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CYLINDRICAL WAVE DIFFRACTION BY AN ABSORBING STRIP

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A solution for the problem of diffraction of a cylindrical sound wave near an absorbing strip introducing the Kutta–Joukowski condition is obtained. The two faces of the strip have impedance boundary conditions. The problem which is solved is a mathematical model for a noise barrier whose surface is treated with acoustically absorbing materials. It is found that the field produced by the Kutta–Joukowski condition will be substantially in excess of that in its absence when the source is near the edge.

1. Introduction

Much interest has been shown in recent years to the problem of noise reduction. Unwanted noise from motorways, railways and airports can be shielded by a barrier which intercepts the line of sight from the noise source to a receiver. To design and performance of noise barriers, particularly, for the reduction of traffic noise, has received considerable attention [1]. An effective way of reducing the noise is to use absorbing linings. Absorbing linings have also been used on noise barriers to improve their efficiency. The rationale for such a noise barrier design is given in RAWLINS [2].

In 1970, it was shown by FFOWCS–WILLIAMS and HALL [3] that the aerodynamic sound scattered by a sharp edge is proportional in intensity to the fifth power of the flow velocity and inversely to the cube of the distance of the source from the edge. Thus, the edge is likely to be the dominant sound source, especially when the source is very close to the edge. Their findings were however based upon the assumption of a potential flow near the sharp edge with velocity becoming infinite there. Instead of that if one wishes to prescribe that the velocity is finite, there are two possible points of view. One way is to abandon lighthill's theory and use linearized Navier–Stoke's equation with a source term as employed by ALBLAS [4]. Before discussing the second option, it is better to introduce the Kutta–Joukowski condition.

JONES [5] adopted this approach and introduced the wake condition to examine to effect of the Kutta–Joukowski condition at the edge of the half-plane. He calculated the field scattered from a line source and observed that for the moving medium the imposition of the Kutta–Joukowski condition does not have much influence on the scattered field away from the diffracting plane. Near the wake this condition produces a much stronger field than elsewhere even when the source is not near the edge. Thus the wave acts as a convenient transmission channel for carrying intense sound away from the source. This problem was further extended to the point source excitation by BALASUBRAMANYAM [6].

Keeping in view the importance of the Kutta–Joukowski condition, diffraction of a cylindrical acoustic wave by an absorbing strip is considered in this paper. It is found that the field produced by this condition (Kutta–Joukowski) will be substantially larger than the field produced in its absence when the source is near the edge. The results for rigid and soft strips can be obtained as special cases of this problem by taking the absorbing parameter $\beta = 0$ and $\beta = \infty$, respectively.

2. Formulation of the problem

We shall consider small amplitude sound waves diffracted by a strip. An absorbing strip is assumed to occupy $y = 0, -l \leq x \leq 0$ as shown in the Fig. 1. The strip is assumed to be of negligible thickness and satisfying absorbent boundary conditions [7]

$$p - u_n z = 0, \tag{2.1}$$

on both sides of its surface. Here p is the acoustic pressure of the surface, u_n is the normal component of the perturbation velocity at a point on the surface of the strip and z is the acoustic impedance of the surface. We shall restrict our consideration to a harmonic time dependence, with the time factor $e^{-i\omega t}$ (ω is low angular frequency) being suppressed throughout.

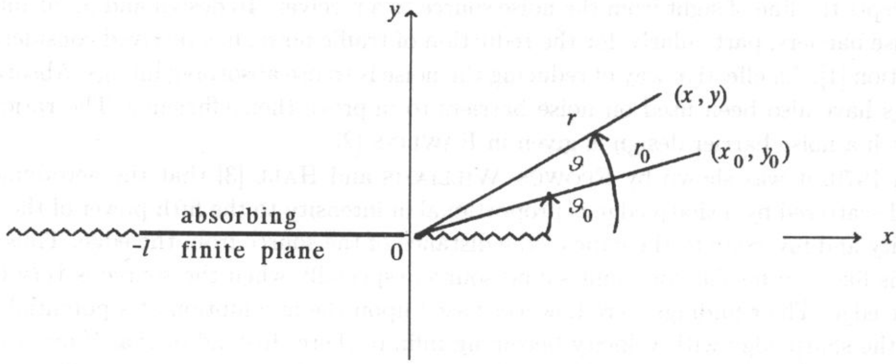

Fig. 1.

