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One-Dimensional Ultrasound Propagation in Stratified Gas

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The system of hydrodynamic-type equations is derived by two-sided distribution function for a stratified gas in gravity field and applied to the problem of ultrasound. The theory is based on the generalized Gross–Jackson kinetic equation, the solution of which is built by means of locally equilibrium distribution function with different local parameters for molecules moving "up" and "down". The problem of propagation of the sound wave from an oscillating plane is explored. The linearized version of the obtained system is studied and compared with other results and experiments for a wide range of Knudsen numbers (Kn). The discrepancy with experiment in attenuation behavior at big Kn range forced us to use generalized kinetic description leading to the Alexeev–Boltzmann equation. Its use essentially improves the results.

Keywords: ultrasound, fluid mechanics; rarefied gas dynamics, kinetic theory.

1. Introduction

The history of acoustic wave propagation theory for gases started from Newton's works and has been almost completed in Stokes' and Kirchhoff's works using the hydro-dynamical (Navier–Stokes–Kirchhoff) approach. The Navier–Stokes equations are derived from Boltzmann kinetics under the strong condition for the Knudsen number Kn $\ll 1$. Therefore this approach is not valid when the mean free path is of comparable order or exceeds the wavelength (characteristic space scale of a gas perturbation) under consideration.

The problem of description of such waves is relevant to either rarefied gas or to high frequencies (ultrasound). It is important, for example, at high altitudes of atmosphere due to the mean free path growth. Other interesting applications appear in microfluidics [1].

Recently the problems of all Knudsen range wave propagation was revisited in connection with general fluid mechanics development by application of nonsingular perturbation method [2–4] to kinetic description. Generalized Boltzmann theories [5–7] also contributed to a progress with respect to this important problem.

In [3, 8] the propagation of one-dimensional gas disturbance was studied on the basis of the method of a piecewise continuous distribution function, launched in a pioneering paper of Lees [9]. In papers [8, 10] it was shown that application of fluid equations based on the BGK model for the sound velocity provides good agreement with experimental data of papers [11, 12]. Since the Bhatnagar–Gross–Krook (BGK) model gives the Prandtl number equals 1, which is incorrect for monatomic gases, a certain judicious choice of mean free path is always required. It is of interest here to examine the higher-order kinetic models of the collision term [13] to assess the model dependence of the results.

In this paper we have considered such class of higher-order models, namely the Gross–Jackson ones [13]. As we may see, the fluid dynamics, based on Boltzmann kinetic equations and the Gross-Jackson model, improves the results at the hydrodynamical realm, but still does not give a satisfactory agreement with the experimental data for attenuation of sound at large Knudsen numbers. To solve this problem we go to the generalized kinetic equations of Alexeev theory [4] in Sec. 5. Our choice is justified by the natural reason: the Alexeev–Boltzmann equation is derived from Bogoliubov chain kinetic equations by the similar nonsingular perturbation method. Deriving the correspondent generalized fluid equations along the algorithm described in previous sections, we again restrict ourselves to the one-dimensional case (the distribution function depends only on one space coordinate, but left the three-dimensional velocity space).

This article is an expanded version of the WESPAC (South Korea) conference presentation. It is organized as follows: for a reader's convenience, in Secs. 2 and 3, we review our paper published at the electronic conference [14], i.e. the fluid dynamics description and correspondent results for linear acoustics based on the simplest Gross–Jackson kinetic equation are given. In Sec. 4 a comparison between results of our previous study and other authors' results is presented in more details (Figs. 1, 2). Figures 3, 4 from Sec. 4 are reproduced from the authors' earlier study [14], however the curves from Struchtrup–Torilhon's paper [17] are added. Section 5 contains new results arising from Alexeev–Boltzmann kinetics that shows strong correlations in the deep Knudsen regime, and finally Conclusions and References are given.

2. Generalized fluid dynamics equations

The derivation of the hydrodynamic-type equations is based on the Gross–Jackson kinetic equation [13], that looks in the one-dimensional case like:

$$\frac{\partial\varphi}{\partial t} + V_z\frac{\partial\varphi}{\partial z} - g\frac{\partial\varphi}{\partial V_z} = \nu\cdot\left(\sum_{I=1}^{3}\langle\chi_I,\varphi\rangle\chi_I + \frac{1}{3}\langle\chi_4,\varphi\rangle\chi_4 - \varphi\right). \tag{1}$$

Here $\varphi(t,z,\mathbf{V}) = (f - f_0)/f_0$ represents a deviation of the distribution function from the Maxwellian; t is time, $\mathbf{V}$ is velocity of a particle of gas with mass m, z is the (vertical) coordinate, $\nu(z) = \nu_0\exp(-z/H)$ is the effective frequency of collisions between particles of the gas at height z, T – temperature, $H = kT/mg$ is a parameter of the gas stratification.

