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PERTURBATION METHODS FOR ACOUSTIC SYSTEMS WITH INTERVAL PARAMETERS

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The aim of the paper is to present a new algebraic system with strictly defined new interval numbers and specifically defined addition and multiplication operations. The new introduced interval numbers are called perturbation interval numbers and are defined as ordered couples of real numbers with specially defined addition and multiplication. It's proved that the new algebraic system is imbedded into the system of real interval numbers and so perturbation interval numbers are generalization of real intervals. Some additional properties as subtraction, inversion and division are presented too, as well as extensions of elementary functions. Applications to numerical methods of interval sound decay and reverberation time analysis are discussed.

Keywords: perturbation numbers, interval parameters, reverberation time.

1. Introduction

Theory of perturbations is a part of science of the great theoretical and practical meaning [1]. Following papers [4–8] the new, unique algebraic system over intervals is presented. Calculations with use of new perturbation numbers lead to applications which are mathematically equivalent with I-order approximations in classical perturbation methods. Advantages of the new algebraic system are as follows: we can omit all complex analytical calculations which are typical for expanding approximated values of solutions in infinite series, we get a great simplification of all arithmetic calculations which appear in analytical formulation and analysis of the problem, most of known numerical algorithms can be simply adapted for the new interval algebraic system without any serious difficulties. With the new algebraic system we get a set of very simple and useful mathematical tools which can be easy used in analytical and computational analysis of complex acoustic problems.

2. Interval perturbation numbers

Remember that an interval number is an ordered couple of reals (a^-, a^+), such that $a^- \leq a^+$. An interval number can be presented in the form $[a^-, a^+]$ or equivalent $\bar{a}$, where real numbers a^-, a^+ are called ends of the interval. With use of middle of the interval named $\breve{a} := 0.5(a^- + a^+)$ and their radius $\mathrm{rad}(\breve{a}) = 0.5(a^+ - a^-)$, we can note the interval number in the form

$$\bar{a} = [\breve{a} - \mathrm{rad}(\bar{a}), \breve{a} + \mathrm{rad}(\bar{a})].$$

The interval number $\bar{a}$ can be in fact the ordered couple of reals $(\breve{a}, \Delta a)_r$, where for simplicity we note $\Delta a := \mathrm{rad}(\bar{a})$ and $\bar{\Delta} := [-\Delta a, \Delta a]$, cf. [3].

2.1. Dependent interval perturbation numbers

Define a new number called further 2-scale perturbation number as ordered 3-couple of real numbers $(x_0, x_1, x_2) \in \mathbb{R}^3$. The set of 2-scale perturbation numbers is denoted by $\mathbb{R}_{2\varepsilon}$. The first element x_0 of the 3-couple is called a main value and the following are the perturbation values or simply the perturbations [4–6].

Let $\zeta, \zeta_1, \zeta_2, \zeta_3 \in \mathbb{R}_{2\varepsilon}$ denote any of perturbation numbers and $\zeta := (x_0, x_1, x_2)$, $\zeta_1 := (y_0, y_1, y_2)$, $\zeta_2 := (z_0, z_1, z_2)$, $\zeta_3 := (v_0, v_1, v_2)$, $x_i, y_i, z_i, v_i \in \mathbb{R}$, $i = 0, 1, 2$. It is called that two perturbation numbers are equal: $\zeta_1 \equiv \zeta_2$ if and only if (iff) $y_i = z_i$ for any $i = 0, 1, 2$.

In the set $\mathbb{R}_{2\varepsilon}$ we introduce the addition $(+_\varepsilon)$ and multiplication $(\bullet_\varepsilon)$ as follows:

$$\begin{aligned} \zeta_1 +_\varepsilon \zeta_2 &= (y_0, y_1, y_2) +_\varepsilon (z_0, z_1, z_2) := (y_0 + z_0, y_1 + z_1, y_2 + z_2), \\ \zeta_1 \bullet_\varepsilon \zeta_2 &= (y_0, y_1, y_2) \bullet_\varepsilon (z_0, z_1, z_2) := (y_0 z_0, y_0 z_1 + y_1 z_0, y_0 z_2 + y_2 z_0). \end{aligned}$$

Notice, that each perturbation number of the form $(a, 0, 0) \in \mathbb{R}_{2\varepsilon}$, $a \in \mathbb{R}$, can be identified with a real number a. We can use this notice to simplify a notion for perturbation operations. Denote $\varepsilon_1 := (0, 1, 0)$ and $\varepsilon_2 := (0, 0, 1)$, respectively. Then for every $\zeta = (x_0, x_1, x_2) \in \mathbb{R}_{2\varepsilon}$ we can write (in short)

$$(x_0, x_1, x_2) = (x_0, 0, 0) +_\varepsilon (0, x_1, 0) +_\varepsilon (0, 0, x_2) = x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_2.$$

