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Acoustic Streaming Caused by Some Types of Aperiodic Sound. Buildup of Acoustic Streaming

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The analysis of streaming caused by aperiodic sound of different types (switched on at transducer sound or sound determined by initial conditions) is undertaken. The analysis bases on analytical governing equation for streaming Eulerian velocity, which is a result of decomposition of the hydrodynamic equations into acoustic and non-acoustic parts. Its driving force (of acoustic nature) represents a sum of two terms; one is the classic one, which, being averaged over the sound period, coincides with the well-known expression. The second one depends on the periodicity of the sound; at the axis of beam propagation it becomes exactly zero after averaging for the strictly periodic sound but differs from zero for other acoustic waves. Both terms are nonlinear and proportional to the standard attenuation due to shear, bulk viscosities and thermal conductivity. Numerical analysis reveals a qualitative agreement with experimental data. Some theoretical conclusions concerning features of streaming caused by sound determined by initial conditions, are made.

Keywords: instantaneous acoustic streaming, radiation force, non-linear sound propagation,
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1. Introduction

The term "acoustic streaming" refers to a bulk movement arising from the transfer of momentum from an acoustic field to a fluid. The well-understood origins of acoustic streaming are nonlinear losses in momentum of acoustic wave during its propagation in a thermoviscous fluid. They cause solenoidal mean mass flow, which arises exclusively in the multi-dimensional flows.

The traditional method for successive separation of different types of motion consists in linear combination of the continuity and momentum equations after their averaging over the sound period [1, 2]. It does not account for energy balance and, therefore, discards thermal conductivity, though it is well-understood that streaming depends on total attenuation involving heat conduction [3]. The temporal average over the sound period of quantity $\partial\rho/\partial t$ is supposed to be zero, where ρ is total density. In the thermoviscous flows, excess density includes, among the acoustic parts, the slowly decreasing part originated from isobaric heating, so that the averaged value of $\partial\rho/\partial t$ is not longer zero. The important inconsistency of classical treatment is supposing that the fluid is incompressible [1]: sound itself can propagate because of fluid compressibility. We can avoid inconsistencies by means of instantaneous combining of initial equations in the differential form using the properties of acoustic, vortex and entropy motions. That allows to decompose specific dynamics equations in a weakly nonlinear flow.

The present study continues the investigations of acoustic streaming and heating, basing on consistent division of conservative equations into specific parts [4–6]. The procedure starts from the determination of all modes (or possible types of fluid motion) as links of hydrodynamic variables independent of time, and pointing out the ways to separate dynamic equations for every mode by linear combining of initial equations. The correspondence in the leading order of classical acoustic radiation force and that obtained by projecting, is demonstrated in the case of strictly periodic sound in Sec. 2. The role of sound aperiodicity in the generation of streaming is discussed in Sec. 3.

2. Dynamic instantaneous equation of acoustic streaming in the thermoviscous unbounded flow

The continuity, momentum and energy equations for a thermoviscous fluid flow in an unbounded space without external forces read:

$$\begin{gathered}
\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}(\rho\mathbf{v}) = 0, \\
\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\boldsymbol{\nabla})\mathbf{v} = \frac{1}{\rho}\left[-\boldsymbol{\nabla}p + \mu\Delta\mathbf{v} + \left(\mu_B + \frac{\mu}{3}\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\mathbf{v})\right], \\
\frac{\partial e}{\partial t} + (\mathbf{v}\boldsymbol{\nabla})e = \frac{1}{\rho}\Bigg[-p\boldsymbol{\nabla}\mathbf{v} + \chi\Delta T + \mu_B\left(\boldsymbol{\nabla}\mathbf{v}\right)^2 \\
+ \frac{\mu}{2}\sum_{i,k=1,2,3}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\boldsymbol{\nabla}\mathbf{v}\right)^2\Bigg].
\end{gathered} \tag{1}$$

Here, $\mathbf{v}$ denotes the Eulerian velocity of fluid, ρ, p are density and pressure, e, T denote internal energy per unit mass and temperature, μ_B, μ, χ are bulk, shear viscosities and thermal conductivity (all supposed to be constants), x_i, t – spacial coordinates and time. Two thermodynamic functions $e(p,\rho)$, $T(p,\rho)$ should complete the system (1).

Basing on the linearized version of Eqs. (1), the dispersion relations can be obtained for three independent "modes" of small-signal disturbances in an unbounded fluid, called the acoustic (two branches), vortex flow (two branches), and thermal (or entropy) modes. They determine the links of excess pressure, density and three components of velocity specific for every mode. On the other hand, each of the field variables contains contributions from each of the three modes, for example, $\mathbf{v} = \mathbf{v}_{ac} + \mathbf{v}_{\mathrm{ent}} + \mathbf{v}_{\mathrm{vort}}$. The method developed by the authors gives a possibility of consequent decoupling of the initial system (1) into specific dynamic equations for every mode, basing on the specific properties of each mode in weakly nonlinear, thermoviscous and diffracting flow in other words, basing on the links inside modes.