The moments of distribution function are defined by:

$$M_I = \langle\chi_I,\varphi\rangle = \frac{1}{\pi^{3/2}}\int \mathrm{d}\mathbf{c}\exp(-c^2)\chi_I(\mathbf{c})\varphi, \tag{2}$$

where $\mathbf{c} = \mathbf{V}/V_T$ is the dimensionless velocity, $V_T = \sqrt{2kT/m}$ denotes the average thermal velocity of particles of gas. The first six eigenfunctions χ_I of the linearized collision operator are:

$$\begin{aligned}
&\chi_1 = 1, && \chi_2 = \sqrt{2}c_Z,\\
&\chi_3 = \sqrt{\frac{2}{3}}\left(\frac{3}{2} - c^2\right), && \chi_4 = \sqrt{5}c_Z\left(1 - \frac{2}{5}c^2\right),\\
&\chi_5 = \frac{1}{\sqrt{3}}\left(c^2 - 3c_Z^2\right), && \chi_6 = \sqrt{\frac{6}{5}}c_Z\left(c^2 - \frac{5}{3}c_Z^2\right).
\end{aligned} \tag{3}$$

The expressions for the moments of distribution function in terms of the gas parameters are:

$$\begin{aligned}
&M_1 = \frac{n - n_0}{n_0}, \qquad M_2 = \sqrt{2}\frac{n}{n_0}\frac{U_Z}{V_T},\\
&M_3 = -\sqrt{\frac{2}{3}}\frac{n}{n_0}\frac{U_Z^2}{V_T^2} - \sqrt{\frac{3}{2}}\frac{n}{n_0}\frac{T - T_0}{T_0},\\
&M_4 = -\frac{3}{\sqrt{5}}\frac{n}{n_0}\frac{T - T_0}{T_0}\frac{U_Z}{V_T} - \frac{2}{\sqrt{5}}\frac{P_{ZZ} - nkT_0}{n_0kT_0}\frac{U_Z}{V_T} - \frac{2}{\sqrt{5}}\frac{n}{n_0}\frac{U_Z^3}{V_T^3} - \frac{4}{\sqrt{5}}\frac{q_Z}{mn_0V_T^3},\\
&M_5 = \frac{1}{\sqrt{3}}\left(\frac{3}{2}\frac{nkT - P_{ZZ}}{n_0kT_0} - 2\frac{n}{n_0}\frac{U_Z^2}{V_T^2}\right),\\
&M_6 = \sqrt{\frac{6}{5}}\left(-\frac{2}{3}\frac{n}{n_0}\frac{U_Z^3}{V_T^3} + \frac{U_Z}{V_T}\frac{3}{2}\frac{nkT - P_{ZZ}}{n_0kT_0} + \frac{2}{n_0V_T^3m}\left(q_Z - \frac{5}{3}\overline{q}_Z\right)\right).
\end{aligned} \tag{4}$$

In linear approach the first three moments are proportional to mass density n, velocity U_Z and temperature T variations respectively. Here P_{ZZ} is the diagonal component of the pressure tensor, q_Z is the vertical component of a heat flux vector, $\overline{q}_Z$ is a similar parameter having dimension of the heat flux.

Following the idea of the method of piecewise continuous distribution functions for a gas in the gravity field [2, 3, 15], let's search a solution φ of the Eq. (1) as a combination of two locally equilibrium distribution functions, each of which gives the contribution in its own area of velocities space:

$$\begin{aligned} \varphi^{\pm}(t,z,\mathbf{V}) &= (f^{\pm}-f_0)/f_0, \\ f^{\pm} &= \frac{n^{\pm}}{\pi^{3/2}V_T^{\pm 3}}\exp\left(-\frac{(\mathbf{V}-\mathbf{U}^{\pm})^2}{V_T^{\pm 2}}\right). \end{aligned} \tag{5}$$

The increase of the number of functional parameters of distribution function results in the fact that the distribution function differs from the local-equilibrium one and describes deviations from the hydrodynamic regime. In the range of small Knudsen numbers $l \ll L$ we automatically have $n^+ = n^-$, $U^+ = U^-$, $T^+ = T^-$ and distribution function reproduces the hydrodynamics of Euler and, within small difference range of the functional 'up' and 'down' parameters: the Navier–Stokes equations. In the range of big Knudsen numbers the theory gives solutions of collisionless problems [9, 15, 18].

If we now multiply the Gross–Jackson kinetic equation (1) by χ_I and integrate over the velocity space, we obtain the fluid dynamic equations:

$$\begin{aligned} &\frac{\partial}{\partial t}M_1 + \frac{V_T}{\sqrt{2}}\frac{\partial}{\partial z}M_2 - \frac{V_T}{\sqrt{2}H}M_2 = 0, \\ &\frac{\partial M_2}{\partial t} + V_T\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{2}}M_1 - \frac{1}{\sqrt{3}}M_3 - \frac{\sqrt{2}}{\sqrt{3}}M_5\right) \\ &\qquad\qquad\qquad\qquad + \frac{V_T}{H}\left(\frac{1}{\sqrt{3}}M_3 + \frac{\sqrt{2}}{\sqrt{3}}M_5\right) = 0, \\ &\frac{\partial M_3}{\partial t} + V_T\frac{\partial}{\partial z}\left(-\frac{1}{\sqrt{3}}M_2 + \sqrt{\frac{5}{6}}M_4\right) - \sqrt{\frac{5}{6}}\frac{V_T}{H}M_4 = 0, \\ &\frac{\partial M_4}{\partial t} + V_T\frac{\partial}{\partial z}\left(\frac{\sqrt{5}}{2}M_1 - \sqrt{\frac{5}{6}}M_3 - \sqrt{\frac{5}{3}}M_5 - \frac{2}{\sqrt{5}}J_1\right) \\ &\qquad + \frac{V_T}{H}\left(-\frac{\sqrt{5}}{2}M_1 + \sqrt{\frac{10}{3}}M_3 + \frac{7}{\sqrt{15}}M_5 + \frac{2}{\sqrt{5}}J_1\right) = -\frac{2}{3}\nu M_4, \end{aligned} \tag{6}$$

$$\frac{\partial M_5}{\partial t} + V_T \frac{\partial}{\partial z}\left(-\sqrt{\frac{2}{3}}M_2 + \frac{2}{\sqrt{15}}M_4 + \frac{3}{\sqrt{10}}M_6\right) - \frac{V_T}{H}\left(\frac{2}{\sqrt{15}}M_4 + \frac{3}{2}\sqrt{\frac{2}{5}}M_6\right) = -\nu M_5,$$

$$\frac{\partial M_6}{\partial t} + \sqrt{\frac{6}{5}}V_T \frac{\partial}{\partial z}\left(J_1 - \frac{5}{3}J_2\right) + \frac{V_T}{H}\frac{3}{\sqrt{10}}M_5 - \frac{V_T}{H}\sqrt{\frac{6}{5}}\left(J_1 - \frac{5}{3}J_2\right) = -\nu M_6, \tag{6 [cont.]}$$

where $J_1 = \langle c_Z^2 c^2, \varphi\rangle$, $J_2 = \langle c_Z^4, \varphi\rangle$.

System (6) of the equations, according to the derivation scheme, is valid at all Knudsen numbers, when it is possible to neglect of higher momenta. It is a system of hydrodynamical type and generalizes the classical equations of a viscous fluid to arbitrary Kn, up to a free molecule flow. However, the system (6) is not closed yet. It is necessary to present values of two integrals $J_{1,2}$ as functions of thermodynamic parameters.

In [8] we have obtained the expressions for the two integrals in the case of a small Mach number:

$$J_1 = \frac{5}{4}\left(\frac{n-n_0}{n_0} + 2\frac{T-T_0}{T_0}\right) + \frac{61}{24}\frac{P_{ZZ} - nkT_0}{n_0 kT_0},$$

$$J_2 = \frac{3}{4}\left(\frac{n-n_0}{n_0} + 2\frac{T-T_0}{T_0}\right) + \frac{13}{8}\frac{P_{ZZ} - nkT_0}{n_0 kT_0}.$$

Substituting $J_{1,2}$ into the system (6), in the small Mach number (linear) case we obtain the system:

$$\begin{aligned}
&\frac{\partial}{\partial t}(n') + V_T\frac{\partial}{\partial z}U' - \frac{V_T}{H}U' = 0,\\
&\frac{\partial}{\partial t}(U') + \frac{V_T}{2}\frac{\partial}{\partial z}(n' + T' + \pi'_{ZZ}) - \frac{V_T}{2H}(T' + \pi'_{ZZ}) = 0,\\
&\frac{\partial}{\partial t}\left(\frac{3}{2}T'\right) + V_T\frac{\partial}{\partial z}(U' + 2q') - \frac{V_T}{H}2q' = 0,\\
&\frac{\partial}{\partial t}(q') + V_T\frac{\partial}{\partial z}\left(\frac{5}{8}T' + \frac{31}{48}\pi'_{ZZ}\right) - \frac{19}{48}\frac{V_T}{H}\pi'_{ZZ} = -\frac{2}{3}\nu q',\\
&\frac{\partial}{\partial t}(\pi'_{ZZ}) + V_T\frac{\partial}{\partial z}\left(\frac{4}{3}U' - \frac{4}{3}q' + 4\overline{q}'\right) - \frac{V_T}{H}\left(-\frac{4}{3}q' + 4\overline{q}'\right) = -\nu\pi'_{ZZ},\\
&\frac{\partial}{\partial t}(\overline{q}') + V_T\frac{\partial}{\partial z}\left(\frac{3}{8}T' + \frac{7}{16}\pi'_{ZZ}\right) - \frac{1}{16}\frac{V_T}{H}\pi'_{ZZ} = -\nu\left(\overline{q}' - \frac{1}{5}q'\right).
\end{aligned} \tag{7}$$

Here the dimensionless viscous stress tensor $\pi'_{ZZ} = \dfrac{P_{ZZ} - nkT}{n_0 k T_0}$ is introduced.

Thus a modification of the procedure for deriving fluid mechanics (hydrodynamic-type) equations from the kinetic theory is proposed, it generalizes the Navier–Stokes equation at arbitrary density or frequency (Knudsen numbers).