From multiplicity formulas it follows that

$$\begin{aligned} \varepsilon_1^2 &= \varepsilon_1 \varepsilon_1 = (0, 1, 0)(0, 1, 0) = (0, 0, 0), \\ \varepsilon_2^2 &= \varepsilon_2 \varepsilon_2 = (0, 0, 1)(0, 0, 1) = (0, 0, 0), \\ \varepsilon_1 \varepsilon_2 &= (0, 0, 0). \end{aligned}$$

Assume now, that the radius of the interval $\bar{z}$ is a two-scale perturbation number and define two symbolic perturbation intervals $\bar{\varepsilon}_1 := [-\varepsilon_1, \varepsilon_1]$ and $\bar{\varepsilon}_2 := [-\varepsilon_2, \varepsilon_2]$. Then we can write a two-scale perturbation interval number as:

$$\bar{z} = \breve{z} + \delta z_1 \bar{\varepsilon}_1 + \delta z_2 \bar{\varepsilon}_2.$$

The set of two-scale perturbation interval numbers is denoted by $\mathbb{R}_{2\varepsilon}$.

Let $\bar{a} = \breve{a} + \delta a_1 \bar{\varepsilon}_1 + \delta a_2 \bar{\varepsilon}_2$, $\bar{b} = \breve{b} + \delta b_1 \bar{\varepsilon}_1 + \delta b_2 \bar{\varepsilon}_2$ denote two dependent intervals in the range of variation of parameters ε_1 and ε_2, however intervals $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ remain independent. Such interval numbers are called further for simplicity 2-perturbation interval numbers (2ε-interval numbers) partially dependent.

Assume, that arithmetic operations for dependent 2ε–interval numbers are defined in the following way

$$\bar{a} * \bar{b} = \text{hull}\{a * b;\ a = \breve{a} + \delta a_1 \varepsilon_1 + \delta a_2 \varepsilon_2,\ b = \breve{b} + \delta b_1 \varepsilon_1 + \delta b_2 \varepsilon_2,\ \varepsilon_1 \in \bar{\varepsilon}_1,\ \varepsilon_2 \in \bar{\varepsilon}_2\},$$

where $* =$ "$+, -, \bullet, /$". So for addition we have

$$\bar{a} + \bar{b} = \breve{a} + \breve{b} + \bar{\varepsilon}_1 \left(\delta a_1 + \delta b_1\right) + \bar{\varepsilon}_2 \left(\delta a_2 + \delta b_2\right).$$

Similarly for subtraction

$$\bar{a} - \bar{b} = \breve{a} - \breve{b} + \bar{\varepsilon}_1 \left(\delta a_1 - \delta b_1\right) + \bar{\varepsilon}_2 \left(\delta a_2 - \delta b_2\right),$$

and for multiplication

$$\bar{a}\bar{b} = \breve{a}\breve{b} + \bar{\varepsilon}_1(\breve{a}\delta b_1 + \breve{b}\delta a_1) + \bar{\varepsilon}_2(\breve{a}\delta b_2 + \breve{b}\delta a_2).$$

Denote further $\bar{1}_{2\varepsilon} := (1, 0_{2\varepsilon})_r = (1,0,0,0)_r$, $\bar{0}_{2\varepsilon} := (0, 0_{2\varepsilon})_r = (0,0,0,0)_r$. Element $\bar{0}_{2\varepsilon}$ has the property as neutral element of addition, while $\bar{1}_{2\varepsilon}$ is the neutral element of multiplication for any 2-perturbation interval $\bar{a} \in \mathbb{R}_{2\varepsilon}$.

One can show that there exist inverse element $\bar{z} = \breve{z} + \delta z_1 \bar{\varepsilon}_1 + \delta z_2 \bar{\varepsilon}_2$ with respect to 2ε-interval number $\bar{a}$, i.e.

$$\bar{a}\bar{z} = \breve{a}\breve{z} + \bar{\varepsilon}_1 \left(\breve{a}\delta z_1 + \breve{z}\delta a_1\right) + \bar{\varepsilon}_2 \left(\breve{a}\delta z_2 + \breve{z}\delta a_2\right) = \bar{1}_{2\varepsilon}.$$

The following must be satisfied

$$\bar{a}^{-1} = \left(\breve{a} + \delta a_1 \bar{\varepsilon}_1 + \delta a_2 \bar{\varepsilon}_2\right)^{-1} = \frac{1}{\breve{a}} - \frac{\delta a_1}{\breve{a}^2}\bar{\varepsilon}_1 - \frac{\delta a_2}{\breve{a}^2}\bar{\varepsilon}_2.$$

