot \exp(-z/2H). \tag{3}$$

This well-known change of variables allows to apply the Fourier analysis in the studies of infinitely small signal flows. Meaning the weakly nonlinear flow $M \ll 1$ (the Mach number is the ratio of particle velocity and sound speed, $M = v/c_0$, c_0 is the infinitely small signal sound velocity in the gas), we keep only second-order nonlinear terms in the right-hand side of the system, while the left-hand one contains the linear quantities. The resulting system along with the correspondent quadratic terms is:

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{\rho_{00}}\left(\frac{\partial}{\partial z} - \frac{1}{2H}\right)P + \frac{gR}{\rho_{00}} = \varphi_1,\\
&\frac{\partial P}{\partial t} + \gamma g H\rho_{00}\left(\frac{\partial V}{\partial z} + \frac{1}{\gamma H}(\gamma/2 - 1)V\right) = \varphi_2,\\
&\frac{\partial R}{\partial t} + \rho_{00}\left(\frac{\partial}{\partial z} - \frac{1}{2H}\right)V = \varphi_3,\\
&\varphi_1 = -\exp(z/2H)\left(V\left(\frac{\partial}{\partial z} + \frac{1}{2H}\right)V - \frac{R}{\rho_{00}^2}\left(\frac{\partial}{\partial z} - \frac{1}{2H}\right)P - \frac{g}{\rho_{00}^2}R^2\right),\\
&\varphi_2 = -\exp(z/2H)\left(V\left(\frac{\partial}{\partial z} - \frac{1}{2H}\right)P + \gamma P\left(\frac{\partial}{\partial z} + \frac{1}{2H}\right)V\right),\\
&\varphi_3 = -\exp(z/2H)\left(R\frac{\partial V}{\partial z} + V\frac{\partial R}{\partial z}\right).
\end{aligned} \tag{4}$$

The motions of infinitely small amplitudes satisfy the system (4) with zero right-hand side. In this case, the Fourier analysis applies. The quantities R, P, V may be represented by the Fourier integrals as follows:

$$R = \int_{-\infty}^{\infty} R_k \cdot \exp(i\omega t - ikz)\,\mathrm{d}k + cc,$$
$$P = \int_{-\infty}^{\infty} P_k \cdot \exp(i\omega t - ikz)\,\mathrm{d}k + cc, \tag{5}$$
$$V = \int_{-\infty}^{\infty} V_k \cdot \exp(i\omega t - ikz)\,\mathrm{d}k + cc.$$

The dispersion relations of this flow are well-known, they are roots of the dispersion relation resulting from the linearized version of Eq. (4), after inserting Eq. (5) in the system (4) and assuming φ_1, φ_2, φ_3 to be zero, we obtain:

$$\mathbf{Det}\begin{vmatrix} i\omega & -\frac{1}{\rho_{00}}\left(ik+\frac{1}{2H}\right) & \frac{g}{\rho_{00}} \\ -i\gamma gHk\rho_{00} + g\rho_{00}(\gamma/2-1) & i\omega & 0 \\ -\rho_{00}\left(ik+\frac{1}{2H}\right) & 0 & i\omega \end{vmatrix} = \mathbf{0},$$

$$\omega_{1(ac)} = \sqrt{\gamma gH\left(k^2+\frac{1}{4H^2}\right)},$$
$$\omega_{2(ac)} = -\sqrt{\gamma gH\left(k^2+\frac{1}{4H^2}\right)}, \tag{6}$$
$$\omega_{st} = 0.$$

The two first data ($\omega_{1(ac)}$, $\omega_{2(ac)}$) correspond to the acoustic, upwards and downwards progressive waves, and the last one (ω_{st}) correspond to the stationary (entropy) type of motion. The formulas above tend to those describing motion over the uniform background without gravity forces when $H \to \infty$, $g \to 0$ but $E_0 = \dfrac{gH}{\gamma - 1} = \text{const}$.

3. Linear definition of modes and governing equation of sound

The dispersion relations (6) determine linear links of specific variations of pressure, density and velocity inside every mode in the Fourier space (k, t). The correspondent links for acoustic and internal modes in (z, t) space are integro-differential. The exact links for acoustic and entropy modes and relative projectors have been established by the author for any, not obligatory large product kH [5, 6]. For example, links connecting the acoustic pressure and velocity for motion taking place in the positive direction of axis OZ looks:

$$P(z,t) = \frac{\rho_{00}}{\pi\sqrt{\gamma g H}} \int_{-\infty}^{\infty} \mathrm{d}z' \Big(g(1-\gamma/2)F(z-z') \\ - \gamma g H F(z-z') \frac{\partial}{\partial z'} \Big) V_z(z',t), \tag{7}$$

where $F(z)$ reflects the dispersive features of a stratified gas,

$$F(z) = \frac{2}{\pi}\left(I_0(z/2H) - L_0(z/2H)\right) = \int_0^{\infty} \mathrm{d}k \frac{\sin(kz)}{\sqrt{k^2 + 1/4H^2}},$$

with I_0, L_0 denoting the modified Bessel function of zero order, and the Struve function, respectively.