Our limited aim is an equation for acoustic streaming valid within the accuracy up to the second order of the number of small parameters. The first one is acoustic Mach number $M = v_0/c_0$, where v_0 is the magnitude of particle velocity, c_0 is the infinitely-small amplitude sound speed. The next small parameters are dimensionless viscosities and thermal conductivity,

$$\beta = \frac{\mu\omega}{\rho_0 c_0^2},$$

$$\beta_B = \frac{\mu_B\omega}{\rho_0 c_0^2},$$

$$\delta = \frac{\chi\omega}{\rho_0 c_0^2}\left(\frac{1}{C_v} - \frac{1}{C_p}\right),$$

where C_p and C_v denote the specific heats per unit mass at constant pressure and constant volume, respectively, ω is characteristic circular frequency od sound, ρ_0 is static density. The weak diffraction presupposes smallness of $\epsilon = (c_0/R_t\omega)^2$, where R_t is a transversal scale of a flow (in the plane (x,z)), for example, radius of a transducer. The beam geometry is considered: a weakly diffracting sound beam propagates along the axis y.

At last, sound is dominative, so that the ratio of particle velocities correspondent to sound and vortex flow, should remain small. All formulae everywhere below in the text, including links of modes and governing equation, are written in the leading order. Following Lighthill, we choose to treat total attenuation $b = 4\beta/3 + \beta_B + \delta$ and M of comparable smallness, and we shall discard the $O(b^2M)$ and $O(M^3)$ terms in all expansions. The resulting model accounts for the combined effects of nonlinearity, dissipation and diffraction on three-dimensional sound waves and vortex flow.

It is convenient to rearrange the formulae in the dimensionless quantities as follows:

$$p' = \frac{p - p_0}{c_0^2 \cdot \rho_0}, \qquad \rho' = \frac{\rho - \rho_0}{\rho_0}, \qquad \mathbf{v}' = \frac{\mathbf{v}}{c_0},$$
$$x' = \frac{\sqrt{\epsilon}\omega}{c_0} x, \qquad y' = \frac{\omega}{c_0} y, \qquad z' = \frac{\sqrt{\epsilon}\omega}{c_0} z, \tag{2}$$
$$t' = \omega t,$$

where p_0 is the static pressure.

Everywhere below in the text, primes at dimensionless quantities are dropped. The acoustic field is represented in general by two branches, progressive in the positive and negative directions of y, and vortex modes (two branches of motion in perpendicular planes $z = 0$ and $x = 0$). There are five eigenvectors of the linearized system (1), including the entropy, or thermal mode. The acoustic wave progressive in the positive direction of axis y, and a sum of two vortex branches possess links as follows:

$$\boldsymbol{\nabla} \times \mathbf{v}_a = \mathbf{0},$$
$$v_{a,y} = \left(1 - 0.5\epsilon\Delta_\perp \int \mathrm{d}y \int \mathrm{d}y - 0.5b\frac{\partial}{\partial y}\right)\rho_a,$$
$$p_a = \left(1 - \delta\frac{\partial}{\partial y}\left(1 + 0.5\epsilon\Delta_\perp \int \mathrm{d}y \int \mathrm{d}y\right)\right)\rho_a, \tag{3}$$
$$\boldsymbol{\nabla}\mathbf{v}_{\mathrm{vort}} = 0,$$
$$\rho_{\mathrm{vort}} = p_{\mathrm{vort}} = 0,$$

where $\boldsymbol{\nabla} = (\sqrt{\epsilon}\partial/\partial x,\ \partial/\partial y,\ \sqrt{\epsilon}\partial/\partial z)$ is the dimensionless divergency,

$$\epsilon\Delta_\perp = \epsilon\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)$$

is a Laplacian that operates in the plane perpendicular to the axis of beam. In evaluations of modes, the series of square root of Laplacian $\Delta = \partial^2/\partial y^2 + \epsilon\Delta_\perp$ is used: $\sqrt{\Delta} \approx \partial/\partial y + 0.5\epsilon\Delta_\perp \int dy$.

The projection of overall velocity into specific vortex part may be proceeded by acting at it by the following row operator:

$$P_{\mathrm{vort},y} = \left(-\sqrt{\epsilon}\left(1 - \epsilon\Delta_\perp \int \mathrm{d}y \int \mathrm{d}y\right)\frac{\partial}{\partial x}\int \mathrm{d}y,\ \epsilon\Delta_\perp \int \mathrm{d}y \int \mathrm{d}y,\right.$$
$$\left. -\sqrt{\epsilon}\left(1 - \epsilon\Delta_\perp \int \mathrm{d}y \int \mathrm{d}y\right)\frac{\partial}{\partial z}\int \mathrm{d}y\right). \tag{4}$$

It warrants the requirements below in the leading order:

$$P_{\mathrm{vort},y}\mathbf{v}_a = 0, \qquad P_{\mathrm{vort},y}\mathbf{v}_{\mathrm{vort}} = v_{\mathrm{vort},y}. \tag{5}$$

That allows to decompose successfully the vortex longitudinal velocity in the linear part of momentum equation, and to account for acoustic quadratic terms in the role of driving force of streaming by acting of $P_{\mathrm{vort},y}$ at the momentum equation. Collecting the $O(M)$ terms on the left, $O(M^2)$ on the right in the system (1), and acting $P_{\mathrm{vort},y}$ at its both sides, we decouple perturbations in the linear part and yield in nonlinear "force" reflecting modes interaction. If rightwards progressive beam is dominant, only the corresponding nonlinear terms are kept. That coincides with the context of acoustic streaming, where a ratio of magnitudes of vortex and acoustic velocities is expected to be small. Interactions of other types of motion are left out of account (between vortex and entropy motions and so on). This means that conclusions are true over temporal and spacial domains, where acoustic perturbations are dominant comparatively to both other slow modes, solenoidal and entropy. The right-side acoustic "force" is automatically solenoidal: $\boldsymbol{\nabla}\mathbf{F}_a = 0$.