The perturbation velocity u of the irrotational sound waves can be expressed in terms of the total velocity potential $\phi_t(x, y)$ by $u = \operatorname{grad} \phi_t$. The resulting pressure in the sound field is given by $p = i\omega\varrho_0\phi_t(x, y)$, where ϱ_0 is the density of the initially undisturbed ambient medium. The primary source is taken to be a line source which is located at the position (x_0, y_0), $y_0 > 0$. Thus, the wave equation satisfied by the total velocity potential ϕ_t in the presence of the line source is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\phi_t = \delta(x - x_0)\delta(y - y_0), \tag{2.2}$$

where $k(= \omega/c)$ is the free space wave number and c is the speed of sound. For analytic convenience k is assumed to be complex and has a small positive imaginary part.

The effect of the strip is described by the boundary conditions

$$\left(\frac{\partial}{\partial y} \pm ik\beta\right)\phi_t(x, 0^\pm) \qquad (-l < x < 0), \tag{2.3}$$

where $\beta(= \varrho_0 c/z)$ is the small absorbing parameter and for acoustic absorption $\operatorname{Re}(\beta) > 0$. We remark that $\beta = 0$ corresponds to the rigid barier and $\beta = \infty$ corresponds to the pressure release barrier.

In order to satisfy the Kutta–Joukowski condition at the edge, JONES [5] introduced a discontinuity in the field at $0 < x < \infty$ and postulated the existence of a wake condition. According to him, ϕ_t is discontinuous, while $\partial\phi_t/\partial y$ remains continuous for $y = 0$, $x > 0$. With the same analysis as used by JONES [5], the boundary conditions can thus be expressed as

$$\frac{\partial}{\partial y}\phi_t(x, y^+) = \frac{\partial}{\partial y}\phi_t(x, y^-) \qquad (x < -l, \quad x > 0, \quad y = 0), \tag{2.4}$$

and

$$\begin{aligned} \phi_t(x, y^+) - \phi_t(x, y^-) &= \alpha e^{i\mu x} \qquad (x > 0, \quad y = 0), \\ \phi_t(x, y^+) - \phi_t(x, y^-) &= \alpha e^{-i\mu x} \qquad (x < -l, \quad y = 0). \end{aligned} \tag{2.5}$$

In Eq. (2.5), α and μ are constants. The constant μ is regarded as known and we shall write

$$\mu = k\cos\vartheta_1, \tag{2.6}$$

where $0 \le \operatorname{Re}\vartheta_1 < \pi$, $\operatorname{Im}\vartheta_1 \ge 0$. While k has a positive imaginary part we shall take $0 < \operatorname{Re}\vartheta_1 < \pi$ and $\operatorname{Im}\vartheta_1 > 0$; eventually we shall be concerned primarily with the case $\operatorname{Re}\vartheta_1 = 0$, $\operatorname{Im}\vartheta_1 > 0$. In Eq. (2.5), α can be determined by means of a Kutta–Joukowski condition. We note that $\alpha = 0$ corresponds to a no wake situation. It is appropriate to split ϕ_t as

$$\phi_t(x, y) = \phi_0(x, y) + \phi(x, y), \tag{2.7}$$

where ϕ_0 is the incident wave which accounts for the inhomogeneous source term and ϕ is the solution of the homogeneous wave Eq. (2.2) that corresponds to the diffracted field. Thus ϕ_0 and ϕ satisfy the following equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\phi_0(x, y) = \delta(x - x_0)\delta(y - y_0), \tag{2.8}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\phi(x, y) = 0. \tag{2.9}$$

In addition we insist that ϕ represents an outward travelling wave as $r = \sqrt{x^2 + y^2} \to \infty$ and satisfies the normal edge condition at the boundary discontinuity [8].

3. Solution of the problem

We define the Fourier transform pair by

$$\begin{aligned}\overline{\phi}(\nu, y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x, y) e^{i\nu x} dx, \\ \phi(x, y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\phi}(\nu, y) e^{-i\nu x} d\nu,\end{aligned} \tag{3.1$_1$}$$

where ν is a complex variable. In order to accomodate three part boundary conditions on $y = 0$, we split $\overline{\phi}(\nu, y)$ as

$$\overline{\phi}(\nu, y) = \overline{\phi}_+(\nu, y) + e^{-i\nu l}\overline{\phi}_-(\nu, y) + \overline{\phi}_1(\nu, y), \tag{3.1$_2$}$$

where

$$\overline{\phi}_+(\nu, y) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \phi(x, y) e^{i\nu x} dx,$$

$$\overline{\phi}_-(\nu, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-l} \phi(x, y) e^{i\nu (x+l)} dx,$$

and

$$\overline{\phi}_1(\nu, y) = \frac{1}{\sqrt{2\pi}} \int_{-l}^{0} \phi(x, y) e^{i\nu x} dx.$$