To simplify the consideration, let us concentrate at motions with characteristic vertical scales much smaller than the specific scale of the background H: $kH \gg 1$. In view of that, the dispersion relations and the following from them formulae may be expanded in the Taylor series in the vicinity of $1/kH = 0$. Finally, links for acoustic modes, upwards and downwards propagating (indexed by 1 and 2, respectively), have the form:

$$\psi_{1(ac)} = \begin{pmatrix} V \\ P \\ R \end{pmatrix}_{1(ac)} = \begin{pmatrix} 1 \\ c_0\rho_{00}\left(1 + \frac{\gamma-2}{2\gamma H}\int \mathrm{d}z\right) \\ \frac{\rho_{00}}{c_0}\left(1 - \frac{1}{2H}\int \mathrm{d}z\right) \end{pmatrix} V_{1(ac)},$$
$$\psi_{2(ac)} = \begin{pmatrix} V \\ P \\ R \end{pmatrix}_{2(ac)} = \begin{pmatrix} 1 \\ -c_0\rho_{00}\left(1 + \frac{\gamma-2}{2\gamma H}\int \mathrm{d}z\right) \\ -\frac{\rho_{00}}{c_0}\left(1 - \frac{1}{2H}\int \mathrm{d}z\right) \end{pmatrix} V_{2(ac)}, \tag{8}$$

where c_0 denotes the infinitely small sound velocity over the isothermal ideal gas of constant background pressure and density:

$$c_0 = \sqrt{\gamma p_0/\rho_0} = \sqrt{\gamma g H}.$$

Vectors $\psi_{1(ac)}$, $\psi_{2(ac)}$ relate to positive and negative signs of acoustic circular frequency (6). Limits of integration should agree with the physical meaning of the problem. The stationary mode, relating to $\omega_{st}=0$, possesses the fixed links of quantities P, R and V as follows:

$$\psi_{st}=\begin{pmatrix}V\\P\\R\end{pmatrix}_{st}=\begin{pmatrix}0\\-\dfrac{c_0^2}{\gamma H}\displaystyle\int \mathrm{d}z\\1\end{pmatrix}R_{st}. \tag{9}$$

In contrast to the motion over the uniform background, the excess pressure of the stationary mode does not equal zero. Matrix operators, projecting the overall field into acoustic modes ($\Pi_{1(ac)}\psi=\psi_{1(ac)}$, $\Pi_{2(ac)}\psi=\psi_{2(ac)}$, $\Pi_{st}\psi=\psi_{st}$), follows from links inside the specific modes (8, 9):

$$\Pi_{1(ac)}=\begin{pmatrix}\dfrac{1}{2} & \dfrac{1}{2\rho_{00}c_0}\left(1-\dfrac{1}{2H}\displaystyle\int \mathrm{d}z\right) & \dfrac{g}{2\rho_{00}c_0}\displaystyle\int \mathrm{d}z\\ \dfrac{\rho_{00}c_0}{2}\left(1-\dfrac{2-\gamma}{2\gamma H}\displaystyle\int dz\right) & \dfrac{1}{2}-\dfrac{1}{2\gamma H}\displaystyle\int \mathrm{d}z & \dfrac{g}{2}\displaystyle\int \mathrm{d}z\\ \dfrac{\rho_{00}}{2c_0}\left(1-\dfrac{1}{2H}\displaystyle\int \mathrm{d}z\right) & \dfrac{1}{2c_0^2}-\dfrac{1}{2c_0^2H}\displaystyle\int \mathrm{d}z & \dfrac{1}{2\gamma H}\displaystyle\int \mathrm{d}z\end{pmatrix},$$

$$\Pi_{2(ac)}=\begin{pmatrix}\dfrac{1}{2} & -\dfrac{1}{2\rho_{00}c_0}\left(1-\dfrac{1}{2H}\displaystyle\int \mathrm{d}z\right) & -\dfrac{g}{2\rho_{00}c_0}\displaystyle\int \mathrm{d}z\\ -\dfrac{\rho_{00}c_0}{2}\left(1-\dfrac{2-\gamma}{2\gamma H}\displaystyle\int \mathrm{d}z\right) & \dfrac{1}{2}-\dfrac{1}{2\gamma H}\displaystyle\int \mathrm{d}z & \dfrac{g}{2}\displaystyle\int \mathrm{d}z\\ -\dfrac{\rho_{00}}{2c_0}\left(1-\dfrac{1}{2H}\displaystyle\int \mathrm{d}z\right) & \dfrac{1}{2c_0^2}-\dfrac{1}{2c_0^2H}\displaystyle\int \mathrm{d}z & \dfrac{1}{2\gamma H}\displaystyle\int \mathrm{d}z\end{pmatrix}, \tag{10}$$

$$\Pi_{st}=\begin{pmatrix}0 & 0 & 0\\ 0 & \dfrac{1}{\gamma H}\displaystyle\int \mathrm{d}z & -g\displaystyle\int \mathrm{d}z\\ 0 & \dfrac{1}{c_0^2}\left(-1+\dfrac{1}{H}\displaystyle\int \mathrm{d}z\right) & 1-\dfrac{1}{\gamma H}\displaystyle\int \mathrm{d}z\end{pmatrix}.$$

The projectors form the full orthogonal basis with properties:

$$\Pi_{1(ac)}\cdot\Pi_{2(ac)}=\Pi_{1(ac)}\cdot\Pi_{st}=...=\Pi_{st}\cdot\Pi_{2(ac)}=0,$$

$$\Pi_{1(ac)}^2=\Pi_{1(ac)},...\Pi_{1(ac)}+\Pi_{2(ac)}+\Pi_{st}=I,$$

where $0,I$ are zero and unit matrix operators.

It is easy to establish an accordance of the projectors to that written on for any product kH [5, 6] assuming the approximate expressions for the following operators in the case $kH \gg 1$:

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}z' F(z'-z)\frac{\partial}{\partial z'} \approx 1,$$

$$\exp(-z/2H)\int_{-\infty}^{z} \mathrm{d}z' \exp(z'/2H)\frac{\partial}{\partial z'} \approx 1, \tag{11}$$

$$\exp(z/2H)\int_{\infty}^{z} \mathrm{d}z' \exp(-z'/2H)\frac{\partial}{\partial z'} \approx 1.$$

