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Modelling and Vibration Control of Planar Systems by the use of Piezoelectric Actuators

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In recent years, a great deal of work has been done on the development of control methodologies for structural and acoustic applications by using piezoelectric actuators. This paper presents the derivation of the models for planar structures with surface-mounted piezoelectric actuators. In the first approach, a parametric system identification procedure is employed to establish a mathematical model of the considered system, on the basis of the data collected from the measurements. The second approach consist in modelling of the fluid-acoustic-structural dynamics in the form of the partial differential equations, derived from physical principles such as forces and moments balance. Assuming axially symmetrical vibrations, on the basis of the models derived, the control algorithm has been developed for the circular plate to which centrally placed PZT actuators were bonded. The results of performed simulations are included and discussed.

Keywords: plates models, vibration control, piezoelectric actuators.

1. Introduction

The main aim of the control systems designed for planar flexible structures is to cancel their vibrations and the related acoustic radiation as much as possible. The most straightforward approach to this problem is Active Vibration Control (AVC) [1, 2]. Various applications such as aircraft, automotive fields and manufacturing industries have proved that AVC system is much superior than the passive vibration control. In recent years, a great deal of work has been done on the development of control methodologies for structural and acoustic applications by using piezoelectric actuators. Actuators are the most significant parts of the controlled structures. Such features as low weight, high efficiency, durability and compact size are important issues. Piezoelectric actuators with proper design satisfy almost all of these requirements, but modelling of these materials is difficult because they are highly integrated with the plate. On the other hand, it is well-known that most control law design methods require an explicit mathematical model of the system to be controlled and the problem of confidence in sufficient fidelity of dynamic models of structures is very important for the system performance. Finding of such a model is an art in itself and often involves several approximations and simplifying assumptions. A suitable model has to create the a proper balance between accuracy and complexity. There are two main approaches to the problem of deriving a model of the considered system:

- *modelling* – based on the physical laws that are supposed to govern the dynamic properties of a given system, a mathematical model can be constructed,
- *identification* – when signals produced by the system in question can be measured and be used to construct a model.

Thus, the derivation of the models for considered here planar structure with the surface-mounted actuators can be also guided in two ways. The first approach consists in modelling the fluid-acoustic-structural dynamics in the form of the partial differential equations derived from physical principles such as the balance of forces and moments [5, 6]. The analysis which will be undertaken here is based upon the idea of finding of the dominating vibrating modes. This mathematical modelling approach has been used for many years, with early references found in [1, 2, 10]. The other way in which a model can be established is commonly referred to as the system identification [9]. In this case, a model of the system can be inferred from a set of data collected during a practical experiment. The techniques described are very flexible and can be used even when complex design problems are treated.

In this paper, a feedback control system formed by a pair of circular piezoactuators (PZT) and four pairs of strain sensors attached to a structure is developed. A clamped circular plate is considered, which is subjected to a periodic excitation. Two methods of obtaining the mathematical model are compared: the first one is the PDE-based equation of motion and the second one is the transfer function obtained by means of the ARX identification method. The obtained models are used to develop the control algorithm for vibration cancellation. The technique used to find the controller parameters is the pole placement. Finally, simulations obtained for the considered plate are presented and discussed.

2. Plate equation of motion

One of the first attempts to use the smart materials technology involved the materials constructed to do the work of electromechanical devices. Since then, many types of sensors and actuators have been developed to measure or excite a system. This section introduces the model of the system studied, derived from the physical principles, which is composed of a clamped plate with a pair of circular piezoactuators (PZT) attached to a structure. The problem of control of plate vibration based on such a model has been examined by several authors. It was also solved by the author of this paper [3–5] for point and distributed actuators and for different boundary conditions of the circular plate located in a finite baffle.