Note, that in any thermoviscous nonlinear flow, acoustic energy loss induces heating, which may input noticeably in the background density and temperature. The correspondent acoustic "source" is proportional to the total attenuation, analogously to the acoustic radiation force of streaming. Heating does not induce bulk movement of a fluid (though there exists secondary weak movement with velocity proportional to the thermal conductivity [5]). This type of slow process is left out of account in the present study.

Acting by $P_{\mathrm{vort},y}$ at the momentum equation results in the dynamic equation for the longitudinal component of vortex flow velocity $v_{\mathrm{vort},y}$:

$$\frac{\partial v_{\mathrm{vort},y}}{\partial t} - \beta\frac{\partial^2 v_{\mathrm{vort},y}}{\partial y^2} + (\mathbf{v}_{\mathrm{vort}}\boldsymbol{\nabla})v_{\mathrm{vort},y} = \frac{\epsilon b}{2}\left(1 - \epsilon\Delta_\perp \int \mathrm{d}y \int \mathrm{d}y\right.$$
$$\left.\int \mathrm{d}y\left(3\frac{\partial}{\partial x}\left(\rho_a\frac{\partial^2\rho_a}{\partial x\partial y}\right) + 3\frac{\partial}{\partial z}\left(\rho_a\frac{\partial^2\rho_a}{\partial z\partial y}\right) - 2\Delta_\perp\int \mathrm{d}y\left(\rho_a\frac{\partial^2\rho_a}{\partial y^2}\right)\right)\right). \tag{6}$$

More details may be found in [6]. The nonlinear term corresponding to the vortex mode itself, $(\mathbf{v}_{\mathrm{vort}}\boldsymbol{\nabla})v_{\mathrm{vort},y}$, though small compared to acoustic ones, is held to remind the hydrodynamic nonlinearity, which is of importance at the latest stages of evolution and restricts the growth of the streaming velocity [7–9]. Nonlinear terms standing by the total attenuation b originate from the series of density $(1+\rho_a)^{-1}$ and thermoviscous links connecting $v_{a,y}$, p_a and ρ_a (3). Links for acoustic mode (3) were used to express all acoustic perturbations in terms of the specific excess density.

An excess acoustic density of the rightwards progressive beam itself satisfies the famous Khokhlov–Zabolotskaya–Kuznetsov equation (the well-known version using the dimensionless retarded time $\tau = t - y$, which is convenient in the boundary regime problems, follows the first one in the brackets; $B/A = (\rho_0/c_0^2)(\partial^2 p/\partial\rho^2)_s$ is the parameter of fluid nonlinearity, evaluated at the unperturbed state):

$$\frac{\partial\rho_a}{\partial t} + \frac{\partial\rho_a}{\partial y} + \frac{\epsilon}{2}\int \Delta_\perp\rho_a\,\mathrm{d}y + (1+B/2A)\rho_a\frac{\partial\rho_a}{\partial y} - \frac{b}{2}\frac{\partial^2\rho_a}{\partial y^2} = 0, \tag{7}$$

$$\left(\frac{\partial}{\partial\tau}\left(\frac{\partial\rho_a}{\partial y} - 0.5(1+B/2A)\frac{\partial\rho_a^2}{\partial\tau} - \frac{b}{2}\frac{\partial^2\rho_a}{\partial\tau^2}\right) - \frac{\epsilon}{2}\Delta_\perp\rho_a = 0\right)$$

which may be derived consistently on the basis of projecting [4]. Let us consider the axial symmetry relatively to the axis y of beam propagation: $\rho_a(x,y,z) = \rho_a(r = \sqrt{x^2+z^2}, y)$, $\Delta_\perp = 1/r\partial/\partial r + \partial^2/\partial r^2$. The Laplacian $\Delta_\perp$ acting at acoustic excess density may be replaced by the following operators:

$$\begin{aligned}\epsilon\Delta_\perp &= \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} + O(M,b),\\ \epsilon\Delta_\perp &= -2\frac{\partial^2}{\partial t\partial y} - 2\frac{\partial^2}{\partial y^2} + O(M,b).\end{aligned} \tag{8}$$

In the leading order, these operators apply not only to the rightward acoustic values V, but also to a product VW, if W also satisfies the wave equation for the rightward progressive sound (8). By consequent replacing of operators in Eq. (6), it is easy to rearrange it into the following equation:

$$\begin{gathered}\frac{\partial v_{\mathrm{vort},y}}{\partial t} - \beta\frac{\partial^2 v_{\mathrm{vort},y}}{\partial y^2} + (\mathbf{v}_{\mathrm{vort}}\boldsymbol{\nabla})v_{\mathrm{vort},y} = F_y = F_{y,\mathrm{class}} + F_{\mathrm{add}},\\ F_{y,\mathrm{class}} = -b\rho_a\frac{\partial^2\rho_a}{\partial y^2},\\ F_{y,\mathrm{add}} = \frac{b}{2}\left(2 - \frac{\partial^2}{\partial t^2}\int \mathrm{d}y\int \mathrm{d}y\right)\\ \left(-\frac{3}{2}\epsilon\left(\frac{\partial\rho_a}{\partial r}\right)^2 + \frac{3}{2}\frac{\partial^2}{\partial t^2}\rho_a^2 + \frac{\partial}{\partial t}\int \mathrm{d}y\left(3\left(\frac{\partial\rho_a}{\partial y}\right)^2 - 2\frac{\partial}{\partial t}\int \mathrm{d}y\rho_a\frac{\partial^2\rho_a}{\partial y^2}\right)\right).\end{gathered} \tag{9}$$