Complete perturbation is a sum of all specific parts, for example, $R = R_{(1,ac)} + R_{(2,ac)} + R_{st}$, and so on. In order to get equations with accuracy M^2, projectors apply to the both sides of system (4). They distinguish the corresponding perturbation in the left-hand linear side and yield in the nonlinear terms of all modes while acting at the right-hand side [5, 6]. Considering the terms relating to the first, "upwards progressive" mode, the nonlinear dynamic equation in the leading order looks similarly to the corresponding equation for a wave propagating over the uniform background (the difference is in the factor $e^{z/2H}$ which tends to 1 if H tends to infinity):

$$\frac{\partial V_{(1,ac)}}{\partial t} + c_0\frac{\partial V_{(1,ac)}}{\partial z} + \frac{\gamma+1}{2}e^{z/2H}V_{(1,ac)}\frac{\partial V_{(1,ac)}}{\partial z} = O(M^3, M^2/kH). \tag{12}$$

In the linear part, dispersive features of the medium, relating to a term of order $1/(kH)^2$, are not accounted for. The linear links of perturbations (8) make the sound isentropic in the leading order. The nonlinear links keeping sound isentropic in the second order, may be established using the links analogous to that for the Riemann wave [3, 7]:

$$\begin{aligned} P_{(1,ac)} &= c_0\rho_{00}\left(1+\frac{\gamma-2}{2\gamma H}\int \mathrm{d}z\right)V_{(1,ac)} + \exp(z/2H)\rho_{00}\frac{\gamma+1}{4}V^2_{(1,ac)}, \\ R_{(1,ac)} &= \frac{\rho_{00}}{c_0}\left(1-\frac{1}{2H}\int \mathrm{d}z\right)V_{(1,ac)} + \exp(z/2H)\frac{3-\gamma}{4}\frac{\rho_{00}}{c_0^2}V^2_{(1,ac)}. \end{aligned} \tag{13}$$

It may be easily verified that the governing equation for the sound with the corrected links (13) is still Eq. (12). The aim of linear projecting is to decompose linear parts of Eq. (4) and to distribute properly their nonlinear right-hand sides between the different dynamic equations.

4. Governing nonlinear equations for sound

4.1. The Riemann waveform – an exclusive example of complete separation of acoustic quantities in the nonlinear flow

It is helpful to remember at this point the Riemann acoustic wave propagating over an ideal gas of background constant density and pressure in absence of attenuation. It is known to be a wave which does not generate other types of motion and is an exact solution of the system of hydrodynamic equations. The links of the rightwards progressive (in the positive direction of axis OZ, denoted by index 1) Riemann wave are as follows:

$$P_{1,R} = \frac{\rho_0 c_0^2}{\gamma}\left(1 + \frac{\gamma-1}{2}\frac{V_{1,R}}{c_0}\right)^{2\gamma/\gamma-1} - \frac{\rho_0 c_0^2}{\gamma},$$
$$R_{1,R} = \rho_0\left(1 + \frac{\gamma-1}{2}\frac{V_{1,R}}{c_0}\right)^{2/\gamma-1} - \rho_0. \tag{14}$$

This coincides with links (13) when $H \to \infty$, $\rho_0(z) \equiv \rho_{00}$. On the other hand, the exact dynamic equation governing the nonlinear Riemann wave is the Earnshaw equation:

$$\frac{\partial V_{1,R}}{\partial t} + c_0\frac{\partial V_{1,R}}{\partial z} + \frac{\gamma+1}{2}V_{1,R}\frac{\partial V_{1,R}}{\partial z} = 0, \tag{15}$$

which corresponds to Eq. (12) when $H \to \infty$, $\rho_0(z) \equiv \rho_{00}$. Note that Eqs. (14), (15) are exact but (12), (13) are obtained with accuracy up to quadratic terms. Making the links inside acoustic mode more precise, one can specify the links (13) within any accuracy. The Riemann wave is an exclusive example of complete separation of acoustic quantities in the nonlinear flow. Similar waveforms of both directions of propagation, exist also in the flow over the medium affected by gravity.

It is well understood that such separation is impossible in the viscous flows, where exists a nonlinear generation of non-acoustic types of motion [3, 8, 9]. The nonlinear losses of acoustic momentum induces the vortex flow (acoustic streaming), and losses of acoustic energy result in acoustic heating [10]. Dynamic equations governing acoustic heating include nonlinear terms standing by the viscosity coefficient, which plays a role of acoustic source of heating [10, 11].

4.2. Links of perturbations and governing equations for an acoustic waveform

For such a waveform to exist (for example, upwards progressive), the links connecting excess pressure, density and velocity (like (14)) must lead to three equivalent equations for velocity (like (15)). In order to correct links and equations up to the terms of order M^2/kH, Eqs. (13) should be completed by the new terms as follows:

$$
\begin{aligned}
P_{1,ac} &= c_0\rho_{00}\left(1+\frac{\gamma-2}{2\gamma H}\int \mathrm{d}z\right)V_{(1,ac)}+\exp\left(z/2H\right)\rho_{00}\frac{\gamma+1}{4}V^2_{(1,ac)} \\
&\qquad + \exp\left(z/2H\right)\overline{P}_1, \\
R_{1,ac} &= \frac{\rho_{00}}{c_0}\left(1-\frac{1}{2H}\int \mathrm{d}z\right)V_{(1,ac)}+\exp\left(z/2H\right)\frac{3-\gamma}{4}\frac{\rho_{00}}{c_0^2}V^2_{(1,ac)} \\
&\qquad + \exp\left(z/2H\right)\overline{R}_1,
\end{aligned}
\tag{16}
$$

where $\overline{P}_1$, $\overline{R}_1$ are suitable quadratic functions of $V_{(1,ac)}$ of order M^2/kH, for example, proportional to

$$
\frac{1}{H}V_{(1,ac)}\int V_{(1,ac)}\,\mathrm{d}z \qquad \text{or to} \qquad \frac{1}{H}\int V^2_{(1,ac)}\,\mathrm{d}z.
$$