In Eq. (3.1)$_2$, $\overline{\phi}_+$ is regular for $\operatorname{Im} \nu > -\operatorname{Im} k$, $\overline{\phi}_-$ is regular for $\operatorname{Im} \nu < \operatorname{Im} k$ and $\overline{\phi}_1(\nu, y)$ is an integral function and is therefore analytic in $-\operatorname{Im} k < \operatorname{Im} \nu < \operatorname{Im} k$. For this we recall that k is complex and ϕ represents an outward travelling wave. The solution of Eq. (2.8) can be written in a straight forward manner as

$$\begin{aligned}\phi_0(x, y) &= \frac{-1}{4i} H_0^{(1)}\left(k\left[(x - x_0)^2 + (y - y_0)^2\right]^{1/2}\right), \\ &= \frac{1}{4\pi i} \int_{-\infty}^{\infty} \frac{e^{-i\nu(x-x_0)+i(k^2-\nu^2)^{1/2}|y-y_0|}}{\sqrt{k^2 - \nu^2}} d\nu.\end{aligned} \tag{3.2}$$

Making change of variables

$$x_0 = r_0 \cos\vartheta_0, \qquad y_0 = r_0 \sin\vartheta_0 \qquad (0 \le \vartheta_0 \le \pi),$$

in Eq. (3.2) and letting $r_0 \to \infty$, we obtain using the asymptotic form for the Hankel function

$$\phi_0 = b e^{-ik(x\cos\vartheta_0 + y\sin\vartheta_0)}, \tag{3.3}$$

where

$$b = \frac{-1}{4i}\sqrt{\frac{2}{\pi k r_0}} e^{i(kr_0 - \pi/4)}, \tag{3.4}$$

and ϑ_0 is the angle measured from the x-axis. Now taking the Fourier transform of Eq. (2.9), we obtain

$$\left(\frac{d^2}{dy^2}+\gamma^2\right)\overline{\phi}(\nu,y)=0, \tag{3.5}$$

where $\gamma=\sqrt{k^2-\nu^2}$ and the ν-plane is cut such that $\operatorname{Im}\gamma>0$. The solution of Eq. (3.5) which satisfies the radiation condition is

$$\overline{\phi}(\nu,y)=\begin{cases}A_1(\nu)e^{i\gamma y} & (y>0),\\ A_2(\nu)e^{-i\gamma y} & (y<0).\end{cases} \tag{3.6}$$

Transforming the boundary conditions (2.3) to (2.5), we have

$$\overline{\phi}_1'(\nu,0^{\pm})=\mp ik\beta\overline{\phi}_1(\nu,0^{\pm})\mp ik\beta\overline{\phi}_0(\nu,0)-\overline{\phi}_0'(\nu,0), \tag{3.7}$$

$$\overline{\phi}_{\pm}'(\nu,0^+)=\overline{\phi}_{\pm}'(\nu,0^-)\ =\ \overline{\phi}_{\pm}'(\nu,0), \tag{3.8}$$

$$\begin{aligned}\overline{\phi}_+(\nu,0^+)-\overline{\phi}_+(\nu,0^-)\ &=\ \frac{i\alpha}{\sqrt{2\pi}(\nu+\mu)},\\ \overline{\phi}_-(\nu,0^+)-\overline{\phi}_-(\nu,0^-)\ &=\ \frac{-i\alpha e^{i\mu l}}{\sqrt{2\pi}(\nu-\mu)},\end{aligned} \tag{3.9}$$

where $'$ denotes differentiation with respect to y. From Eqs. $(3.1)_2$, (3.6) and (3.8), we can write

$$\begin{aligned}&\overline{\phi}_+'(\nu,0)+\overline{\phi}_-'(\nu,0)e^{-i\nu l}+\overline{\phi}_1'(\nu,0^+)\\ &\qquad\qquad =\ i\gamma\left[\overline{\phi}_+(\nu,0^+)+\overline{\phi}_-(\nu,0^+)e^{-i\nu l}+\overline{\phi}_1(\nu,0^+)\right],\\ &\overline{\phi}_+'(\nu,0)+\overline{\phi}_-'(\nu,0)e^{-i\nu l}+\overline{\phi}_1'(\nu,0^-)\\ &\qquad\qquad =\ -i\gamma\left[\overline{\phi}_+(\nu,0^-)+\overline{\phi}_-(\nu,0^-)e^{-i\nu l}+\overline{\phi}_1(\nu,0^-)\right],\end{aligned} \tag{3.10}$$