2.1. Distributed control force

The actuators of consideration are piezoelectric elements bonded to the surface of the structure. Let us assume that the structure under study is the vibrating circular plate of radius a and thickness h, having clamped edges. The circular piezo-actuators of radius a_1 and thickness h_p are bonded to both sides of the plate as depicted in Fig. 1a.

a)

b)

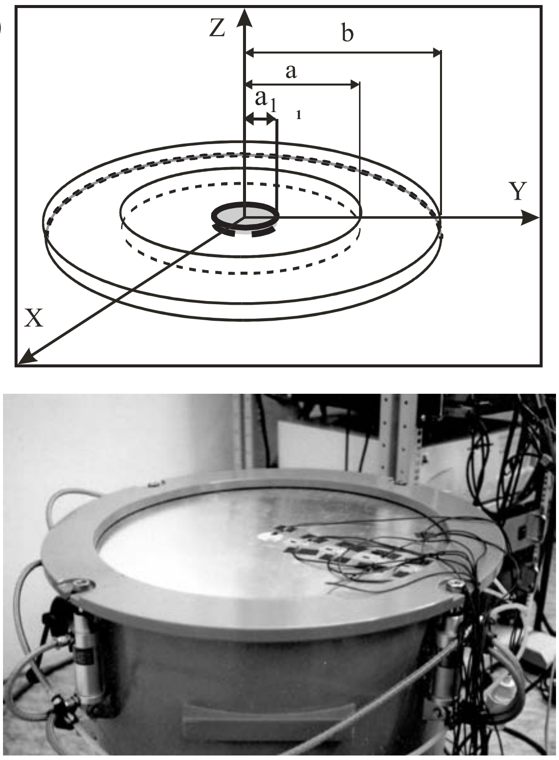


Fig. 1. a) Circular plate with circular piezoelectric ceramic patch of radius a_1, located in a finite baffle of radius b; b) Experimental set-up.

The external moment generated by the piezo-element (PZT) in response to an applied voltage $u(t)$ is given by [4]:

$$M_p = \kappa d_{31} u(t) H(a_1 - r). \tag{1}$$

In the above expression, $H(a_1 - r)$is the Heaviside function and d_{31} is the piezoelectric strain constant. The parameter κ results from integration of the free stresses generated by the piezo-actuator and in the case of having a pair of piezoelectric elements bonded to the plate, it takes the form:

$$\kappa = \frac{E_p(2h + h_p)}{2(1 - \nu_p)}, \tag{2}$$

where ν_p is Poisson's ratio of the piezo-actuator. The equivalent external load can be calculated as follows:

$$f_s(r,\ t) = \frac{\partial^2 M_p}{\partial r^2} + \frac{2}{r}\frac{\partial M_p}{\partial r}, \qquad 0 \le r \le a_1. \tag{3}$$

One can obtain

$$f_s(r,t) = \kappa d_{31} u(t) \left[\delta'(a_1 - r) + \frac{2}{r}\delta(a_1 - r)\right], \tag{4}$$

by denoting the Dirac delta distribution and its derivative by $\delta(.)$ and $\delta'(.)$.

2.2. Plate equation

Assuming axially symmetrical vibrations we can write the governing differential equation of forced motion of the plate as follows [5, 6]:

$$B\nabla^4 w(r,t) + R\frac{\partial}{\partial t}[\nabla^4 w(r,t)] + \gamma\frac{\partial}{\partial t}w(r,t) + \rho h\frac{\partial^2}{\partial t^2}w(r,t) = f_w(r,t) + f_s(r,t) + f_p(r,t), \tag{5}$$

where $B = Eh^3/12(1-\nu^2)$ is the bending stiffness of the plate, E, ν and R are the Young's modulus, Poisson's ratio and the Kelvin–Voigt damping coefficient for the plate, ρ is density of the combined structure, and γ is the viscous air-damping coefficient. The external surface forces can be expressed as follows:

- $f_w(r,t)$ is a surface force, modelling the external excitation, generated by a loudspeaker:

$$f_w(r,t) = F_0 e^{-i\omega t}, \qquad 0 \le r \le a, \tag{6}$$

- $f_s(r,t)$ is the control force due to application of voltage to the PZT elements, described by (4).

The system model is formulated by taking into account the coupling effect between the structure and the acoustic medium, so the third component of the right-hand side of Eq. (5), $f_p(r,t)$, represents the acoustic fluid-load acting on the plate as an additional force.