It is easy to prove that links (16) with zero $\overline{P}_1$ and $\overline{R}_1$ make the mode isentropic up to the terms of order M^2. Taking into account the dynamic Eq. (12), the Eqs. (4) transform into the following system:

$$
\begin{aligned}
\xi + e^{z/2H}\frac{1}{\rho_{00}}\frac{\partial\overline{P}_1}{\partial z} &= -\frac{e^{z/2H}}{2H}\left(\frac{3+\gamma}{2\gamma}V^2_{(1,ac)}+\frac{\partial V_{(1,ac)}}{\partial z}\int V_{(1,ac)}\,\mathrm{d}z\right), \\
\xi + e^{z/2H}\frac{1}{\rho_{00}c_0}\frac{\partial\overline{P}_1}{\partial t} &= -\frac{e^{z/2H}}{2H}\left(\frac{3\gamma^2+\gamma-6}{4\gamma}V^2_{(1,ac)}\right. \\
&\qquad \left. -(2-\gamma)\frac{\partial V_{(1,ac)}}{\partial z}\int V_{(1,ac)}\,\mathrm{d}z\right), \\
\xi + e^{z/2H}\frac{c_0}{\rho_{00}}\frac{\partial\overline{R}_1}{\partial t} &= -\frac{e^{z/2H}}{2H}\left(\frac{\gamma-3}{4}V^2_{(1,ac)}-\frac{\partial V_{(1,ac)}}{\partial z}\int V_{(1,ac)}\,\mathrm{d}z\right),
\end{aligned}
\tag{17}
$$

where

$$
\xi = \frac{\partial V_{(1,ac)}}{\partial t}+c_0\frac{\partial V_{(1,ac)}}{\partial z}+e^{z/2H}\frac{\gamma+1}{4}\frac{\partial V^2_{(1,ac)}}{\partial z}.
$$

Calculating the difference of the first and the second equations, one gets:

$$
\begin{aligned}
c_0\frac{\partial\overline{P_1}}{\partial z}-\frac{\partial\overline{P_1}}{\partial t} &\approx 2c_0\frac{\partial\overline{P_1}}{\partial z} \\
&= \frac{c_0\rho_{00}}{2H}\left(\frac{3\gamma^2-\gamma-12}{4\gamma}V^2_{z(1,ac)}+(\gamma-3)\frac{\partial V_{z(1,ac)}}{\partial z}\int V_{z(1,ac)}\,\mathrm{d}z\right)
\end{aligned}
$$

and therefore,

$$
\overline{P}_1 = \frac{\rho_{00}}{4H}\int\left(\frac{3\gamma^2-\gamma-12}{4\gamma}V^2_{(1,ac)}+(\gamma-3)\frac{\partial V_{(1,ac)}}{\partial z}\int V_{(1,ac)}\,\mathrm{d}z\right)\mathrm{d}z,
$$

which yields the expression for $\overline{R}_1$ and dynamic equation for velocity of sound:

$$\overline{R}_1 = -\frac{\rho_{00}}{4Hc_0^2}\int\left(\frac{\gamma+9}{4}V_{(1,ac)}^2 + (\gamma+1)\frac{\partial V_{(1,ac)}}{\partial z}\int V_{(1,ac)}\,\mathrm{d}z\right)\mathrm{d}z,$$

$$\frac{\partial V_{(1,ac)}}{\partial t} + c_0\frac{\partial V_{(1,ac)}}{\partial z} + e^{z/2H}\left(\frac{\gamma+1}{4}\frac{\partial V_{(1,ac)}^2}{\partial z} + \frac{3(\gamma+1)}{16H}V_{(1,ac)}^2 + \frac{\gamma-1}{4H}\frac{\partial V_{(1,ac)}}{\partial z}\int V_{(1,ac)}dz\right) = 0. \tag{18}$$

It is easy to verify that the links and relative equation for the downwards directed wave are the following:

$$P_{2,ac} = -c_0\rho_{00}\left(1 + \frac{\gamma-2}{2\gamma H}\int\mathrm{d}z\right)V_{(2,ac)} + \exp{(z/2H)}\rho_{00}\frac{\gamma+1}{4}V_{(2,ac)}^2 + \exp{(z/2H)}\overline{P}_2,$$

$$R_{2,ac} = -\frac{\rho_{00}}{c_0}\left(1 - \frac{1}{2H}\int\mathrm{d}z\right)V_{(2,ac)} + \exp{(z/2H)}\frac{3-\gamma}{4}\frac{\rho_{00}}{c_0^2}V_{(2,ac)}^2 + \exp{(z/2H)}\overline{R}_2,$$

$$\overline{P}_2 = \frac{\rho_{00}}{4H}\int\left(\frac{3\gamma^2-\gamma-12}{4\gamma}V_{(2,ac)}^2 + (\gamma-3)\frac{\partial V_{(2,ac)}}{\partial z}\int V_{(2,ac)}\,\mathrm{d}z\right)\mathrm{d}z, \tag{19}$$

$$\overline{R}_2 = -\frac{\rho_{00}}{4Hc_0^2}\int\left(\frac{\gamma+9}{4}V_{(2,ac)}^2 + (\gamma+1)\frac{\partial V_{(2,ac)}}{\partial z}\int V_{(2,ac)}\,\mathrm{d}z\right)\mathrm{d}z,$$

$$\frac{\partial V_{(2,ac)}}{\partial t} - c_0\frac{\partial V_{(2,ac)}}{\partial z} + e^{z/2H}\left(\frac{\gamma+1}{4}\frac{\partial V_{(2,ac)}^2}{\partial z} + \frac{3(\gamma+1)}{16H}V_{(2,ac)}^2 + \frac{\gamma-1}{4H}\frac{\partial V_{(2,ac)}}{\partial z}\int V_{(2,ac)}\,\mathrm{d}z\right) = 0.$$

In the frames of the accepted accuracy, the specific density and pressure of stationary modes are not affected by the sound and are governed by equations:

$$\frac{\partial R_{st}}{\partial t} = \frac{\partial P_{st}}{\partial t} = 0,$$

with $V_{st} = 0$.