After eliminating $\overline{\phi}_1'(\nu,0^+)$ from $(3.7)_1$ and $(3.10)_1$, $\overline{\phi}_1'(\nu,0^-)$ from Eqs. $(3.7)_2$ and $(3.10)_2$ and adding the resulting expressions, we arive at

$$\begin{aligned}&\overline{\phi}_+'(\nu,0)+\overline{\phi}_-'(\nu,0)e^{-i\nu l}-i\gamma N(\nu)J_1(\nu,0)\\ &\qquad\qquad =\ \overline{\phi}_0'(\nu,0)-\frac{\alpha\gamma}{2\sqrt{2\pi}}\left(\frac{1}{(\nu+\mu)}-\frac{e^{-i(\nu-\mu)l}}{(\nu-\mu)}\right),\end{aligned} \tag{3.11}$$

where

$$N(\nu)=1+\frac{k\beta}{\gamma},\qquad J_1(\nu,0)=\frac{1}{2}\left[\overline{\phi}_1(\nu,0^+)-\overline{\phi}_1(\nu,0^-)\right].$$

In a similar way by eliminating $\overline{\phi}_1(\nu,0^+)$ from Eqs. $(3.7)_1$ and $(3.10)_1$, $\overline{\phi}_1(\nu,0^-)$ from $(3.7)_2$ and $(3.10)_2$, and subtracting the resulting equations, we obtain

$$\begin{aligned}&\overline{\phi}_+(\nu,0^+)+\overline{\phi}_-(\nu,0^+)e^{-i\nu l}-\frac{N(\nu)J_1'(\nu,0)}{ik\beta}\\ &\qquad\qquad =\ \overline{\phi}_0(\nu,0)+\frac{i\alpha}{2\sqrt{2\pi}}\left[\frac{1}{\nu+\mu}-\frac{e^{-i(\nu-\mu)l}}{\nu-\mu}\right],\end{aligned} \tag{3.12}$$

where

$$J_1'(\nu,0)=\frac{1}{2}\left[\overline{\phi}_1'(\nu,0^+)-\overline{\phi}_1'(\nu,0^-)\right].$$

From Eqs. (3.3) and (3.11), we have

$$\begin{aligned}\overline{\phi}_+'(\nu,0)+\overline{\phi}_-'(\nu,0)e^{-i\nu l}-i\gamma N(\nu)J_1(\nu,0)\\ +\frac{\alpha\gamma N(\nu)}{2\sqrt{2\pi}}\left[\frac{1}{\nu+\mu}-\frac{e^{-i(\nu-\mu)l}}{\nu-\mu}\right]-\frac{\alpha k\beta}{2\sqrt{2\pi}}\left[\frac{1}{\nu+\mu}-\frac{e^{-i(\nu-\mu)l}}{\nu-\mu}\right]\\ =\frac{-kb\sin\vartheta_0}{\sqrt{2\pi}(\nu-k\cos\vartheta_0)}\left[1-e^{-i(\nu-k\cos\vartheta_0)l}\right].\end{aligned} \tag{3.13}$$

For the solution of Eq. (3.13), we make the following factorizations

$$\gamma=(k+\nu)^{1/2}(k-\nu)^{1/2}=K_+(\nu)K_-(\nu), \tag{3.14}$$

and

$$N(\nu)=N_+(\nu)N_-(\nu), \tag{3.15}$$

where $N_+(\nu)$ and $K_+(\nu)$ are regular for $\operatorname{Im}\nu>-\operatorname{Im}k$ and $N_-(\nu)$, and $K_-(\nu)$ are regular for $\operatorname{Im}\nu<\operatorname{Im}k$. The factorization (3.15) has been discussed by NOBLE [9, p. 164] and is directly quoted here as

$$N_\pm(\nu)=1-\frac{i\beta}{\pi}\left((\nu/k)^2-1\right)^{1/2}\cos^{-1}(\pm\nu/k). \tag{3.16}$$