To approximate the plate dynamics, a Fourier–Bessel expansion of the plate displacement is used to discretize the infinite-dimensional system (Eq. (5)).

We need to formulate a reduced-order model. The plate displacement can be approximated by:

$$w^N(r,t) = \sum_m^N s_m(t) w_m(r), \tag{7}$$

where N is considered to be a finite number, suitably large for the accurate modelling of the system dynamics and $w_m(r)$ are the eigenfunctions. The eigenfunctions satisfy the orthogonality condition and can be normalized as follows:

$$\iint_S w_m^2(r)\,\mathrm{d}S = \pi a^2. \tag{8}$$

Inserting the above expansions into the Eq. (5), multiplying both sides by the orthogonal eigenfunction $w_n(r)$, and integrating over the surface of the structure S, the governing equation of motion can be re-expressed as in the state-space representation [5, 6]:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{V}\mathbf{z}(t), \tag{9}$$

where the dot denotes differentiation with respect to time, $\mathbf{x}$ is the $(n \times 1)$ state vector, $\mathbf{u}$ is the $(m \times 1)$ control vector, and $\mathbf{A}$ is the $(n \times n)$ state matrix, $\mathbf{B}$ is the $(n \times m)$ control input matrix, $\mathbf{V}$ is the $(1 \times n)$ disturbance matrix, described as follows:

$$\begin{gathered}\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -(\mathbf{I}+\mathbf{E})^{-1}\mathbf{\Omega}^2 & -(\mu_2 + \mu_1\mathbf{\Omega}^2)(\mathbf{I}+\mathbf{E})^{-1} \end{bmatrix}, \\ \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ (\mathbf{I}+\mathbf{E})^{-1}\mathbf{K}_s \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} \mathbf{0} \\ (\mathbf{I}+\mathbf{E})^{-1}\mathbf{K}_w \end{bmatrix}.\end{gathered} \tag{10}$$

In the above expression $\mathbf{I}$ denotes the identity matrix, $\mathbf{K}_s$ and $\mathbf{K}_w$ are the coefficient vectors, $\mathbf{E}$ represents a fluid-plate interaction matrix, $\mathbf{\Omega} = \mathbf{diag}[\omega_1, \omega_2, .., \omega_N]$, $\mu_1 = R/B$ and $\mu_2 = \gamma/\rho h$. It is assumed that the response of the considered plate to the applied force distribution is measured by a set of linearly independent point sensors, situated at locations $\mathbf{r}$ on the plate. The output equation is as follows:

$$y(t) = \mathbf{C}\mathbf{x}(t), \tag{11}$$

where

$$C = \begin{bmatrix} w_1(r_1) & \cdots & w_N(r_1) & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ w_1(r_{Nc}) & \cdots & w_N(r_{Nc}) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & w_1(r_1) & \cdots & w_N(r_1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & w_1(r_{Nc}) & \cdots & w_N(r_{Nc}) \end{bmatrix}, \tag{12}$$

N_k and N_v denote the numbers of displacement and velocity sensors respectively, $w_i(r_j)$ is a value of i-th eigenfunction at j-th measurement point.

There is a need to reduce the model order of the system to derive a control strategy. Usually one is concerned with the control strategy which is based on controlling a low number of critical structural modes, where the risk of performance-reducing control spillover resulting from the presence of untargeted modes exists. However, lower orders of system model are usually preferable for reasons of reduced complexity, which lead to simplified implementation. There are some ways in which the reduction of order system could be approached. Instead of a simple truncation of the higher model modes, in this paper the *Balanced Stochastic Truncation* method [11] has been used. This method produces the same state-space realizations, but of a lower order, maintaining the system dynamics properties as far as possible.