4.3. Numerical examples

4.3.1. Validity of expansions of integrals in the series

The question concerning validity of expansion of the linear operators (11) needs additional examinations. Generally, the larger is k compared to $1/H$, the smaller is the difference. Every perturbation localized in the space possesses in its spectrum zero wavenumber, among other, though the larger input has a wavenumber of order L^{-1}, where L is a characteristic scale of perturbation. In order to examine how precise the approximate expressions (11) are, four exemplary waveforms, modulated impulses $V = V_0 \exp(-(z/H - 5)^2)\cos(10z/H)$, $V = V_0 \exp(-5 \cdot 10^4 (z/H - 0.02)^2)\cos(5 \cdot 10^3 z/H)$, and single Gauss pulses $V = V_0 \exp(-10(z/H-5))$, $V = V_0 \exp(-5 \cdot 10^4 (z/H - 0.02))$ are considered. Numerical results of applying of the first and second operators from the set of (11) to all waveforms are shown consistently in Fig. 1.

The discrepancy of the initial waveform and results of integration are quite small even for waveforms at Figs. 1a and 1c with characteristic scale of order H,

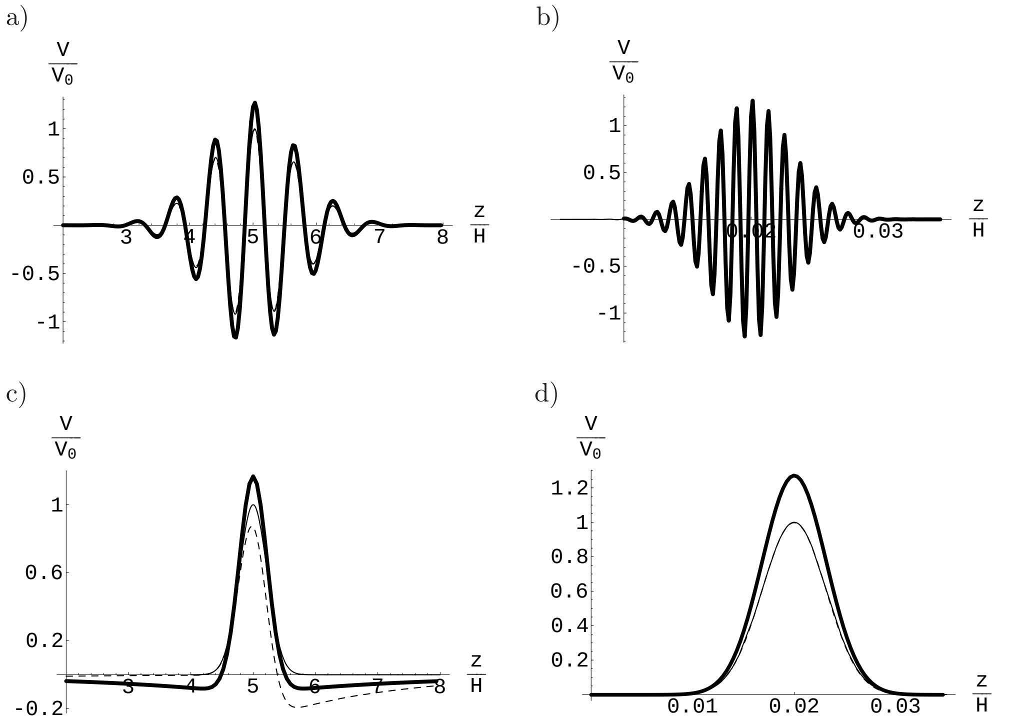

Fig. 1. The waveform V/V_0 as a function of vertical distance z/H (solid line), $\frac{1}{\pi V_0}\int\limits_{-\infty}^{\infty} dz' F(z'-z)\frac{\partial V(z')}{\partial z'}$ (bold line), $\frac{1}{V_0}\exp(-z/2H)\int\limits_{-\infty}^{z} dz' \exp(z'/2H)\frac{\partial V(z')}{\partial z'}$ (dotted line). At the sets a, b, and d, the dotted and solid curves are very close to each other.

and therefore, for the ones being out of frames of consideration. Note that typical quantity of H in the atmosphere is about 10 km, so that such perturbations are actually very extended. In Figs. 1 a, b, d the difference between original waveform and the second integral from the set (11) is indistinguishable. Much larger difference exhibits the first integral of the set (11). This deviation may be explained partially by the limited accuracy of special functions forming $F(z)$. The calculations were performed with the help of *Mathematica*. Such linear variations in waveform caused by a linear operator may be investigated also by means of spectral analysis.

To demonstrate the difference between two last integrals in the set (11),

$$\frac{1}{V_0}\exp(-z/2H)\int_{-\infty}^{z} dz' \exp(z'/2H)\frac{\partial V(z')}{\partial z'}$$

and

$$\frac{1}{V_0}\exp(z/2H)\int_{\infty}^{z} dz' \exp(-z'/2H)\frac{\partial V(z')}{\partial z'},$$

respectively, the initial waveform $V = V_0\exp(-20(z/H-3)^2)$ is considered. The Fig. 2a represents the first and second integrals (fine and large-scale dotted lines, respectively). The initial waveform V/V_0 and

$$\frac{1}{\pi V_0}\int_{-\infty}^{\infty} dz' F(z'-z)\frac{\partial V(z')}{\partial z'}$$

are plotted by solid normal and boldface lines, respectively, in Fig. 2b.