Our method gives a reasonable agreement with the experimental data in the case of a homogeneous gas [8].

3. A limiting case of gas oscillations at high frequencies of collisions (small Knudsen numbers)

Let us consider a system in the hydrodynamic limit $\nu \to \infty$. It follows from the last three equations of the system (7) that q, π_{ZZ}, $\overline{q} \ll \rho$, T, U. Next assume $\nu^{-1} = 0$ in the zero-order of the parameter ν^{-1}. We put q, π_{ZZ}, $\overline{q} = 0$ and the system (7) tends to the linearized Euler's system:

$$\frac{\partial}{\partial t}(n') + V_T \frac{\partial}{\partial z} U' - \frac{V_T}{H} U' = 0,$$

$$\frac{\partial}{\partial t}(U') + \frac{V_T}{2}\frac{\partial}{\partial z}(n' + T') - \frac{V_T}{2H}(T') = 0,$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}T'\right) + V_T \frac{\partial}{\partial z}(U') = 0.$$

The small nonzero moments q, π_{ZZ}, $\overline{q}$ yield the next order of the parameter ν^{-1}. Then from the last three equations of the system (6), one obtains the following relations:

$$q' = -\frac{15}{16}\frac{V_T}{\nu}\frac{\partial}{\partial z}(T'), \quad \pi'_{ZZ} = -\frac{4}{3}\frac{V_T}{\nu}\frac{\partial}{\partial z}(U'), \quad \overline{q}' = -\frac{9}{16}\frac{V_T}{\nu}\frac{\partial}{\partial z}(T'). \tag{8}$$

Further substitution of (8) into the first three equations of the system (7) gives the linear version of Navier–Stokes equations:

$$\begin{aligned}
&\frac{\partial}{\partial t}(n') + V_T \frac{\partial}{\partial z} U' - \frac{V_T}{H} U' = 0,\\
&\frac{\partial}{\partial t}(U') + \frac{V_T}{2}\frac{\partial}{\partial z}(n' + T') - \frac{2}{3}\frac{\partial}{\partial z}\left(\frac{V_T^2}{\nu}\frac{\partial}{\partial z}(U')\right)\\
&\qquad - \frac{V_T}{2H}(T') + \frac{2}{3}\frac{V_T^2}{\nu H}\frac{\partial}{\partial z}(U') = 0,\\
&\frac{\partial}{\partial t}\left(\frac{3}{2}T'\right) + V_T \frac{\partial}{\partial z}(U') - \frac{15}{8}\frac{\partial}{\partial z}\left(\frac{V_T^2}{\nu}\frac{\partial}{\partial z}(T')\right) + \frac{15}{8}\frac{V_T^2}{\nu H}\frac{\partial}{\partial z}(T') = 0.
\end{aligned} \tag{9}$$

Comparing the mentioned expressions for the viscosity factor η and the coefficient of heat conductivity κ with corresponding items in Eq. (9) we obtain

$$\eta = -\frac{n_0 k T_0}{\nu}, \qquad \kappa = -\frac{15}{4}\frac{n_0 k T_0}{\nu}.$$

Finding the Prandtl number and taking into account the molecular thermal capacity of the ideal gas under constant pressure $C_p = 5/2$, we obtain

$$\mathrm{Pr} = \frac{\eta C_P}{\kappa} = \frac{2}{3},$$

what coincides with the Prandtl number of ideal gas. So using Gross–Jackson model in contrast to the BGK one, it gives us the right value of Prandtl number. In higher orders of the theory, from the system (6) the linearized Burnett's equations follow.

4. Comparison with experiment and results of other evaluations

One of important verifications of fluid dynamics system concerns the problem of sound propagation. Let's consider a limiting case of homogeneous medium and compare it with the classic experimental data of [11, 12]. A generating plate oscillated in the direction of its normal with frequency w. The pressure of the gas was changed from 0.001 Torr to the normal atmospheric pressure. We assume plane wave solutions of the form:

$$\psi = \widetilde{\psi} \exp\{-i(wt - kz)\},$$

where ψ is the complex amplitude of the wave, w is its frequency, and k is its wave number. We obtain the dispersion relation

$$\frac{18}{125}\widetilde{k}^6 + \left(-\frac{46}{25}ir - \frac{39}{25} + \frac{3}{5}r^2\right)\widetilde{k}^4 + \left(-\frac{2}{3}ir^3 + \frac{524}{75}ir - \frac{58}{15}r^2 + \frac{58}{15}\right)\widetilde{k}^2 \\ - 1 - \frac{8}{3}ir + \frac{7}{3}r^2 + \frac{2}{3}ir^3 = 0. \qquad (10)$$

Here the dimensionless wave number $\widetilde{k} = kC_0/w$ and the Reynolds number $r = \nu/w$ are introduced, where w is the frequency of a wave, k – the (vertical) component of the wave vector and $C_0 = \sqrt{5/6}\,V_T$ – the adiabatic sound speed of linear wave. The Reynolds number and the Knudsen number are obviously linked:

$$\mathrm{Kn} = \frac{\lambda}{\lambda_b} = \frac{w}{\nu}\frac{V_T}{2\pi C_0} = \sqrt{\frac{6}{5}}\frac{1}{2\pi r}.$$

Let $\widetilde{k} = \beta + i\alpha$, then

$$\psi = \widetilde{\psi} \exp\left(-iw\left(t - \frac{\beta}{C_0}z\right)\right) \exp\left(-w\frac{\alpha}{C_0}z\right)$$

and the real part $\beta = \mathrm{Re}(k)C_0/w = C_0/C$ – the inverse non-dimensional phase velocity, α – the factor of attenuation. The dispersion relation (10) is the binary cubic equation with coefficients parametrized by r.