In many applications, it is convenient to use acoustic pressure instead of excess density. In the dimensionless quantities, they are equal in the leading order in accordance to links (3), so that ρ_a may be replaced by p_a in Eqs. (7), (9). In its form (9), the governing equation exhibits that absorption, nonlinearity and divergence are the origins of streaming. It is useful to establish the equivalence of the acoustic force from the right-hand side of Eq. (9) and the well-known one for periodic acoustic wave coming from [1]. For the strictly periodic sound, averaged acoustic radiation force may be rewritten in the form as follows (overbaring denotes temporal average over sound period, 2π in dimensionless variables, $\overline{F_y} = \frac{1}{2\pi}\int\limits_t^{t+2\pi} F_y \, \mathrm{d}t$):

$$\overline{F_{y,\text{periodic}}} = \overline{F_{y,\text{class}}} + \overline{F_{y,\text{add}}} = b\left(-\overline{\rho_a \frac{\partial^2 \rho_a}{\partial y^2}} - \frac{3\epsilon}{2}\overline{\left(\frac{\partial \rho_a}{\partial r}\right)^2}\right). \tag{10}$$

Formula (10) differs from the classic result $\overline{F_{y,\text{class}}}$ by the last term in brackets, which is less or equal to zero for any dependence ρ_a on r but exactly equals to zero at the axis of beam propagation $r = 0$: $\frac{\partial \rho_a}{\partial r} = 0$. So that, at the axis of beam, classic and instantaneous radiation forces coincide for the periodic sound. Outside the beam, the instantaneous formula gives somewhat smaller quantities. That confirms some experimental data [10].

3. Examples of radiation force and induced by it streaming

3.1. Buildup of acoustic streaming

In spite of a number of simplifying conditions during derivation of Eq. (9), the complexity of mutual solution of Eqs. (7), (9) is obvious. A hard lock to pick is nonlinearity in the both equations. There are no general analytical methods to solve the KZK equation (7); moreover, there is still absent the general analytical method to solve the KZ equation (non-viscous limit of KZK), except the one considering the periodic Gaussian beams in the paraxial area [11].

The limited aim of the present study is to give simple illustrations of the applicability of the dynamic Eq. (9) in the flows differing from the strictly periodic ones. The exactly periodic ones are covered by the well-known formula for radiation force (Eq. (10)). The first question is what to insert in the role of acoustic excess density in the right-hand side of (9). Strictly speaking, it should be a solution of the KZK equation (7). The difficulty of considering non-periodic sound is also in solution of the KZK equation under the corresponding boundary regime at transducer which is an aperiodic function.

Let us suppose, that the plane wave propagating in the positive direction of axis y, may be taken in the role of acoustic source. Hence, the effect of nonlinearity is considered only by the quadratic form of the acoustic radiation force, and the effect of diffraction and attenuation are considered by two corresponding multipliers ϵ and b standing by it at the right-hand side of Eq. (9). That is supposed to be valid at least at the distances not very far from the transducer, but not very close where acoustic field is rapidly oscillating. The choice of possibly simple sound in the role of an origin of streaming is very important in view of complexity of the formula for acoustic force, involving integration with respect to y four times.

The second question is account for hydrodynamic nonlinearity of the convective term in the right-hand side of Eq. (9). There is a number of theoretical and well-agreed with them experimental investigations underlying the importance of taking into account of hydrodynamic nonlinearity [7–9, 12]. It makes the streaming velocity not to grow infinitely. At the early stages, linear theory and experiment dealing with the periodic sound, agree well [12]. The effect of shear viscosity on the streaming velocity grows with distance from a transducer. So that, at the beginning of evolution, the radiation force is simply a partial derivative of the streaming velocity with respect to time:

$$\frac{\partial v_{\mathrm{vort},y}}{\partial t} = F_y. \tag{11}$$

There is unfortunately poor literature concerning the investigations of streaming establishment. We will refer to the papers [12, 13], where some experimental data of time history of the axial streaming in water are given when ultrasound is instantaneously or gradually switched on. A set of measurements was undertaken using a laser Doppler velocimeter. The transducer transmitting 5.05 MHz is 10 mm in diameter.