Thus, substitution of Eqs. (3.14) and (3.15) in Eq. (3.13) yields

$$\begin{aligned}\overline{\phi}_+'(\nu,0)+\overline{\phi}_-'(\nu,0)e^{-i\nu l}+S_+(\nu)S_-(\nu)J_1(\nu,0)\\ +\frac{i\alpha S_+(\nu)S_-(\nu)}{2\sqrt{2\pi}}\left[\frac{1}{\nu+\mu}-\frac{e^{-i(\nu-\mu)l}}{\nu-\mu}\right]-\frac{\alpha k\beta}{2\sqrt{2\pi}}\left[\frac{1}{\nu+\mu}-\frac{e^{-i(\nu-\mu)l}}{\nu-\mu}\right]\\ =\frac{-kb\sin\vartheta_0}{\sqrt{2\pi}(\nu-k\sin\vartheta_0)}\left[1-e^{-i(\nu-k\cos\vartheta_0)l}\right].\end{aligned} \tag{3.17}$$

In Eq. (3.17), $S_+(\nu)$ $[=K_+(\nu)N_+(\nu)]$ is regular for $\operatorname{Im}\nu>-\operatorname{Im}k$ and $S_-(\nu)$ $[=K_-(\nu)N_-(\nu)]$ is regular for $\operatorname{Im}\nu<\operatorname{Im}k$. The unknown functions $\overline{\phi}_+'(\nu,0)$ and $\overline{\phi}_-'(\nu,0)$ in Eq. (3.17) have been determined using the procedure discussed by NOBLE [9, p. 166] and are given by

$$\begin{aligned}\overline{\phi}_+'(\nu,0)&=\frac{-kb\sin\vartheta_0}{\sqrt{2\pi}}\left(S_+(\nu)G_1(\nu)+T(\nu)S_+(\nu)C_1\right)\\ &\quad+\frac{\alpha}{2\sqrt{2\pi}}\left(\frac{k\beta}{(\nu+\mu)}-\frac{iS_+(\mu)S_+(\nu)}{(\nu+\mu)}+\frac{T(\nu)S_+(\nu)}{(k+\mu)}C_3\right),\\ \overline{\phi}_-'(\nu,0)&=\frac{-kb\sin\vartheta_0}{\sqrt{2\pi}}\left(S_-(\nu)G_2(-\nu)+T(-\nu)S_-(\nu)C_2\right)\\ &\quad+\frac{\alpha}{2\sqrt{2\pi}}\left(\frac{k\beta}{(\mu-\nu)}-\frac{iS_+(\mu)S_-(\mu)}{(\mu-\nu)}+\frac{T(-\nu)S_-(\nu)}{(k+\mu)}C_3\right).\end{aligned} \tag{3.18}$$

In Eqs. (3.18),

$$
\begin{aligned}
S_+(\nu) &= \sqrt{k+\nu}N_+(\nu), \qquad S_-(\nu) = e^{i\pi/2}\sqrt{\nu-k}N_-(\nu),\\
C_1 &= \frac{S_+(k)}{[1-T^2(k)S_+^2(k)]}\left[G_2(k)+G_1(k)T(k)S_+(k)\right],\\
C_2 &= \frac{S_+(k)}{[1-T^2(k)S_+^2(k)]}\left[G_1(k)+G_2(k)T(k)S_+(k)\right],\\
C_3 &= \frac{-iS_+(\mu)S_+(k)}{[1-T^2(k)S_+^2(k)]}\left[T(k)S_+(k)-e^{i\mu l}\right],
\end{aligned}
$$

$$
\begin{aligned}
G_1(\nu) &= \frac{1}{\nu-k\cos\vartheta_0}\left[\frac{1}{S_+(\nu)}-\frac{1}{S_+(k\cos\vartheta_0)}\right]-R_1(\nu)e^{ikl\cos\vartheta_0},\\
G_2(\nu) &= \frac{1}{\nu+k\cos\vartheta_0}\left[\frac{1}{S_+(\nu)}-\frac{1}{S_+(-k\cos\vartheta_0)}\right]e^{ikl\cos\vartheta_0}-R_2(\nu),
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
R_{1,2}(\nu) &= \frac{E_{-1}[W_{-1}\{-i(k\pm k\cos\vartheta_0)l\}-W_{-1}\{-i(k+\nu)l\}]}{2\pi i(\nu\mp k\cos\vartheta_0)}\\
T(\nu) &= \frac{1}{2\pi i}E_{-1}W_{-1}\{-i(k+\nu)l\},\\
E_{-1} &= 2\sqrt{l}e^{ikl-3i\pi/4},\\
W_{-1}(m) &= \Gamma\left(\frac{1}{2}\right)e^{m/2}(m)^{-3/4}W_{-1/4,-1/4}(m),
\end{aligned}
$$

[$m=-i(k+\nu)l$ and $W_{i,j}$ is a Whittaker function].