Figure 2 presents the Bode diagram for the model of 16-th order and for two models reduced with the use of BST algorithm of 8-th and 4-th orders, in the first measurement point located at $r = 0.18$ [m] circle on the plate. It can be seen that the frequency characteristics of the first and second mode of the considered

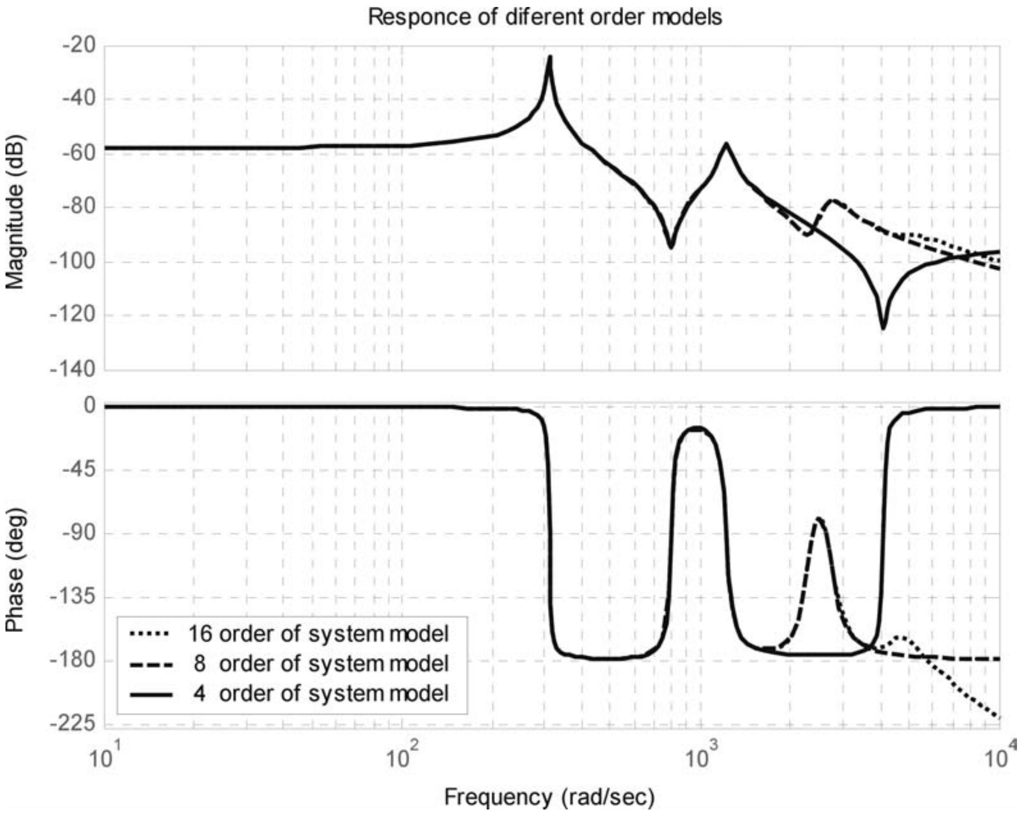


Fig. 2. Bode diagram for the model of the 16-th order and for two models reduced with the use of BST (*Balanced Stochastic Truncation*) algorithm.

plate cover each other and for lower values of frequency, the model order does not influence the system response. However, for the third resonance frequency which is closed to 500 Hz, the differences between curves can be easily noticed and the phase characteristic becomes more complicated. It means that system dynamics can be approximated with enough fidelity, even by means of the 4-th order model if the operating frequency is low (below 500 Hz).

Nowadays, sampled-data control systems dominate the control industry. The frequency-domain bilinear transform plays a role of bridging the gap between the continuous model and their discrete counterparts.

To compare such a model with a discrete system model, constructed with the use of identification technique from a set of data collected during a practical experiment, the popular Tustin transform can be used to convert a continuous system model into the equivalent discrete one. It can be seen in Fig. 3 that this A/D conversion does not generate any significant error up to the frequency of 5000 rad/sec.