3.1.1. Instantaneous switching of ultrasound

Some experiments of [13] reveal the history of streaming induced by instantaneous switching of ultrasound. In the majority of experiments in water, the frequency is approximately $f = 5$ MHz, that corresponds to the sound period of $T = 2 \cdot 10^{-7}$ sec. For characteristic times of sound switching on of order 10^{-6} sec or less, an acoustic pressure or its excess density (the dimensionless quantities are equal in the leading order) may be approximated by slowly varying envelope multiplied by periodic function such like $\sin(t - y)$ at the axis of beam propagation $r = 0$:

$$p_a(t, y, r = 0) = P_0(1 - \exp(-n\tau)) \sin(\tau), \qquad \tau > 0, \tag{12}$$

where $\tau = t - y$ is the dimensionless retarded time, n is responsible for the rate of sound increasing, P_0 denotes amplitude. The parameter n may be chosen arbitrarily, it determines the characteristic time of amplitude increase $T_0 = \lambda/(nc)$, where λ denotes the characteristic wavelength. Two exemplary initial waveforms are plotted in Fig. 1 for two values of n: $n = 0.05$ and $n = 0.001$, corresponding to the characteristic times of acoustic amplitude growth $T_0 = 4 \cdot 10^{-6}$ sec, $T_0 = 2 \cdot 10^{-4}$ sec, respectively. For the simple waveform (12), the parts of longitudinal radiation force at the axis of beam propagation may be immediately calculated by means of *Mathematica*. The limits of integration for the second part are $[0, y]$.

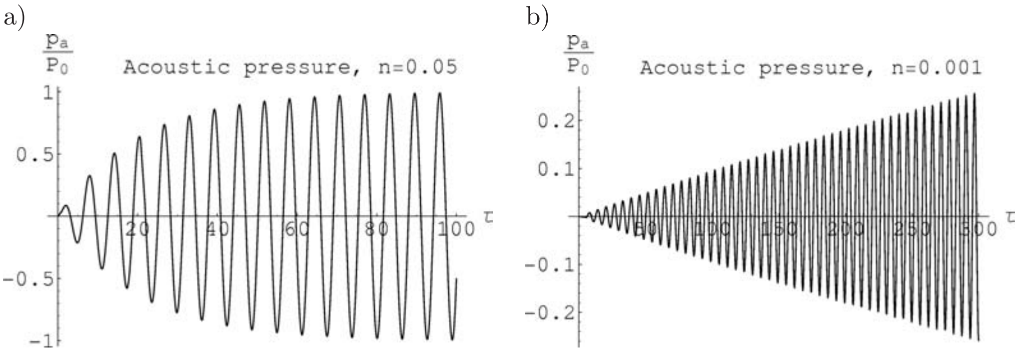

Fig. 1. The normalized acoustic pressures p_a/P_0 as functions of retarded time $\tau = t - y$ for different n: 0.05 and 0.001.

Figures 2, 3 show the parts of acoustic radiation pressure and relative streaming for this kind of acoustic source. $\overline{F_{y,\text{class}}}$ and the corresponding part of stream-

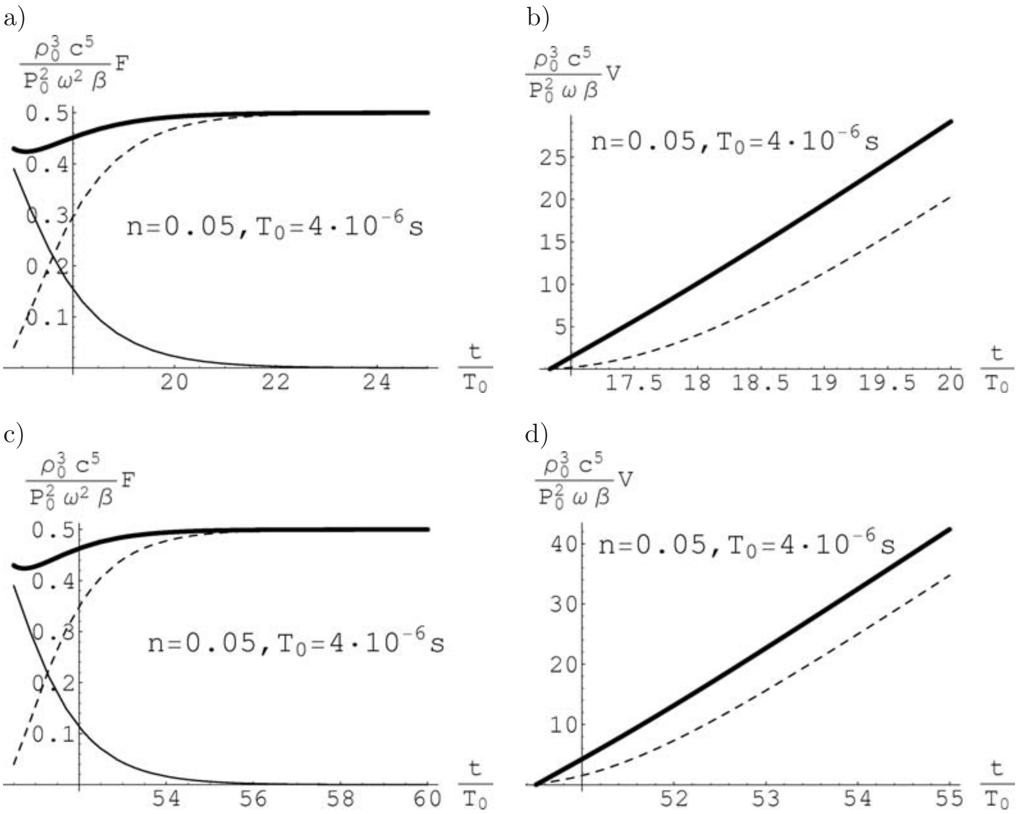

Fig. 2. The dimensionless radiation force and velocity of streaming for $n = 0.05$ and $Y_0 = 0.1$ m (2a, 2b) or $Y_0 = 0.3$ m (2c, 2d). $\overline{F_{y,\text{class}}}$, $\overline{V_{y,\text{class}}}$ are plotted by the dotted line, and $\overline{F_{y,\text{add}}}$, $\overline{V_{y,\text{add}}}$ are plotted by the solid thin line, and overall force or summary velocity of streaming are plotted by the bold line.