In Figs. 1, 2 a comparison of our results of numerical calculation of dimensionless sound speed and attenuation factor depending on r for the Gross–Jackson

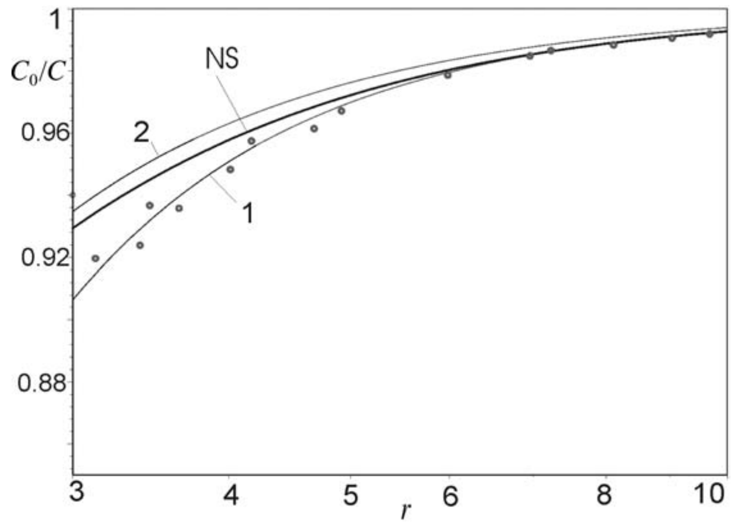

Fig. 1. The inverse non-dimensional phase velocity as a function of the inverse Knudsen number. The results of this paper for Gross–Jackson model – 1 are compared to BGK model [8] – 2, Navier–Stokes – NS and the experimental data of Meyer–Sessler and Schotter [11, 12] – circle.

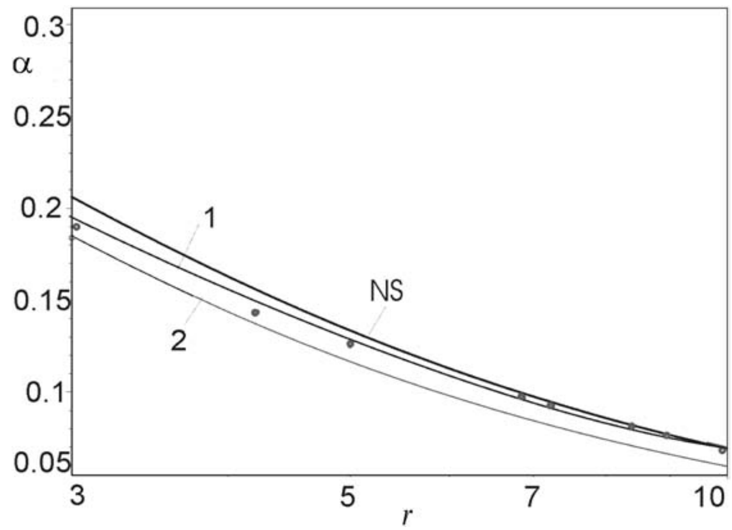

Fig. 2. The attenuation factor of the linear disturbance as a function of the inverse Knudsen number. Notation – see Fig. 1.

model is made with our results for BGK-model [8], Navier–Stokes and experimental data. Our results for the Gross–Jackson model give good consistency with the experiments at $r > 3$. In Figs. 3, 4 our results of numerical calculation of the sound characteristics are compared with the results obtained by other authors for wider range of the Knudsen numbers.

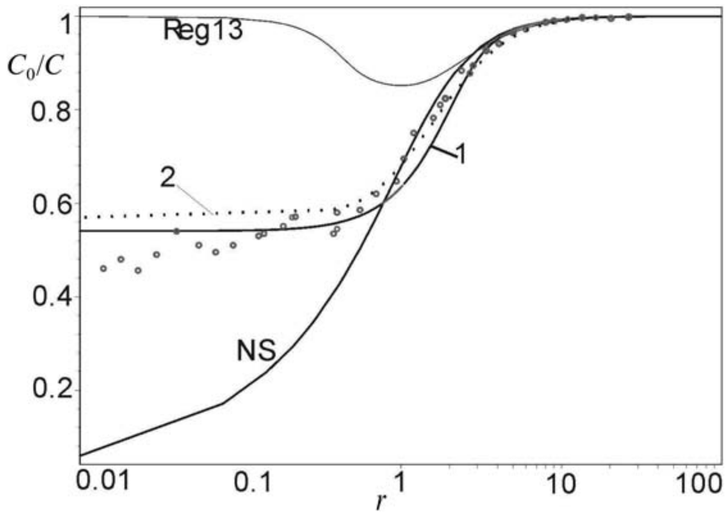

Fig. 3. The inverse non-dimensional phase velocity as a function of the inverse Knudsen number. The results of this paper – 1 are compared to the results of Chen–Rao–Spiegel [4] – 2, Navier–Stokes – NS, Struchtrup–Torrilhon [17] – Reg13 and the experimental data [11, 12] – circles.