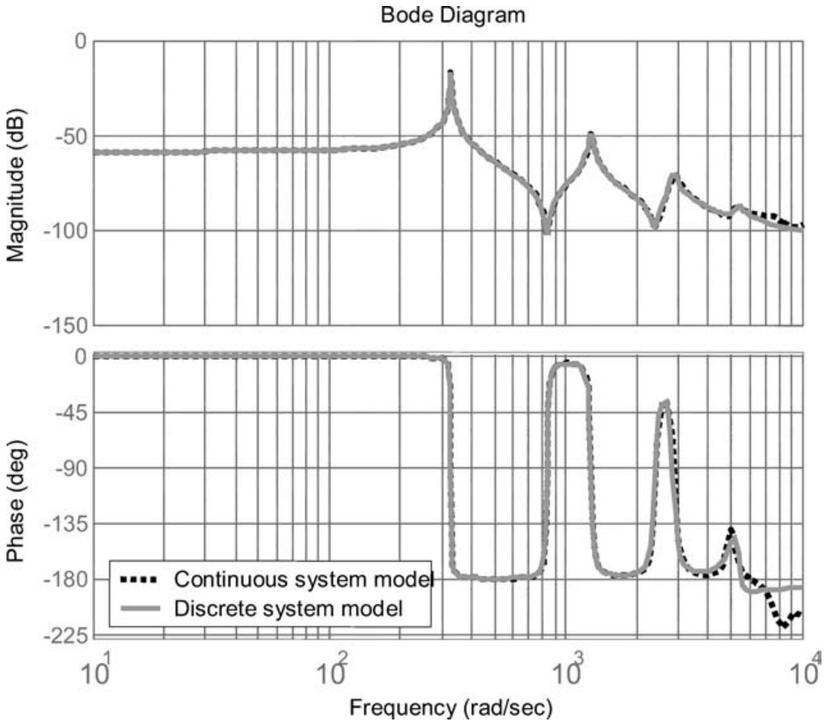


Fig. 3. Bode diagram of the system for discrete and continuous model of 16-th order.

3. System identification

Instead of using analytical methods, a parametric system identification procedure can be employed to establish a mathematical model of the considered system. The relationship between the input sample $u(t)$ and output signal $y(t)$ can be described by a transfer function. The commonly used system identification model which will be considered here is referred to as the Auto-Regressive model with eXogenous (ARX) input parameters [10]. It is the goal of the identification algorithm to identify the numerical values of the model coefficients.

Transfer function of linear discrete system is based on a *difference equation*:

$$\begin{aligned} y(k) + a_1 y(k-1) + ... + a_n y(k-n) &= b_1 u(k-d) \\ &+ ... + b_m u(k-d-m+1) + \eta(k), \end{aligned} \tag{13}$$

where y and u represent the output and input data, d is the number of samples before the input affects the system output and $\eta(k)$ is some disturbance, which appears in the plant output, representing nonmeasurable influences on the process.

After reorganizing Eq. (13) one could get the following form:

$$y(k) = \boldsymbol{\theta}^{\mathrm{T}} \boldsymbol{\varphi}(k) + \eta(k), \qquad \forall k, \tag{14}$$

where

$$\boldsymbol{\varphi}(k) = [-y(k-1), ..., -y(k-n),\ u(k-d), ..., u(k-d-m+1)]^{\mathrm{T}}, \tag{15}$$

$$\boldsymbol{\theta} = [a_1, ..., a_n,\ b_1, ..., b_m]^{\mathrm{T}}. \tag{16}$$

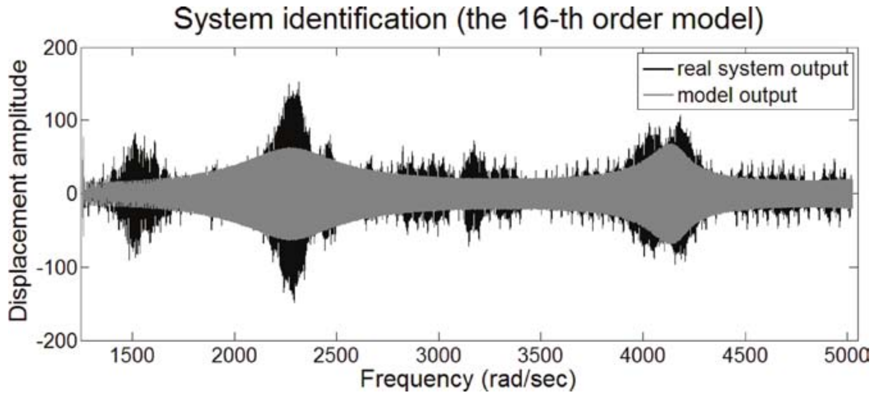
Vector $\boldsymbol{\varphi}$ represents the measured input and output, $\boldsymbol{\theta}$ – the unknown model coefficients, which should be obtained using the chosen identification algorithm. The simplest and most intuitive method for parameter identification is the least squares method (LMS). This classical method provides good performance over a relatively small range of uncertainty and is extensively used for linear control techniques.

