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THE ACOUSTIC POWER OF A CIRCULAR PLATE EXCITED BY NON-UNIFORM SURFACE PRESSURE

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This work is focused on the analysis of the influence of a non-uniformly distributed surface excitation on sound radiation by a clamped circular plate embedded in a flat rigid baffle. It is assumed that the vibrations and radiations are axisymmetric and time-harmonic, moreover, the losses of energy into the plate's material are neglected. The influence of the surrounding medium on the vibration of the plate is not taken into account. The distribution of the exciting pressure is assumed in the form of a function whose shape can be modeled by fitting suitable values of parameters. The approximation of the distribution of the forcing pressure by the use of an analytical function makes it possible to consider practical cases. Integral formulae for describimg the active power have been obtained. The solution of the equation of motion is achieved by employing an expansion into series of eigenfunctions. An analysis of the influence of the parameters characterizing the surface excitation on the radiation power has also been presented. The low and high frequency elementary form asymptotics can be obtained by using the reached herein formulae and the known approximated expressions.

Key words: acoustic radiation power, circular plate vibration, external non-uniform excitation.

Notations

a	radius of the plate,
B	bending stiffness of the plate,
c	propagation velocity of the wave in a fluid medium,
$F(r,t)$	function describing the distribution of the exciting external pressure,
h	thickness of the plate,
I_n	n-th order modified Bessel function,
J_n	n-th order Bessel function,
k	structural wavenumber,
k_0	wave number,
$p(\mathbf{r})$	acoustic pressure,
P	total sound power,
P_a	active power,

r	radial variable of a point on the plate surface in polar coordinates,
S	area of the plate S_n – n-th order Struve function,
t	time,
$v(\mathbf{r})$	vibration velocity of the plate's points,
E	Young's modulus,
$\eta(r,t)$	function describing the transverse displacements of the plate's points,
ν	Poisson's ratio,
ρ	density of the plate's material,
ρ_0	rest density of the fluid medium,
ω	frequency.

1. Introduction

The vibrating surfaces being constructional elements are likely to produce some harmful noise. The plates or the membranes excited to vibration by an external pressure produce sound waves which may be considered as an undesirable effect. The sound radiation by the plates is essentially dependent on the excitation and boundary configurations. In order to reduce the harmful noise, it is necessary to know the dependence of the magnitudes discribing the radiation on the parameters determining the external excitation, as well as the boundary configurations.

In paper [1], the problem of sound radiation by a circular plate excited by surface pressure distributed non-uniformly over the surface was signalled. W. RDZANEK, using the Kirchhoff–Love linear model of a plate and the Cauchy's theorem, obtained an asymptotic expression for the sound power of the circular plate excited to vibration by means of external pressure, uniformly distributed on the circle's surface [2]. Making use of the Cauchy's theorem, the mutual impedance of two circular plates of high frequencies was obtained by P. WITKOWSKI [3].

However, an analysis of the sound power radiated by a circular plate excited to vibration by a non-uniformly distributed external surface pressure has not yet been presented in the literature.

The main aim of this study is to analyse the influence of the shape of the external excitation on the sound power. The distribution of the excited pressure, known from experiment, can be approximated by means of an analytical function. The magnitudes describing the sound radiation obtained in this paper can be used in practical cases.

2. Assumptions of the analysis

A circular plate whose radius equals a is embedded in a flat, rigid baffle and is surrounded by a gaseous lossless medium. There are no energy losses in the plate's material.

Assuming that the density of the medium is small in comparison with that of the plate's material, the influence of the medium on the plate's radiation can be neglected. Moreover, it is assumed that the vibrations are axisymmetric and time-harmonic. The transverse displacement of the plate's points in accordance with the above presented assumptions can be formulated as $\eta(r,t) = \eta(r)e^{-i\omega t}$, where ω and $\eta(r)$ are the frequency, and the function describing the dependence the amplitude of a displacement on the radial variable, respectively.