Having in hand the inverse operation we can define the division operation, as

$$\frac{\bar{a}}{\bar{b}} := \bar{a}\bar{b}^{-1}$$

and in the consequence

$$\frac{\bar{a}}{\bar{b}} = \frac{\breve{a}}{\breve{b}} + \left(\frac{\delta a_1}{\breve{b}} - \frac{\breve{a}\delta b_1}{\breve{b}^2}\right)\bar{\varepsilon}_1 + \left(\frac{\delta a_2}{\breve{b}} - \frac{\breve{a}\delta b_2}{\breve{b}^2}\right)\bar{\varepsilon}_2.$$

Following previous results we can formulate properties of the new symbols $\overline{\varepsilon}_1$ and $\overline{\varepsilon}_2$, which are called interval perturbation. We can write

$$\begin{aligned}
\overline{\varepsilon}_i + \overline{\varepsilon}_i &= 2\overline{\varepsilon}_i, & i &= 1,2,\\
\overline{\varepsilon}_i - \overline{\varepsilon}_i &= \overline{0}_{2\varepsilon}, & i &= 1,2,\\
\overline{\varepsilon}_i\,\overline{\varepsilon}_j &= \overline{0}_{\varepsilon 2}, & i,j &= 1,2,\\
\overline{1}_{\varepsilon 2}/\overline{\varepsilon}_i & \quad\text{not feasible for} & i &= 1,2,\\
\frac{\overline{a}+\alpha\overline{\varepsilon}_1+\beta\overline{\varepsilon}_2}{\overline{a}+\alpha\overline{\varepsilon}_1+\beta\overline{\varepsilon}_2} &= \overline{1}_{\varepsilon 2} \quad \text{for any} & \alpha,\beta &\in R,\ \ \overline{a}\neq\overline{0}_{\varepsilon 2}.
\end{aligned}$$

2.2. Independent interval perturbation numbers

Let $\bar{a} = \breve{a} + \delta a_1\overline{\varepsilon}_1 + \delta a_1\overline{\varepsilon}_2$, $\overline{b} = \breve{b} + \delta b_1\overline{\varepsilon}_1 + \delta b_1\overline{\varepsilon}_2$ denote now two independent intervals in the range of variation of parameters ε_1 and ε_2. Notice that classical algebraic operations for independent interval numbers are usually defined as follows:

$$\overline{a} * \overline{b} = \text{hull}\left\{a * b;\ a\in\overline{a},\ b\in\overline{b}\right\}$$

where $* =$ "$+,-,*,/$". Such interval numbers are called further for simplicity 2-perturbation independent interval numbers (2ε-interval numbers).

So for algebraic operations we have

$$\begin{aligned}
\overline{a}+\overline{b} &= \breve{a}+\breve{b}+\overline{\varepsilon}_1\left(\delta a_1+\delta b_1\right)+\overline{\varepsilon}_2\left(\delta a_2+\delta b_2\right),\\
\overline{a}-\overline{b} &= \breve{a}-\breve{b}+\overline{\varepsilon}_1\left(\delta a_1+\delta b_1\right)+\overline{\varepsilon}_2\left(\delta a_2+\delta b_2\right).
\end{aligned}$$

Denote further that the neutral elements are the same as for dependent intervals. Notice, that addition and multiplication are identical to those defined usually for dependent 2ε-interval numbers, the essential difference appears in subtraction.

Notice that addition and subtraction operations are identical with the same operations for usual interval numbers, the difference appeared only in multiplication.

Following previous results we can formulate properties of the new symbols $\overline{\varepsilon}_1$ and $\overline{\varepsilon}_2$, which can be called interval perturbation. We can write

$$\begin{aligned}
\overline{\varepsilon}_i + \overline{\varepsilon}_i &= 2\overline{\varepsilon}_i, & i &= 1,2,\\
\overline{\varepsilon}_i - \overline{\varepsilon}_i &= \overline{2}_{2\varepsilon}, & i &= 1,2,\\
\overline{\varepsilon}_i\,\overline{\varepsilon}_j &= \overline{0}_{\varepsilon 2}, & i,j &= 1,2,\\
\overline{1}_{\varepsilon 2}/\overline{\varepsilon}_i & \quad\text{not feasible for} & i &= 1,2,\\
\frac{\overline{a}+\alpha\overline{\varepsilon}_1+\beta\overline{\varepsilon}_2}{\overline{a}+\alpha\overline{\varepsilon}_1+\beta\overline{\varepsilon}_2} &= \overline{1}_{\varepsilon 2} \quad \text{for any} & \alpha,\beta &\in R,\ \ \overline{a}\neq\overline{0}_{\varepsilon 2}.
\end{aligned}$$

3. Extended interval two-scale perturbation functions

Two-scale perturbation interval value functions are defined for two-scale perturbation interval arguments as extensions of classical elementary functions. Properties of two-scale perturbation interval are analyzed in details in [7].