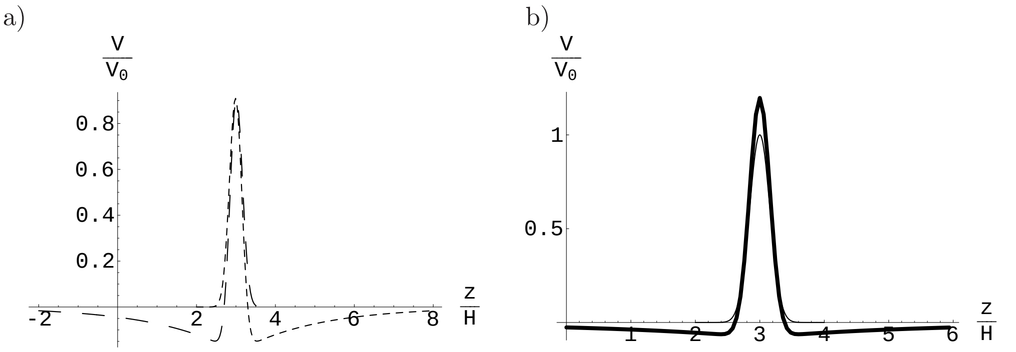

Fig. 2. a) $\frac{1}{V_0}\exp(-z/2H)\int_{-\infty}^{z} dz' \exp(z'/2H)\frac{\partial V(z')}{\partial z'}$ and $\frac{1}{V_0}\exp(z/2H)\int_{\infty}^{z} dz' \exp(-z'/2H)\frac{\partial V(z')}{\partial z'}$ (fine dotted line and large-scaled dotted line, respectively) as functions of vertical distance z/H; b) $\frac{V}{V_0} = \exp(-20(z/H-3)^2)$ (solid line) and $\frac{1}{\pi V_0}\int_{-\infty}^{\infty} dz' F(z'-z)\frac{\partial V(z')}{\partial z'}$ (boldface line).

4.3.2. Decomposition of initial waveform into specific parts

Using operators (10), any initial perturbation may be decomposed into specific parts, acoustic and stationary. In the linear flow, the subdivision is exact in the frames of the problem ($kH \gg 1$). In the weakly nonlinear flow, the projection may be used as approximate estimation. The analysis reveals that even for the large Mach numbers of order 1, modes with linear links keep the property to propagate mainly in one chosen direction according to the roots of dispersion relation (6) [5].

The examples of subdivision of pure perturbation of velocity

$$V/V_0 = \sin(2\pi z/H)$$

or pressure

$$P/(\rho_{00}c_0V_0) = \sin(2\pi z/H)$$

for $0.5H < z < 1.5H$ and zero outside this interval, into specific parts, are illustrated by the Figs. 3, 4 below. Pure velocity perturbation decouples into two acoustic branches, and pure pressure perturbation into all three types of motion, each with fixed links of pressure, density and velocity according to (8), (9).

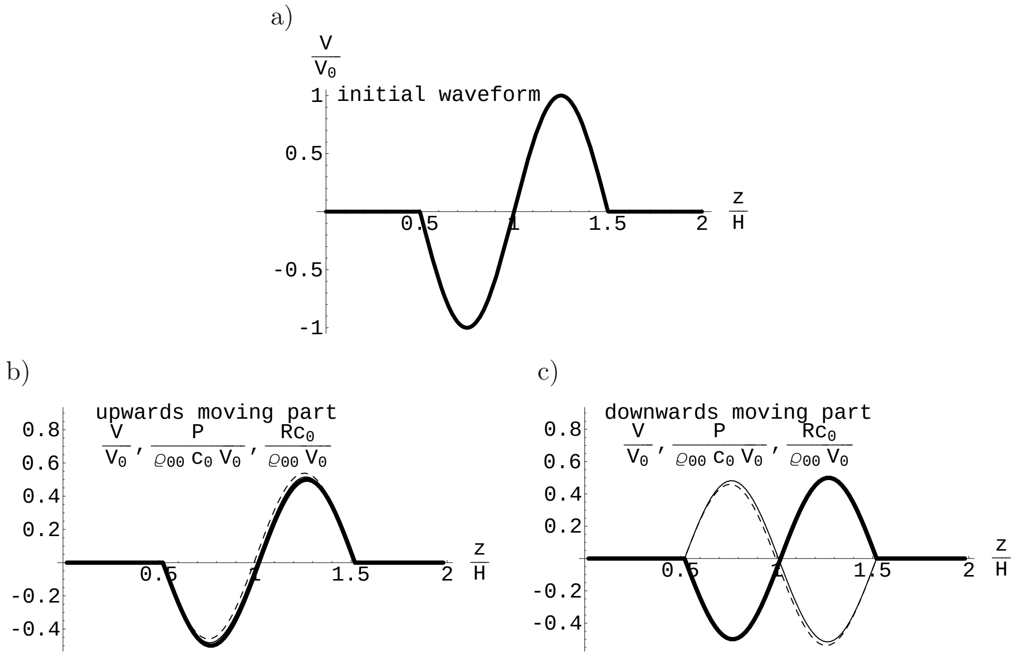

Fig. 3. a) Initial perturbation $\frac{V}{V_0}$; b), c) The excess pressure, density and velocity of relative acoustic branches. $\frac{V}{V_0}$ is plotted by the boldface line, $\frac{Rc_0}{\rho_{00}V_0}$ is plotted by the dotted line, and $\frac{P}{\rho_{00}V_0c_0}$ by the solid line.