The considerations are based on the Kirchhoff–Love theory of a perfectly elastic plate what implies that the transverse displacement and thickness of the plate are very small in comparison with its radius. The plate is forced to vibrations by means of a pressure distributed non-uniformly over the whole surface. This pressure is time – harmonic. The external excitation is axisymmetric and can be presented in the following mathematical form: $F(r,t) = F(r)e^{-i\omega t}$. Here $F(r)$ denotes the distribution of the pressure amplitude on the plate's surface.

The equation of motion in the case of the source under consideration can be presented in the following form

$$\left(k^{-4}\nabla^4 - 1\right)\eta(r) = \frac{F(r)}{\rho h\omega^2}, \tag{1}$$

where ρ and h are the density of the plate's material and its thickness, respectively. In Eq. (1), $k^4 = \omega^2\rho h/B$ is the structural wavenumber, $B = Eh^3/\left[12(1-\nu^2)\right]$ denotes the plate's bending stiffness, ν is the Poisson's ratio, E is the Young's modulus [4].

A solution of the equation of motion can be reached by the expansion into a series of eigenfunctions which leads to the form of an infinite sum

$$\eta(r) = \sum_{n=1}^{\infty} c_n\eta_n(r), \tag{2}$$

where

$$\eta_n(r) = J_0(k_n r) + \frac{J_1(\gamma_n)}{I_1(\gamma_n)} I_0(k_n r), \tag{3}$$

J_n is the Bessel's function of n-th order and I_n is the Bessel's modified function of n-th order and the particular values of γ_n are solutions of the following equation $J_0(\gamma_n)I_1(\gamma_n)+J_1(\gamma_n)I_0(\gamma_n) = 0$. Within the limits $r \in (0,a)$. The functions $\eta_n(r)r \in (0,a)$ are orthogonal and satisfy the following differential homogeneous equation $\left(k_n^{-4}\nabla^4 - 1\right)\eta_n(r) = 0$, where $k_n = \gamma_n/a$ [1, 7, 8]. Moreover, they satisfy the boundary conditions associated with the clamped edge.

Now the constants in the formula (2) can be expressed in the following integral form

$$c_n = \frac{\gamma^4}{(\gamma_n^4 - \gamma^4)J_0^2(\gamma_n)} \frac{1}{a^2\rho h\omega^2} \int_0^a F(r)\eta_n(r)r\,\mathrm{d}r, \tag{4}$$

where $\gamma = ka$.

The function $F(r)$ is assumed in the form of a polynomial whose powers are even numbers to make possible an exact computation of the integrals in formula (4). Moreover, it is assumed that: $F(0) = F_0$, $F(a) = F_1$, where F_0, F_1 denote the values of the pressure amplitude in the plate's center and at its edge, respectively.

The function for the description of the distribution of the exciting pressure can be written finally in the following form

$$F(r) = F_0 \left(1 + (p - s - 1) \left(\frac{r}{a}\right)^2 + s \left(\frac{r}{a}\right)^4 \right), \tag{5}$$

where $p = F_1/F_0$ and the parameter s in Eq. (5) is introduced to modify the shape of the surface excitation.

There are a few shapes of excitation for different values of p and s. They are presented in Fig. 1.