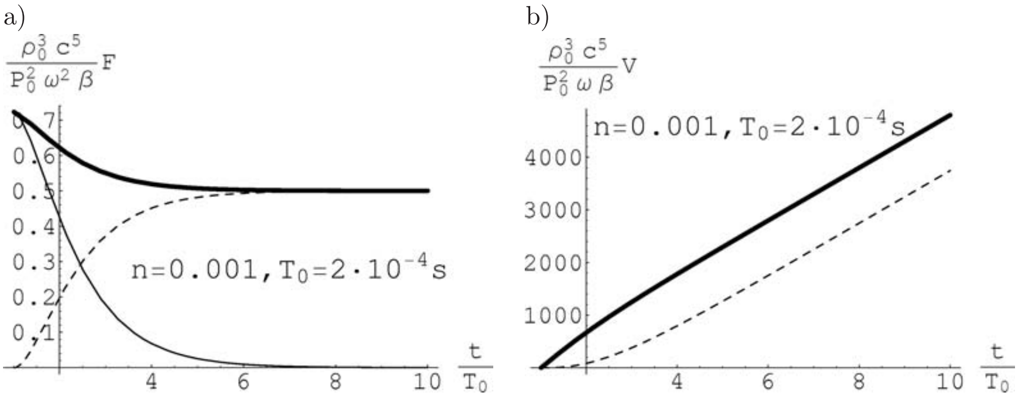

Fig. 3. The dimensionless radiation force and velocity of streaming for $n=0.001$ and $Y_0=0.3$ m.

ing velocity are plotted by the dotted line, and $\overline{F_{y,\mathrm{add}}}$ with its part of velocity are plotted by the solid thin line, and overall force or summary velocity of streaming are plotted by the bold line. All quantities are averaged over the interval $[t, t+2\pi]$. Both figures show the dimensionless quantities (F, t and V denote dimensional values).

The velocity of streaming is calculated simply by integration of radiation force over the time interval $[y = Y_0/\lambda, t]$. The rate of the streaming velocity increases with the increase of distance from the transducer Y_0. That is in agreement with Fig. 3 from [13]. Experimental data reveal that the second partial derivative $\partial V^2/\partial t^2$ at any distance is not positive. Simple evaluations for the part of streaming velocity corresponding to the classical force give:

$$\frac{\partial^2 \overline{V_{\mathrm{class}}}}{\partial t^2} \approx \frac{\partial \overline{F_{y,\mathrm{class}}}}{\partial t} \approx n \cdot \exp(-n\tau)(1 - \exp(-n\tau)) > 0.$$

The account for the deviation from periodicity by the second part of radiation force, $\overline{F_{y,\mathrm{add}}}$, provides the proper sign. For calculations, typical data for water are taken: $c = 1491$ m/s, $b = 50 \cdot 10^{-15} cf/\pi$.

3.1.2. Gradually switched ultrasound

In some series of measurements described in [13], the voltage applied at the transducer grows linearly with different rates. Since the applied voltage is proportional to the acoustic intensity, the acoustic pressure may be assumed as

$$p_a(t, y, r = 0) = P_0 \sqrt{k(t-y)} \sin(t-y), \tag{13}$$

where $\sqrt{k(t-y)}$ is a slowly varying function with a dimensionless rate of growth k. Integrals in (9) may be approximately evaluated. At the axis of beam $r = 0$, one obtains:

$$\overline{\frac{\partial}{\partial t}\int\limits_0^y p_a \frac{\partial^2 p_a}{\partial y^2}\,\mathrm{d}y} = \frac{k(t-y)}{2\pi}\int\limits_0^y \sin(t-y)\frac{\partial^2 \sin(t-y)}{\partial y^2}\,\mathrm{d}y\Big|_t^{t+2\pi} + O(k), \tag{14}$$

and so on. Finally, the parts of radiation force averaged over interval $[t, t+2\pi]$ are:

$$\overline{F_{y,\mathrm{class}}} = 0.5bP_0^2 k(t-y), \qquad \overline{F_{y,\mathrm{add}}} = 1.5P_0^2 ky. \tag{15}$$

At the early stages of streaming, the velocity is a simple integral of radiation force, in accordance to (11):

$$\frac{\partial v_{\mathrm{vort},y}}{\partial t} = \overline{F_y} = 0.5bP_0^2 k(t-y) + 1.5bP_0^2 ky, \tag{16}$$

$$v_{\mathrm{vort},y} = v_{\mathrm{vort},y,\mathrm{class}} + v_{\mathrm{vort},y,\mathrm{add}} = \int\limits_y^t \left(\overline{F_{y,\mathrm{class}}} + \overline{F_{y,\mathrm{add}}}\right)\mathrm{d}t,$$

$$v_{\mathrm{vort},y}(\tau = t-y, y) = 0.25bkP_0^2\tau(\tau + 5y).$$

Hence, the rate of streaming increase depends not only on $\tau = t - y$, but also on y: larger y provides larger rate of streaming increase. This conclusion agrees with experimental data of [13] (Fig. 3 of it) as well as with data on streaming caused by planar ultrasound [12] represented by Fig. 5 of this paper, though the detail comparison is rather impossible in frames of very simple illustrative consideration and lack of details of the sound used in experiments of [12].