In order to establish a mathematical model using this approach, the experimental data from four strain gauges located along the plate radius have been acquired, with sampling time of 0.0001 sec (10 kHz) on the multi-channel system, working under supervision of a real-time operating system (RTAI) [8, 9]. As previously, the choice of the order of the system model plays a crucial role in the performance of the system and requires a careful trade-off between good description of the data and model complexity. Some results of system identification for different orders of ARX models are shown in Fig. 4 and Fig. 5.

Because the PZT actuators applied for the considered plate can excite adequate vibrations for frequency of 200–1600 Hz, the obtained output signals were bonded into this range.

The accuracy of the described identification method depends mostly on choosing the model order: for higher order, higher precision of the model is obtained. On the other hand, if the order of system model increases, the process of designing controller for vibration suppression is more complex.

a)



b)

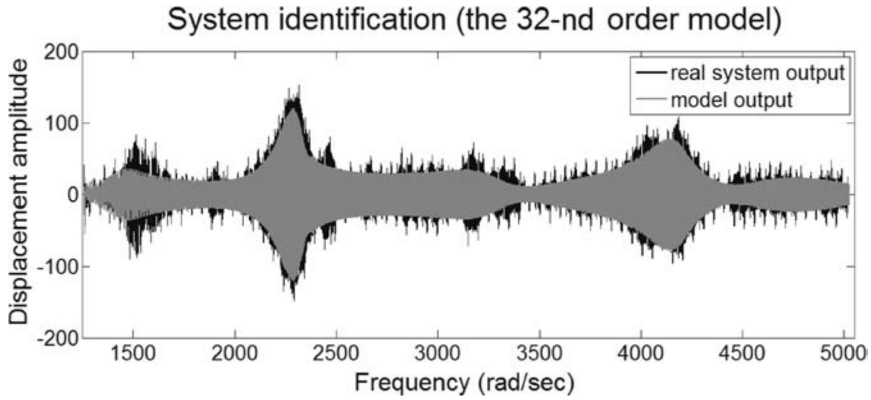


Fig. 4. The experiment data in frequency domain obtained at second measurement point for two different orders of ARX plate model response: a) ARX model of 16-th order; b) ARX model of 32-nd order.

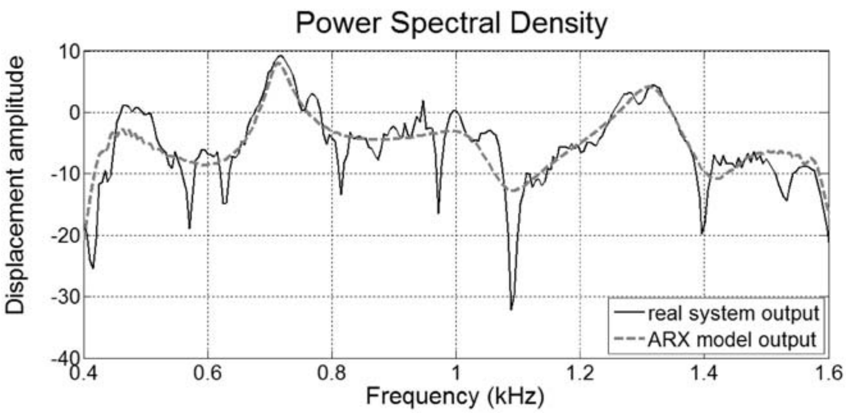


Fig. 5. The power spectral density of the experimental data and for the ARX plate model of 64-th order.