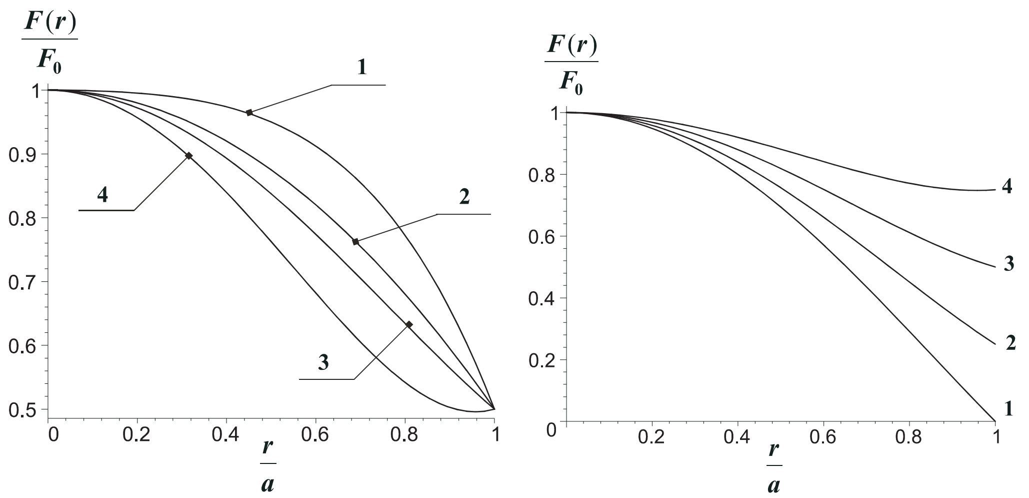

Fig. 1. Left panel: the distribution of the external excited pressure when $p = 0.5$ for 1: $s = -0.4$, 2 : $s = 0$, 3: $s = 0.2$, 4: $s = 0.6$. Right panel: the distribution of the external excited pressure when $s = 0.3$ for 1: $p = 0$, 2: $p = 0.25$, 3: $p = 0.5$, 4: $p = 0.75$.

Assuming that in Eq. (5) $p = 1$ and $s = 0$, a special case of excitation can be obtained, namely an uniformly pressure distribution on the whole surface.

3. The solution of the equation of motion

The following boundary conditions associated with the clamped edge have to be satisfied by the function which is a solution to the equation of motion: $\eta(a) = 0$, $\mathrm{d}\eta(r)/\mathrm{d}r|_{r=a} = 0$.

The expansion into series of eigenfunctions leads to a solution of the equation of motion in the form of the infinite sum (2), in which the coefficients c_n are determined by formula (4). Moreover, this solution satisfies the boundary conditions required.

Taking into account formula (5) describing the shape of the surface excitation, the desirable coefficients can be presented as

$$c_n = \frac{2W_0\gamma^4\left(pJ_1(\gamma_n)I_1(\gamma_n)\gamma_n^2 + \gamma_n(1-p-s)D_2(\gamma_n) + 4sD_3(\gamma_n)\right)}{\left(\gamma_n^4-\gamma^4\right)\gamma_n^3 I_1(\gamma_n)J_0^2(\gamma_n)}, \tag{6}$$

where $D_k(\gamma_n) = J_k(\gamma_n)I_1(\gamma_n) + J_1(\gamma_n)I_k(\gamma_n)$, $k = 2$, 3 and $W_0 = F_0/\rho h\omega^2$.

The infinite sum (2) together with formulae (6) and (3), where the latter describes the adapted system of the orthogonal functions, present the solution of the equation of motion for the plate under consideration.

4. The integral formulae for the active sound power

The total sound power is defined as

$$P = \frac{1}{2}\int\limits_S p(\mathbf{r})v^*(\mathbf{r})\,\mathrm{d}S, \tag{7}$$

where

$$p(\mathbf{r}) = \frac{-ik_0c\rho_0}{2\pi}\int\limits_S v(\mathbf{r}_0)\frac{\exp(ik_0|\mathbf{r}-\mathbf{r}_0|)}{|\mathbf{r}-\mathbf{r}_0|}\,\mathrm{d}S, \tag{8}$$

is the sound pressure, v and v^* are the vibration velocity of the plate's points and its conjugated value, respectively, $|\mathbf{r}-\mathbf{r}_0|$ is the distance between points of the source and that in the soundfield, S is the plate's area, ρ_0, c denote the rest density of the fluid medium and the propagation velocity, respectively, $k_0 = 2\pi/\lambda$ is the acoustic wavenumber withλ denoting the length of the wave in the fluid medium [9].