3.2. Sound determined by initial conditions

Meaning more complex than plane wave type of sound, it is reasonable to account first of all for diffraction in the acoustic pressure participating in forming of the radiation force. The attenuation of water is small in comparison to gases. As the first approach, the coefficient standing by the force accounts for viscous effects. The radiation force contains only nonlinear acoustic terms and considers nonlinearity, both in its form and the nonlinear distortions of sound itself. In spite of complexity of the problem we will not account for the nonlinear distortions of sound, but only of its diffraction. The exact solution of the linear parabolic equation

$$\frac{\partial p_a}{\partial t} + \frac{\partial p_a}{\partial y} + \frac{\epsilon}{2}\int \Delta_\perp p_a\,\mathrm{d}y = 0 \tag{17}$$

for the beams with initial condition ($t = 0$):

$$p_a(y, r, t=0) = F(y)\exp(-r^2) \tag{18}$$

is the following:

$$p_a(y, r, t) = \int\limits_{-\infty}^{\infty} \frac{A_\kappa e^{i\kappa(y-t)}}{1 + 2i\epsilon t/\kappa}\exp\left(-\frac{r^2}{1 + 2i\epsilon t/\kappa}\right)\mathrm{d}\kappa, \tag{19}$$

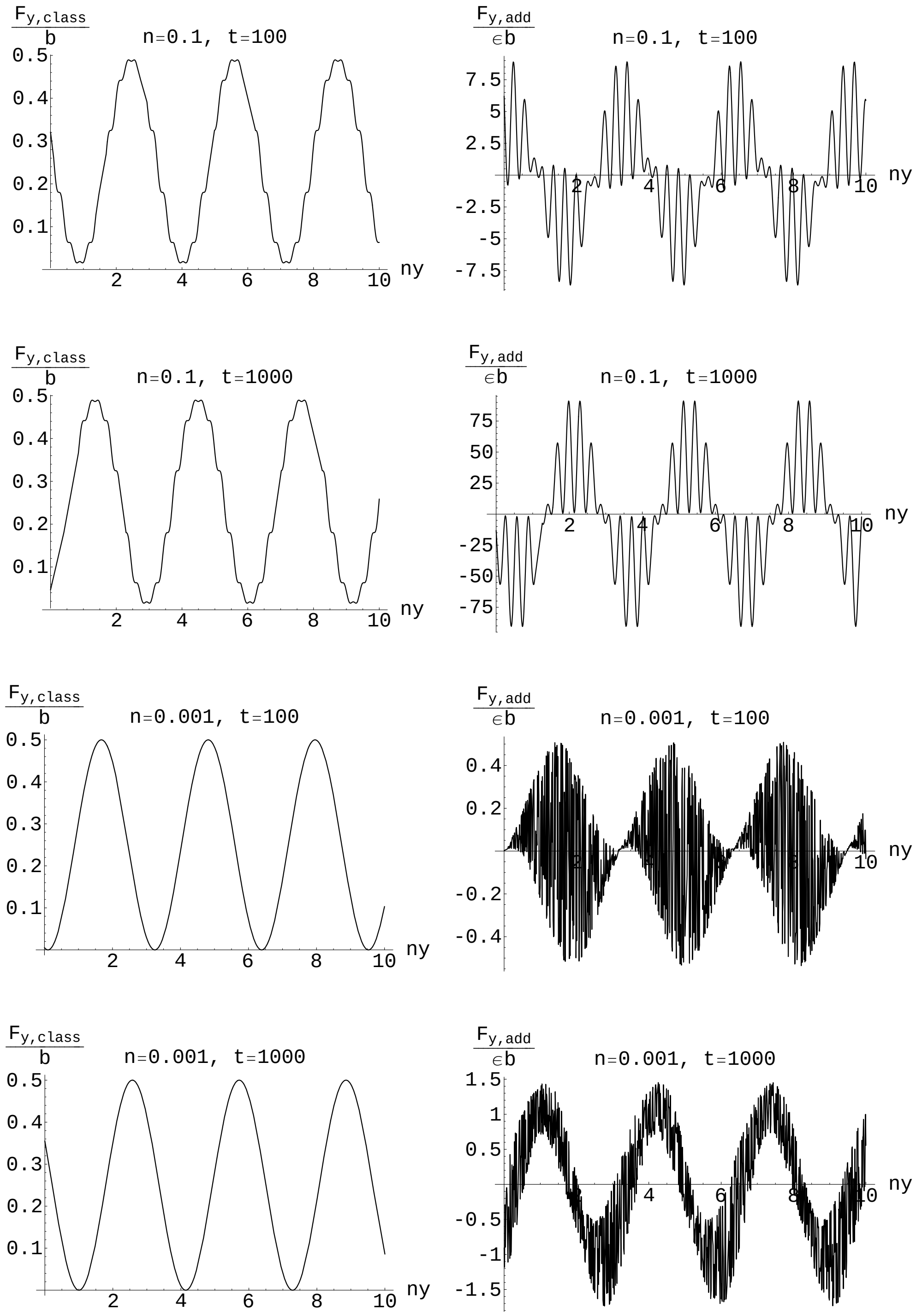

Fig. 4. Dimensionless radiation force and velocity of streaming for $n = 0.001$ and $Y_0 = 0.3$ m.