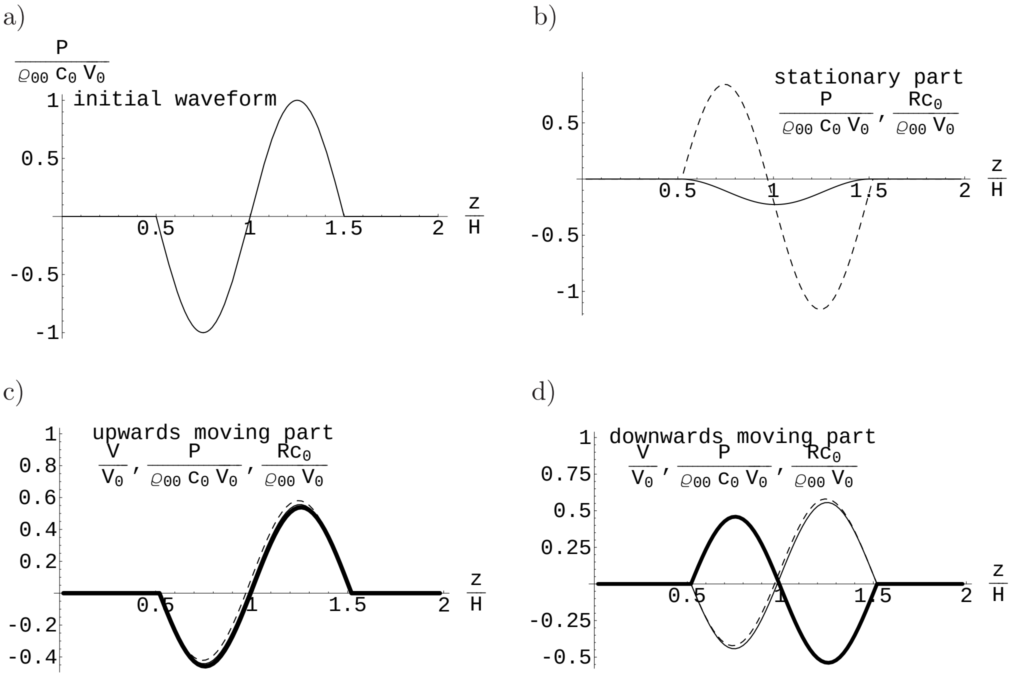

Fig. 4. a) Initial perturbation $\frac{P}{\rho_{00} c_0 V_0}$; b), c), d) The excess pressure, density and velocity of relative stationary and acoustic branches. $\frac{V}{V_0}$ is plotted by the boldface line, $\frac{Rc_0}{\rho_{00} V_0}$ is plotted by the dotted line, and $\frac{P}{\rho_{00} V_0 c_0}$ by the solid line.

4.3.3. Nonlinear sound dynamics

Weakly nonlinear dynamics of upwards moving sound is governed by Eq. (18). All three terms in the brackets are nonlinear. The first one leads in the motion over the background of constant density to $\frac{\gamma+1}{4}\frac{\partial V^2_{(1,ac)}}{\partial z}$ while $H \to \infty$, and the two other tend to zero. Since $e^{z/2H}$ is a growing function of z, the general conclusion is that nonlinear effects grow while the wave propagates upwards. The dynamics of two positive Gauss pulses of different amplitudes

$$\frac{V}{V_0}(z, t=0) = 0.1 \exp(-10(z/H-2)^2)$$

and

$$\frac{V}{V_0}(z, t=0) = 0.3 \exp(-10(z/H-2)^2)$$

as functions of the retarded non-dimensional coordinate $Z = \dfrac{z - tc_0}{H}$ at different times $\dfrac{c_0}{H}t$ is shown by Fig. 5.

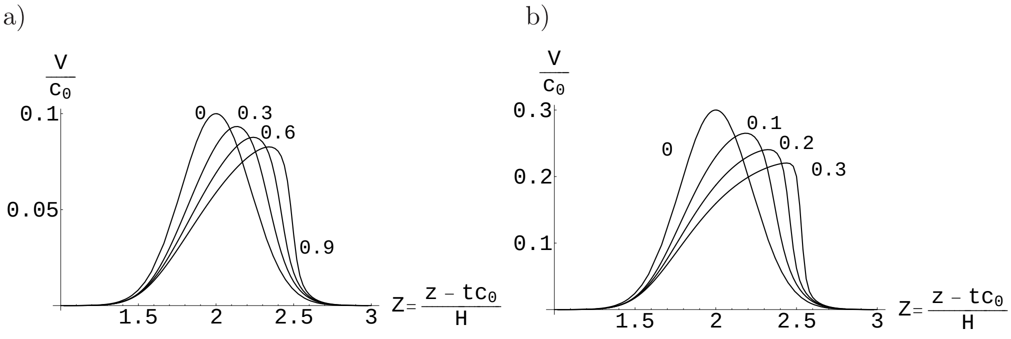

Fig. 5. a) Dynamics of the Gauss pulse $\dfrac{V}{V_0} = 0.1\exp(-10(z/H - 2)^2)$ at dimensionless times $t\dfrac{c_0}{H} = 0, 0.3, 0.6, 0.9$ as a function of the retarded coordinate $Z = \dfrac{z - tc_0}{H}$; b) Dynamics of the Gauss pulse $\dfrac{V}{V_0} = 0.3\exp(-10(z/H - 2)^2)$ at dimensionless times $t\dfrac{c_0}{H} = 0, 0.1, 0.2, 0.3$ as a function of the retarded coordinate $Z = \dfrac{z - tc_0}{H}$.

Figure 6 explains the relative role of two last nonlinear terms in the brackets of Eq. (18) for the same initial pulses and two analogous negative. The solid