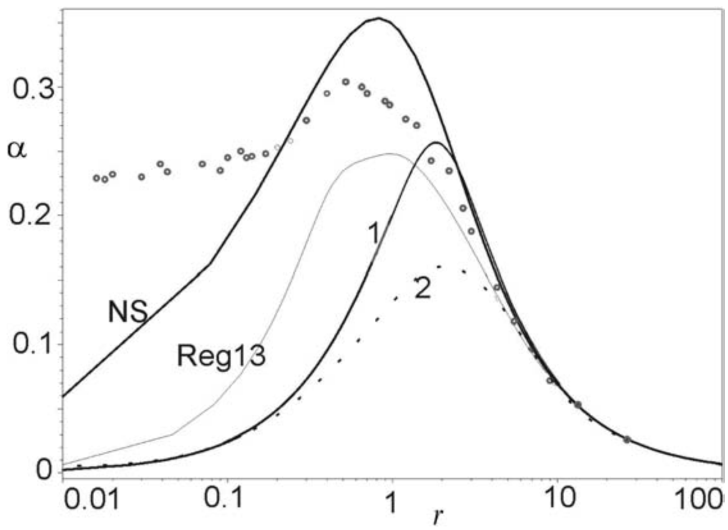

Fig. 4. The attenuation factor of the linear disturbance as a function of the inverse Knudsen number. Notation – see Fig. 3.

The Navier–Stokes-based prediction is qualitatively wrong at big Knudsen numbers. Our results for phase speed give a good consistency with the experiments at all Knudsen numbers. However, our results for the attenuation of ultrasound are good (as we can see comparing them with experiment) only for numbers r up to the order of unity. Note that our results look a bit better than those based on Navier–Stokes, Chen–Rao–Spiegel [4] and the regularization of Grad's method [17]. Unlike the BGK model, using of the Gross–Jackson model yields right coefficients of viscosity and heat conductivity.

From the linearized Eq. (7) we can obtain the integral quantity of perturbed component of pressure tensor and a vertical component of a heat flux vector:

$$\pi_{zz} = -e^{-\nu t}\left[V_T \frac{\partial}{\partial z}\int\left(\frac{4}{3}U - \frac{4}{3}q + 4q_z\right)e^{\nu t}\,\mathrm{d}t\right],$$

$$q_z = -e^{-2/3\nu t}\left[V_T \frac{\partial}{\partial z}\int\left(\frac{5}{8}T + \frac{31}{48}\pi_{zz}\right)e^{2/3\nu t}\,\mathrm{d}t\right].$$

In the first terms one recognizes the Navier–Stokes ones. The nonlinearity and boundary effect could be incorporated e.g. as in [20].

5. Generalized kinetic equation

We see that fluid dynamics equations, based on the Boltzmann equation in all the mentioned theories, result in curves which crucially do not agree with the experimental results for attenuation factor at big Knudsen numbers.

To solve this problem we go to the generalized kinetic equations, derived by Alexeev from the Bogoliubov kinetic chain system by means of nonsingular perturbation theory [4, 19]. This theory, even in its simplest version of one kinetic equation, accounts correlations between particles that are essential in the Kn regime because of a "memory" that is brought through the interval between transducer and receiver when the number of collisions is small.

This kinetic equation has the form:

$$\frac{Df}{Dt} - \frac{D}{Dt}\left(\tau\frac{Df}{Dt}\right) = J^B, \tag{11}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V}\frac{\partial}{\partial \mathbf{r}} + \mathbf{F}\frac{\partial}{\partial \mathbf{V}}$$