4. Feedback control

Two fundamental, different ways of modeling of thin plate vibrations resulted in the derivation of two models, which can be used in vibration control. There are several control strategies that could be developed on the basis of the obtained models [1, 2]. The aim of the project is to design a control system to modify the response of the plate in some desired fashion. If an active vibration cancellation system is required, its bandwidth should be roughly specified. Usually a few vibration modes of the system carry the most important information about the system behavior. It means that such models, which generally involve too many degrees of freedom to be directly useful for only a few degrees of freedom, are not desirable. Since we are basically interested in damping of the dominating vibrating modes and a low-order controller is generally preferred for physical implementation reasons, we take into account the first four modes only.

For the considered system, one possible approach is to determine a control signal by applying the pole-placement procedure. It enables to design a controller which will place the poles of the system at desired locations in the complex plane (Fig 6). It can be done either numerically by solving the diofantine equation [5, 8],

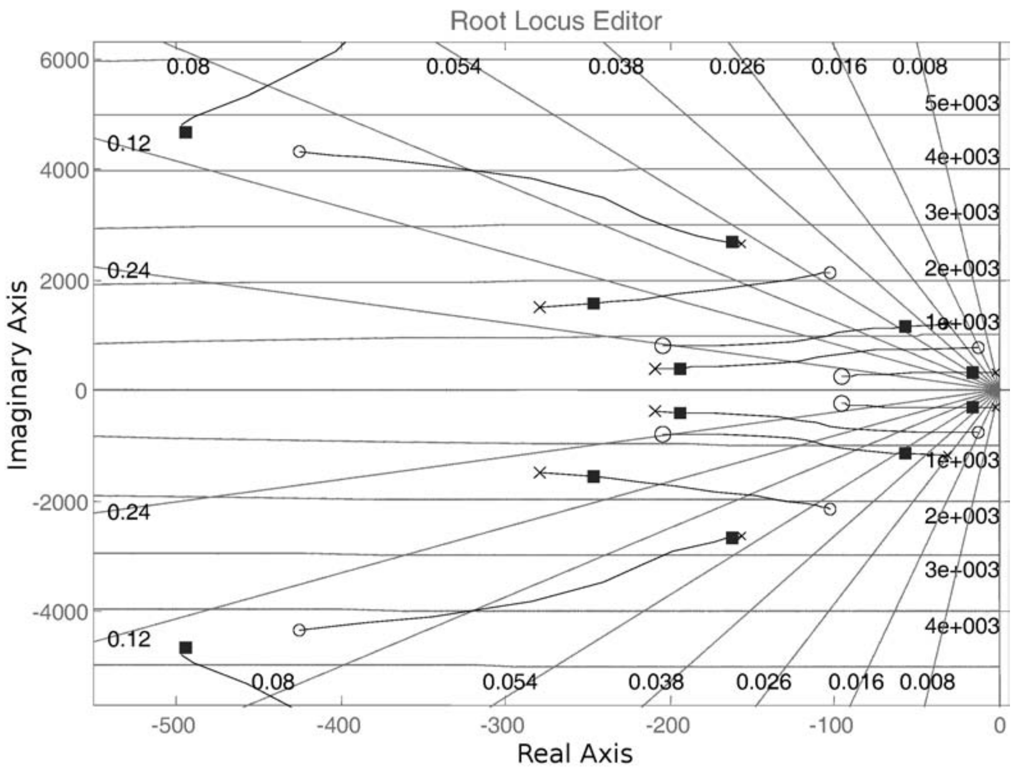


Fig. 6. Poles and zeros of a closed-loop system; ○ – zeros; × – poles.

or graphically by using the MATLAB *Root Locus Tool*. In both approaches, the new location of the close-loop poles of the system in question is obtained by setting the number and values of poles and zeros of the designed controller. Assuming two complex poles and two complex zeros corresponding to the desired controller dynamics (Fig. 6), after a series of attempts we obtained the fourth-order controller with the values given in Table 1.

Table 1. The values of poles and zeros of the designed controller.