where $A_\kappa = \dfrac{1}{2\pi}\int\limits_{\infty}^{\infty} F(y)e^{-i\kappa y}\,\mathrm{d}y$. The waveforms for initially periodic in space waveforms are not periodic in time at $t > 0$ due to diffraction. The series of acoustic pressure in ϵ gives a strictly periodic part of order ϵ^0 and the aperiodic one of order ϵ^1:

$$p_a(y,r,t) = p_{per} + \epsilon p_{aper}. \tag{20}$$

Let us consider the radiation force at the axis of beam symmetry. The additional part in the leading order equals:

$$F_{y,\mathrm{add}} = \frac{b}{2}\left(2 - \frac{\partial^2}{\partial t^2}\int \mathrm{d}y \int \mathrm{d}y\right)\left(3\frac{p_{\mathrm{per}}p_{\mathrm{aper}}}{\partial t^2} + \frac{\partial}{\partial t}\int \mathrm{d}y\right.$$
$$\left.\left(6\frac{\partial p_{\mathrm{per}}}{\partial y}\frac{\partial p_{\mathrm{aper}}}{\partial y} - 2\frac{\partial}{\partial t}\int \mathrm{d}y\left(p_{\mathrm{per}}\frac{\partial^2 p_{\mathrm{aper}}}{\partial y^2} + p_{\mathrm{aper}}\frac{\partial^2 p_{\mathrm{per}}}{\partial y}\right)\right)\right). \tag{21}$$

The exemplary form at the axis of beam $F(y) = P_0\sin(y)$:

$$p_{\mathrm{per}} = P_0\sin(t-y), \qquad p_{\mathrm{aper}} = 2P_0 t\cos(t-y) \tag{22}$$

results in the following part averaged over the sound period:

$$\overline{F_{y,\mathrm{add}}} = P_0^2\cos(2(t-y)).$$

The parts of waveform with the frequency of beating n are following:

$$\begin{aligned} p_{\mathrm{per}} &= P_0\sin(t-y)\sin(n(t-y)),\\ p_{\mathrm{aper}} &= -\frac{P_0 t}{n^2-1}\left((n+1)\sin((n-1)(t-y)) - (n-1)\sin((n+1)(t-y))\right). \end{aligned} \tag{23}$$

They correspond to the spatially periodic radiation force. The plots below show the averaged over the sound period, both parts of longitudinal radiation force at the axis of beam. The deviation from the classical formula increases with time. Smaller n guarantees also larger deviation.

The decoupling of the acoustic pressure into two parts according to the expansions into series of ϵ, allows to evaluate input of aperiodic part but essentially restricts the time interval over which the expansion is valid: $t \le 1/\epsilon$. In dimensional quantities, it results in the upper boundary of about $2\cdot 10^{-4}$ sec for typical data of experiments of water, but at any longitudinal coordinate y. The value of ϵ in these conditions is about $9\cdot 10^{-4}$.

4. Conclusions

In the last decades, attention to the aperiodic sound and the phenomena caused by it grows. Some experiments appeared (including medical and technical applications) dealing with aperiodic sources: series of pulses or modulated by slow function sound [12–14]. In spite of growing interest in acoustic streaming, only a few papers discussed the establishment process both experimentally and theoretically. The possibilities of analytical methods in the study of such multidimensional dependences are superior to that of experimental as well as numerical investigations. Analytic approach provides usually more flexibility, is less time-consuming, and unlike other methods, is not constrained by fixed and limited set of values or various parameters. The mathematical difficulties in studies of nonlinear multidimensional fluid dynamics are well-known.

The analytical method proposed by the authors allows to consider dynamics of streaming caused by loss in momentum of different types of aperiodic sound. The investigation of the present paper concerns the gradually or instantaneously switched sound and the sound determined by initial conditions. The longitudinal radiation force at the axis of beam and corresponding streaming velocity are considered for simplicity, though the dynamic Eq. (9) describes dynamics in the paraxial area as well. The rough illustrations discovering the role of aperiodicity in this study exploit the linear plane sound and the solution of linear wave parabolic equation for a sound beam, though effects of nonlinearity, diffraction and attenuation should be necessarily considered while deriving the radiation force and corresponding governing equation for acoustic streaming velocity. The conclusion is that the instantaneous (after averaging over the sound period) formula of the radiation force differs from the classical one by a term, which consists of a negative value proportional to the divergence ϵ (it equals zero at the axis of beam), and the other term being exactly zero after averaging over the sound period for periodic sound and different from zero for any other waveforms. In this last part, the slightly different from periodic sound is hardly expected to produce a noticeable difference compared to the classic formula. The difference increases with deviation of sound from periodic. The numerical analysis reveals a qualitative agreement with the experimental data concerning buildup of acoustic streaming.

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