is the substantial (particle) derivative, $\mathbf{V}$ and $\mathbf{r}$ are the velocity and radius vector of the particle, respectively; τ – relaxation time, F – external field force, J^B is the collision Boltzmann integral. The application of the method described in the previous sections gives the system of equations:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial z}(\rho U) - \tau\frac{\partial^2}{\partial^2 t}\rho - 2\tau\frac{\partial^2}{\partial t\partial z}(\rho U) - \tau\frac{\partial^2}{\partial^2 z}(P_{zz} + \rho U^2) = 0,\\
&\frac{\partial}{\partial t}\rho U + \frac{\partial}{\partial z}(P_{zz} + \rho U^2) - \tau\frac{\partial^2}{\partial^2 t}(\rho U) - 2\tau\frac{\partial^2}{\partial t\partial z}(P_{zz} + \rho U^2)\\
&\qquad\qquad\qquad\qquad - \tau\frac{\partial^2}{\partial^2 z}(2\overline{q}_z + 3UP_{zz} + \rho U^3) = 0,\\
&\left(\frac{\partial}{\partial t} - \tau\frac{\partial^2}{\partial^2 t}\right)\left(\frac{\rho U^2}{2} + \frac{3}{2}nkT\right)\\
&\qquad\qquad + \left(\frac{\partial}{\partial z} - 2\tau\frac{\partial^2}{\partial t\partial z}\right)\left(\frac{\rho U^3}{2} + U\frac{3}{2}nkT + UP_{zz} + q_z\right)\\
&\quad - \tau\frac{\partial^2}{\partial^2 z}\left(\frac{\rho U^4}{2} + 2U(q_z + \overline{q}_z) + U^2\left(\frac{3}{2}nkT + \frac{5}{2}P_{zz}\right) + \left\langle\frac{m}{2}\xi_z^2\xi^2\right\rangle\right) = 0,\\
&\left(\frac{\partial}{\partial t} - \tau\frac{\partial^2}{\partial^2 t}\right)(\rho U^2 + P_{zz}) + \left(\frac{\partial}{\partial z} - 2\tau\frac{\partial^2}{\partial t\partial z}\right)(\rho U^3 + 3P_{zz}U + 2\overline{q}_z)\\
&\qquad\qquad - \tau\frac{\partial^2}{\partial^2 z}\left(\rho U^4 + 8U\overline{q}_z + 6P_{zz}U^2 + \langle m\xi_z^4\rangle\right) = \nu(nkT - P_{zz}),\\
&\quad\left(\frac{\partial}{\partial t} - \tau\frac{\partial^2}{\partial^2 t}\right)(\rho U^3 + 2P_{zz}U + 3nkTU + 2q_z) + \left(\frac{\partial}{\partial z} - 2\tau\frac{\partial^2}{\partial t\partial z}\right)(\rho U^4)\\
&\qquad + \left(\frac{\partial}{\partial z} - 2\tau\frac{\partial^2}{\partial t\partial z}\right)(4U(q_z + \overline{q}_z) + U^2(3nkT + 5P_{zz}) + \langle m\xi_z^2\xi^2\rangle)\\
&\qquad - \tau\frac{\partial^2}{\partial^2 z}\left(U^2(6q_z + 14\overline{q}_z) + 2\langle m\xi_z^4\rangle U + 3\langle m\xi_z^2\xi^2\rangle U + \langle m\xi_z^3\xi^2\rangle\right)\\
&\qquad - \tau\frac{\partial^2}{\partial^2 z}(\rho U^5 + U^3(3nkT + 9P_{zz})) = -2\nu q_z - 2\nu U(P_{zz} - nkT),\\
&\left(\frac{\partial}{\partial t} - \tau\frac{\partial^2}{\partial^2 t}\right)(\rho U^3 + 3P_{zz}U + 2\overline{q}_z)\\
&\qquad\qquad + \left(\frac{\partial}{\partial z} - 2\tau\frac{\partial^2}{\partial t\partial z}\right)(\rho U^4 + 8U\overline{q}_z + 6P_{zz}U^2 + \langle m\xi_z^4\rangle)\\
&\qquad\qquad - \tau\frac{\partial^2}{\partial^2 z}\left(\rho U^5 + 10U^3P_{zz} + 20U^2\overline{q}_z + 5\langle m\xi_z^4\rangle U + \langle m\xi_z^5\rangle\right)\\
&\qquad\qquad\qquad\qquad = -2\nu\overline{q}_z - 3\nu U(P_{zz} - nkT),
\end{aligned}
\tag{12}
$$

where $\boldsymbol{\xi} = \mathbf{V} - \mathbf{U}$ is the relative velocity, $\rho = nm$ is mass density. It is easy to recognize extra terms following from the correlations: the terms contain the factor τ and the integrals

$$J_1 = \frac{m}{2}\langle \xi_z^2 \xi^2 \rangle, \qquad J_2 = \frac{m}{2}\langle \xi_z^4 \rangle,$$
$$J_3 = \frac{m}{2}\langle \xi_z^5 \rangle, \qquad J_4 = \frac{m}{2}\langle \xi_z^3 \xi^2 \rangle. \tag{13}$$

The values of integrals (13) as functions of thermodynamic parameters of the system (12) are evaluated as in [8]:

$$J_1 = -\frac{5\rho}{2}\left(\frac{kT_0}{m}\right)^2 + \frac{11}{4}\frac{kT_0}{m}P_{ZZ} + \frac{9}{4}\left(\frac{k}{m}\right)^2 \rho T_0 T,$$

$$J_2 = -\frac{3\rho}{2}\left(\frac{kT_0}{m}\right)^2 + \frac{9}{4}\frac{kT_0}{m}P_{ZZ} + \frac{3}{4}\left(\frac{k}{m}\right)^2 \rho T_0 T,$$

$$J_3 = 6\frac{kT_0}{m}\overline{q}_Z + \frac{kT_0}{m}q_Z + 4\rho_0 U\left(\frac{kT_0}{m}\right)^2 - 4\rho U\left(\frac{kT_0}{m}\right)^2,$$

$$J_4 = 6\frac{kT_0}{m}\overline{q}_Z + 3\frac{kT_0}{m}q_Z + 6\rho_0 U\left(\frac{kT_0}{m}\right)^2 - 6\rho U\left(\frac{kT_0}{m}\right)^2.$$

Repeating the procedure from the previous section, we derive the dispersion relation that allows us to plot the corresponding dependences for the improved description numerically and compare it against experiment and other theories [4, 17] – see Figs. 5, 6.