The active power can be expressed by the Hankel representation as

$$P_a = \pi\rho_0ck_0^2\int\limits_0^1 \frac{W(x)W^*(x)x\,\mathrm{d}x}{\sqrt{1-x^2}}, \tag{9}$$

where

$$W(x) = i\omega\int\limits_0^a \eta(r)J_0(k_0rx)r\,\mathrm{d}r \tag{10}$$

is the function describing the radiation directivity of the source in the axisymmetric case.

In the light of formulae (3) and (6), the function (10) can be presented as a following sum

$$W(x) = \frac{4i\omega W_0a^2\gamma^4}{\beta^3}\sum_{n=1}^{\infty}e_nw_n(x), \tag{11}$$

in which $\beta = k_0 a$,

$$e_n = \frac{J_0(\gamma_n)\gamma_n^2}{2W_0\gamma^4} c_n, \tag{12}$$

$$w_n(x) = \frac{\alpha_n \delta_n J_0(\beta x) - x J_1(\beta x)}{\delta_n^4 - x^4}. \tag{13}$$

In Eq. (13), the following notations are introduced: $\alpha_n = J_1(\gamma_n)/J_0(\gamma_n)$ and $\delta_n = \gamma_n/\beta$.

Using the expression (11), the active sound power can be formulated as

$$P_a = P_0 \frac{16\gamma^8}{\beta^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e_n e_m \theta_{nm}, \tag{14}$$

where $P_0 = \pi\rho_0 c\omega^2 a^2 W_0^2$ should be regarded as a unit of the acoustic power and

$$\theta_{nm} = \int_0^1 \frac{w_n(x) w_m(x) x \, \mathrm{d}x}{\sqrt{1 - x^2}}. \tag{15}$$

Neglecting the acoustic damping and the energy losses into plate's material, the e_n coefficients have real values. In this case the total active power can be expressed by the modal self and mutual active power only.

In order to use formula (14) for the computations of the active power for some practical cases, the sums have to be reduced to a finite numbers of terms. The number of terms depends on the required accuracy and on the value of β.

5. Approximation formulae

Basing on the integral formula (14), the asymptotics for the active radiation power of the source under consideration can be obtained. Since the approximated expressions for the integrals (15) valid for both the high and low frequencies are known, formula (14) can be convenient for obtaining the asymptotics for the active radiation power.

For $\beta \geq 10$, $\delta_n < 1$ and $n = m$ the asymptotics for the integrals (15) can be written as [1]

$$\theta_{nm} = \frac{1}{4\delta_n^4} \left\{ \frac{1}{2} \frac{1 + \alpha_n^2}{\sqrt{1 - \delta_n^2}} + \frac{1}{2} \frac{1 - \alpha_n^2}{\sqrt{1 + \delta_n^2}} + \frac{2\delta_n^4 \left[(1 - \alpha_n^2 \delta_n^2) \cos\phi + 2\alpha_n \delta_n \sin\phi \right]}{\beta^{3/2} \sqrt{\pi} (1 - \delta_n^4)^2} \right\}, \tag{16}$$

where $\phi = 2\beta + \pi/4$.

In the case of $n \neq m$ the asymptotics can be expressed in the following form: [10]

$$\theta_{nm} = \frac{1}{4\delta_n^2\delta_m^2}\left\{\frac{\alpha_n\delta_n - \alpha_m\delta_m}{\beta(\delta_n^4 - \delta_m^4)}\left(\delta_n^2\left(\frac{1}{\sqrt{1-\delta_m^2}} - \frac{1}{\sqrt{1+\delta_m^2}}\right)\right.\right.$$
$$\left. + \delta_m^2\left(\frac{1}{\sqrt{1-\delta_n^2}} - \frac{1}{\sqrt{1+\delta_n^2}}\right)\right)$$
$$\left. + \frac{2\delta_n^2\delta_m^2[(1-\alpha_n\alpha_m\delta_n\delta_m)\cos\phi + (\alpha_n\delta_n + \alpha_m\delta_m)\sin\phi]}{\sqrt{\pi}\beta^{3/2}(1-\delta_n^4)(1-\delta_m^4)}\right\}. \tag{17}$$

An approximated expression for the active sound power valid for $\beta \geq 10$ can be found by reducing the number of terms in the sums in Eq. (14) and taking into account the relations (16) and (17). Moreover, for each term in the sums, the following inequality $\delta_n < 1$ must be satisfied [1]. The expression for describing the sound power for high frequencies formulated in this paper is very useful for some numerical calculations.