Gain	Poles	Zeros
200	-209 ± 392 i	-96 ± 250 i
	-279 ± 1500 i	-205 ± 800 i

On the Bode magnitude plot presented in Fig. 7, we can notice that the chosen location of the poles and zeros of the designed controller, causes a significant reduction in the resonance peaks of the dominating vibrating modes.

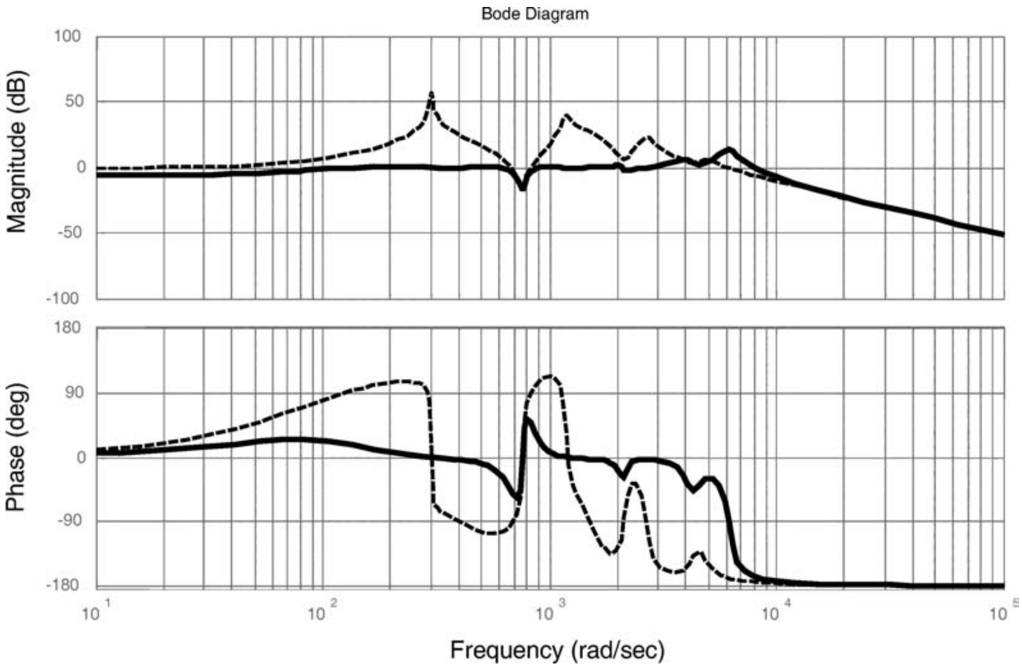


Fig. 7. Compared Bode magnitude plots of the open-loop and closed-loop system - - - open-loop; —— closed-loop.

It can be seen that the uncontrolled plate response vibrates significantly, while the controller causes that plate dominant modes have been reduced very well.

5. Conclusions

One of the main objectives of control-oriented system modeling is to estimate the models that are suitable for controllers design. In the case of thin plates, the AVC (Active Vibration Control) technology will probably face some changes. Traditional equations of motion used in the concepts of vibration suppression are not likely to provide accurate estimates of planar sources in the whole frequency range. Instead, detailed experimental modeling can be used to obtain all relationships between planar structures and the surroundings, in order to achieve significantly better performances.

These two main approaches are studied in this paper. On the basis of the conducted tests it can be stated that the considered models, the theoretical and experimental one, are very similar but each of them has its own advantages and drawbacks. Modern actuators, such as smart materials (PZT), tend to have non-linear and complex characteristics, their own transfer functions with hysteresis, which might influence the object model, and which is not considered in the theoretical model. In case of experimental models, all features of sensors and actuators are taken into account, however the sensibly accurate models are of at least twice higher order than the theoretical ones. The investigations show that a somewhat simplified theoretical model concept can be useful for the active vibration control and it reaches reasonable accuracy. Also, the studies indicate that a well-established experimental model is capable of predicting the behavior of the vibrating plates with superior accuracy but the design of the efficient control is found to be difficult for such high order models.

Designing of the control requires the appropriate and accurate assessment of the situation which involves reaching the compromise between the accuracy and the effective implementation. It seems that any well-established tools might be helpful in predicting the adequate solutions.

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