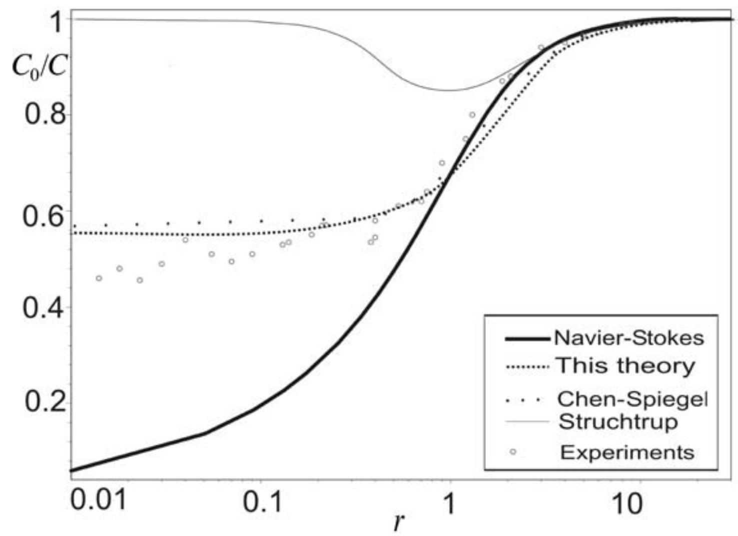

Fig. 5. The inverse non-dimensional phase velocity as a function of the inverse Knudsen number. Comparison of different theories of sound propagation with experiment.

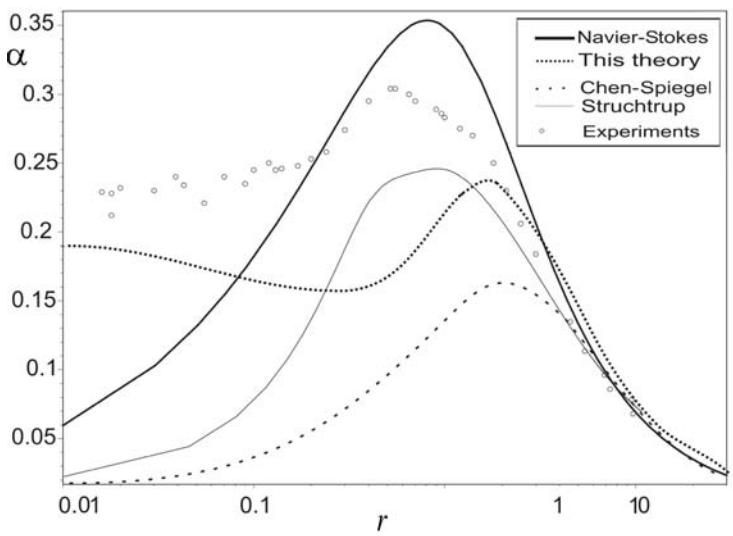

Fig. 6. The attenuation factor of the linear disturbance as a function of the inverse Knudsen number. Comparison of different theories of sound propagation with experiment.

One can also compare our results with those obtained by Alexeev himself for the same problem of the linear one-dimensional sound (Fig. 9 of [19]), which are obtained by momentum expansion. The attenuation factor behaves similarly in Kn∼1 region but our results are obviously better in the deep Kn (collisionless) range.

6. Conclusions

In this paper we have proposed a one-dimensional theory of linear disturbances in a gas, propagating through regions with crucially different Kn numbers. The regime of the propagation dramatically changes from a typically hydrodynamic to the free-molecular one. We have also studied the three-dimensional case [10, 15]. Generally, the theory is based on the Gross–Jackson kinetic equation, the solution of which is built by means of locally equilibrium distribution function with different local parameters for molecules moving "up" and "down". Equations for six moments yield the closed fluid mechanics system. The use of the Gross–Jackson kinetic equation gives us right Prandtl number and right description at small Knudsen numbers. But it does not help much to determine the intermediate values of Kn. In papers [14, 16, 17] it was shown, that the inclusion of the higher eigenfunctions of the collision operator would allow to move into the range of higher Knudsen numbers, but the number of the equations significantly increases. On the other hand, we can increase the order of equations. Hence we apply our method to the generalized Boltzmann equation of Alexeev [4, 19], observing a good progress and we hope, that such "joint" theory development would give a better agreement with the experimental data for attenuation at arbitrary Knudsen numbers in nonlinear acoustics as well.

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