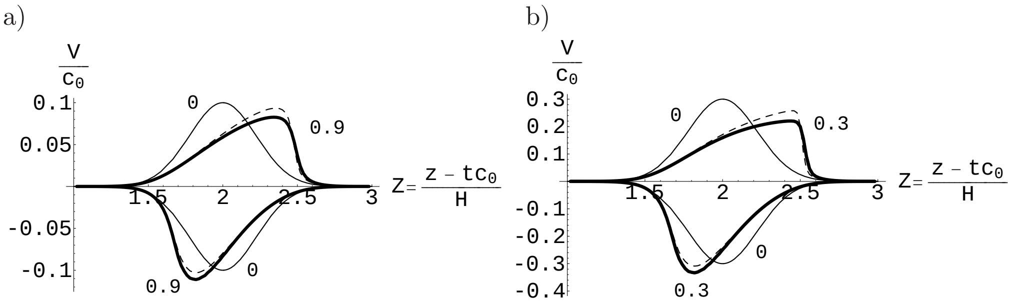

Fig. 6. a) Dynamics of the Gauss pulses $\dfrac{V}{V_0} = 0.1\exp(-10(z/H - 2)^2)$ and $\dfrac{V}{V_0} = -0.1\exp(-10(z/H - 2)^2)$ at dimensionless times $t\dfrac{c_0}{H} = 0,\ 0.9$ as functions of the retarded coordinate $Z = \dfrac{z - tc_0}{H}$, boldface line accounts for all nonlinear terms in (18), the dotted one accounts for only first one; b) Dynamics of the Gauss pulses $\dfrac{V}{V_0} = 0.3\exp(-10(z/H - 2)^2)$ and $\dfrac{V}{V_0} = -0.3\exp(-10(z/H - 2)^2)$ at dimensionless times $t\dfrac{c_0}{H} = 0,\ 0.3$ as functions of the retarded coordinate $Z = \dfrac{z - tc_0}{H}$, boldface line accounts for all nonlinear terms in (18), the dotted one accounts for the first one only.

line denotes the initial waveform, the dotted one accounts for exclusively first nonlinear term in the brackets, and the bold line accounts for all three terms.

The difference in distortions of the analogous pulses of different polarity may be explained by the presence of the factor $\exp(z/2H)$ which makes distortions stronger for larger z.

5. Conclusions

The method of decomposing of dynamic nonlinear equations from the overall system of conservation equations was proposed and applied by the author in the variety of fluid dynamics problems, concerning interaction of sound and non-acoustic types of motion in the one- or multi-dimensional flow [5, 6, 11–13]. It bases on properties of the motions of infinitely small amplitude following from the fixed links of gasdynamic perturbations of infinitely small amplitude.

Both the successive separation of acoustic and stationary parts of the general perturbation, and decomposition of specific dynamic equations from the general system, proceed with the help of linear matrix operators. The idea of projecting relatively to the fluid dynamics problems was also worked out by the author. Projectors may be considerably simplified in the case of large characteristic wavenumber k: $1/k \ll H$, as well as links of wave perturbations inside every mode (Eqs. (8)–(10)). The numerical examples of the investigation concerning the examinations of the simplified and exact operators, present a satisfactory agreement of the waveforms even for fairly extended ones: $1/k \sim H$.

It is demonstrated that the conclusions of the linear flow theory apply to the weakly nonlinear motion. Thus the consideration above concerns the weakly nonlinear flows with $M \ll 1$ and $1/kH \ll 1$. The separation of any perturbation into acoustic and stationary branches at any moment may be successfully undertaken by means of linear projectors.

The nonlinear corrections to specific links inside acoustic modes of order M^2/kH are derived. They support constant entropy of sound within this accuracy. The corresponding dynamic equations for the acoustic branches individually become written on. The illustrations of nonlinear distortions of a single waveform reveal the main features of dynamics compared to the nonlinear distortions of wave in the medium with constant background density and pressure. The individual role of all nonlinear terms is studied: taking into account these of order M^2/kH makes the positive peaks lower but the negative ones-deeper. The dispersive deformations are not taken into account in the dynamic equations for sound, in their linear part, because the expansion of dispersion relations $\omega(k)$ (6) into the series includes terms of order $1/k^2H^2$, which are outside the frames of consideration. But links inside modes (8), (9), nonlinear corrections to them and nonlinear parts of dynamic equations (18), (19) lead to dispersion.

The results may be easily applied in studies of a fluid different from an ideal gas (by involving the correspondent equations of state), in presence of a constant force other than the gravity force.

References

[1] Brekhovskikh L.M., Godin A.O., *Acoustics of layered media*, Springer-Verlag, Berlin 1990.

[2] Chu B.-T., Kovasznay L.S.G., *Nonlinear interactions in a viscous heat-conducting compressible gas*, Journ. Fluid. Mech., **3**, 494–514 (1958).

[3] Rudenko O.V., Soluyan S.I., *Theoretical foundations of nonlinear acoustics*, Plenum, New York 1977.

[4] Gordin A.V., *Mathematical problems of hydrodynamical weather prediction. Analytical aspects. Gydrometeoizdat* [in Russian], Leningrad 1987.

[5] Perelomova A.A., *Nonlinear dynamics of vertically propagating acoustic waves in a stratified atmosphere*, Acta Acustica, **84**, 1002–1006 (1998).

[6] Perelomova A.A., *Nonlinear dynamics of directed acoustic waves in stratified and homogeneous liquids and gases with arbitrary equation of state*, Archives of Acoustics, **25**, 4, 451–463 (2000).

[7] Riemann B., *The collected works of Bernard Riemann*, Dover, New York 1953.

[8] Naugolnykh K., Ostrovsky L., *Nonlinear processes in acoustics*, Cambrige University Press, 1998.

[9] Hamilton M., Morfey C., *Model equations*, [in:] Nonlinear Acoustics, Hamilton M., and Blackstock D. [Eds.], pp. 41–63, Academic Press, 1998.

[10] Makarov S., Ochmann M., *Nonlinear and thermoviscous phenomena in acoustics, Part I*, Acustica, **82**, 579–606 (1996).

[11] Perelomova A., *Modelling of acoustic heating induced by different types of sound*, Archives of Acoustics, **33**, 2, 151–160 (2008).

[12] Perelomova A., *Acoustic streaming induced by the non-periodic sound in a viscous medium*, Archives of Acoustics, **31**, 4 (Supplement), 35–40 (2007).

[13] Perelomova A., *Influence of vortices on a progressive quasi-plane acoustic wave*, Archives of Acoustics, **31**, 4, 489–501 (2006).