When $\beta < \gamma_n, \gamma_m$, the integrals θ_{nm} can be replaced by their asymptotics given in paper [12].

Having reduced the sums in Eq. (14) to a finite number of terms and making use of the low frequencies asymptotic for integrals (15), the active power valid for low frequencies will be formulated.

6. Discussions and conclusions

The formula for describing the active power radiated by a circular plate excited to vibrations by means of non-uniform surface pressure has been obtained. The distribution of the external pressure given analytically by the function (5) can be modified by means of the parameters p and s.

By choosing some suitable values for the p and s parameters, it is possible to approximate the shape of excitation known from experiment by means of the function $F(r)$. In this way, the formulae for the description of the active sound power for a number of practical cases have been obtained. By employing the integral expression (14) reduced to a finite number of terms in the sums, the asymptotic expressions for the active radiation power can be formulated. The dependences of the active power on the parameter β have been plotted in Fig. 2 for some values of p and s. Some of the most simple cases of the surface excitation have been chosen for a numerical analysis, namely the excitation spreading from the center of plate to its edge (see Figs. 1).

Figures 2 show that for some of the determined values of β parameter, the active power tends to infinte values. This results from neglecting the acoustic damping and energy losses in the plate's material. Apart from that, the assumed model of the plate

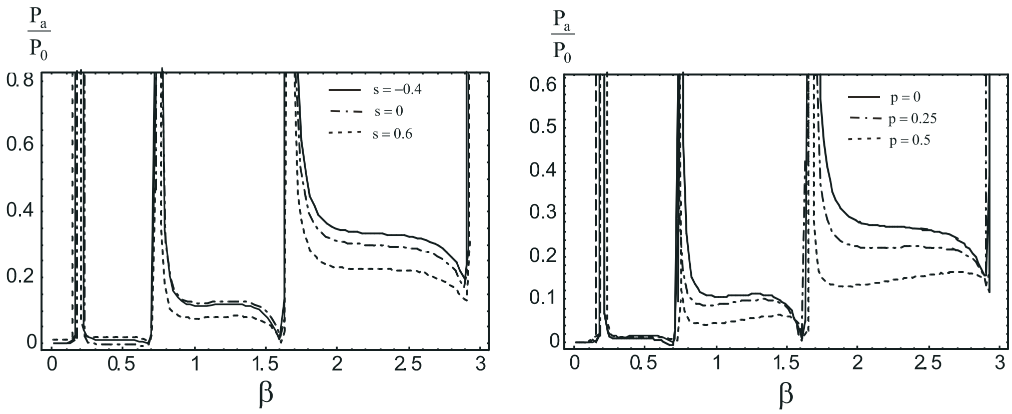

Fig. 2. The active sound power as a function of the parameter β. It has been assumed that: the plate radius and plate thickness are equal to 0.25 m and $1 \cdot 10^{-3}$ m, respectively, the Young's modulus, Poisson ratio and density of the plate material amount to $2.2 \cdot 10^{11}$ N/m^2, 0.3, 7860 kg/m^3, respectively, and the propagation velocity equals 343 m/s. Moreover, for the left panel $p = 0.5$ has been taken and for the right one $s = 0.3$.

does not enable taking into account the nonlinear effects which are very intensive in the case of nearby resonance frequencies.

When $\beta \geq 10$, we can insert expressions (16) and (17) into the integrals θ_{nm} in formula (14). In this way, it is possible to obtain a formula for the active power in an elementary form valid for high frequencies. The latter is very useful for some numerical calculations.The approximation formula for the active power valid for low frequencies can be obtained by employing the expressions for the integrals θ_{nm} from paper [12] and reducing the sums in expression (14) to a small number of terms.

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