Let $D \subset \mathbb{R}_{2\varepsilon}$ be an arbitrary subset. Suppose that we have a rule $\overline{f}_{2\varepsilon}$ which assigns to each element $\overline{z} \in D$ exactly one element $\overline{w}$ of $\mathbb{R}_{2\varepsilon}$. Then we say that $\overline{f}_{2\varepsilon}$ is an extended function defined on D with values in $\mathbb{R}_{2\varepsilon}$. We will denote that function as $\overline{f}_{2\varepsilon} : D \to \mathbb{R}_{2\varepsilon}$ or $\overline{w} = \overline{f}_{2\varepsilon}(\overline{z})$.

To illustrate how we can construct generalizations of usual real functions we use a simple function. We discuss now an extension of a simple exponential function $\exp(x)$, $x \in R$. With polynomials and rational functions it is one of the simplest elementary functions. How can we understand the notion $\exp(\overline{z})$, where $\overline{z} = \breve{z} + \delta z_1 \overline{\varepsilon}_1 + \delta z_2 \overline{\varepsilon}_2 \in \mathbb{R}_{2\varepsilon}$? Notice that we can expand $\exp(x)$ into a classical series

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad x \in R, \tag{1}$$

which is convergent for all $x \in R$. Define the new function $\exp_{2\varepsilon}(\overline{z})$, for $\overline{z} = \breve{z} + \delta z_1 \overline{\varepsilon}_1 + \delta z_2 \overline{\varepsilon}_2 \in \mathbb{R}_{2\varepsilon}$ as

$$\overline{\exp}_{2\varepsilon}(\overline{z}) := 1 + \frac{\overline{z}}{1!} + \frac{\overline{z}^2}{2!} + \frac{\overline{z}^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{\overline{z}^k}{k!}, \qquad \overline{z} \in \mathbb{R}_{2\varepsilon}. \tag{2}$$

Following equation (1) and (2) we can write

$$\overline{\exp}_{2\varepsilon}(\overline{z}) := (1 + \overline{\varepsilon}_1 \delta z_1 + \overline{\varepsilon}_2 \delta z_2) \exp(\breve{z}), \qquad \overline{z} \in \mathbb{R}_{2\varepsilon}.$$

We can prove the generalized convergence of the sequence (2) for every $\overline{z} \in \mathbb{R}_{2\varepsilon}$. In the similar way one can define more complicated multidimensional functions, [7].

4. Interval sound decay and reverberation time

Consider the sound decay and reverberation time of rectangular rooms with some interval uncertainty in parameters. Then following standard considerations we get that the numerical value for the sound velocity is of interval type, cf. [2]

$$\overline{T}_{2\varepsilon} = 0.163 \frac{\overline{V}_{2\varepsilon}}{4\overline{m}\overline{V} - \overline{S} \ln_{2\varepsilon}\left(\overline{1}_{2\varepsilon} - \overline{\alpha}_{2\varepsilon}\right)}.$$

Assume that $\overline{\alpha}_{2\varepsilon}$ is a pure 2ε-interval perturbation of the form

$$\overline{\alpha}_{2\varepsilon} = \overline{\alpha}_1 \overline{\varepsilon}_1 + \overline{\alpha}_2 \overline{\varepsilon}_2, \qquad k = 1, 2, \ldots, n,$$

then we get

$$\ln_{2\varepsilon}\left(\overline{1}_{2\varepsilon} - \overline{\alpha}_{2\varepsilon}\right) = \overline{\alpha}_1 \overline{\varepsilon}_1 + \overline{\alpha}_2 \overline{\varepsilon}_2.$$

This results in interval version of Sabine's reverberation formula

$$\overline{T}_{2\varepsilon} = 0.163 \frac{\overline{V}_{2\varepsilon}}{4\overline{m}\overline{V} + \overline{S}\left(\overline{\alpha}_1 \overline{\varepsilon}_1 + \overline{\alpha}_2 \overline{\varepsilon}_2\right)}.$$

5. Conclusions

This paper investigates the feasibility of predicting the interval perturbations of acoustic signals travelling within an indoor environment. With the new interval algebraic system we get a set of very simple and useful mathematical tools which can be easy used in analytical and computational parts of analysis of complex perturbation acoustic problems. one can omit all complex analytical calculations which are typical for expanding approximated values of solutions in infinite series. It works for expanding unknown values – solutions as well as for perturbed coefficients of the problem; one get a great simplification of all calculations in mathematical analysis area which appear in analytical formulation and analysis of the problem; most of known classical results of the theory of differential equations can be simply adapted for the new system of calculations without any serious difficulties.

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