Now from Eqs. (3.1) and (3.6), we obtain

$$
\begin{aligned}
A_1(\nu)-A_2(\nu) &= e^{-i\nu l}\left[\overline{\phi}_-(\nu,0^+)-\overline{\phi}_-(\nu,0^-)\right]\\
&\quad+\left[\overline{\phi}_1(\nu,0^+)-\overline{\phi}_1(\nu,0^-)\right]+\left[\overline{\phi}_+(\nu,0^+)-\overline{\phi}_+(\nu,0^-)\right],\\
A_1(\nu)+A_2(\nu) &= \frac{1}{i\gamma}\Big\{\left[\overline{\phi}_1'(\nu,0^+)-\overline{\phi}_1'(\nu,0^-)\right]\\
&\quad+\left[\overline{\phi}_+'(\nu,0^+)-\overline{\phi}_+'(\nu,0^-)\right]+\left[\overline{\phi}_-'(\nu,0^+)-\overline{\phi}_-'(\nu,0^-)\right]e^{-i\nu l}\Big\}.
\end{aligned}
\tag{3.20}
$$

Using Eqs. (3.8) and (3.9) in Eqs. (3.20) and then adding and subtracting the resulting expressions we get

$$
A_1(\nu) = \frac{i\alpha}{2\sqrt{2\pi}}\left[\frac{1}{\nu+\mu}-\frac{e^{-i(\nu-\mu)l}}{\nu-\mu}\right]+J_1(\nu,0)+\frac{J_1'(\nu,0)}{i\gamma}, \tag{3.21}
$$

$$
A_2(\nu) = \frac{-i\alpha}{2\sqrt{2\pi}}\left[\frac{1}{\nu+\mu}-\frac{e^{-i(\nu-\mu)l}}{\nu-\mu}\right]-J_1(\nu,0)+\frac{J_1'(\nu,0)}{i\gamma}. \tag{3.22}
$$

Substituting the values of $J_1(\nu,0)$ and $J_1'(\nu,0)$ from Eqs. (3.11) and (3.12) into Eqs. (3.21) and (3.22), we obtain

$$\begin{aligned}A_1(\nu) = \frac{i\alpha}{2\sqrt{2\pi}}\left[\frac{1}{\nu+\mu}-\frac{e^{-i(\nu-\mu)l}}{\nu-\mu}\right]+\frac{1}{i\gamma N(\nu)}\Bigg[\overline{\phi}'_+(\nu,0)+\overline{\phi}'_-(\nu,0)e^{-i\nu l}\\ -\overline{\phi}'_0(\nu,0)+\frac{\alpha\gamma}{2\sqrt{2\pi}}\left\{\frac{1}{\nu+\mu}-\frac{e^{-i(\nu-\mu)l}}{\nu-\mu}\right\}\Bigg]\\ +\frac{ik\beta}{i\gamma N(\nu)}\Bigg[\overline{\phi}_+(\nu,0^+)+\overline{\phi}_-(\nu,0^+)e^{-i\nu l}-\overline{\phi}_0(\nu,0)\\ -\frac{i\alpha}{2\sqrt{2\pi}}\left\{\frac{1}{\nu+\mu}-\frac{e^{-i(\nu-\mu)l}}{\nu-\mu}\right\}\Bigg],\end{aligned}\tag{3.23}$$

$$\begin{aligned}A_2(\nu) = \frac{-i\alpha}{2\sqrt{2\pi}}\left[\frac{1}{\nu+\mu}-\frac{e^{-i(\nu-\mu)l}}{\nu-\mu}\right]-\frac{1}{i\gamma N(\nu)}\Bigg[\overline{\phi}'_+(\nu,0)+\overline{\phi}'_-(\nu,0)e^{-i\nu l}\\ -\overline{\phi}'_0(\nu,0)+\frac{\alpha\gamma}{2\sqrt{2\pi}}\left\{\frac{1}{\nu+\mu}-\frac{e^{-i(\nu-\mu)l}}{\nu-\mu}\right\}\Bigg]\\ +\frac{ik\beta}{i\gamma N(\nu)}\Bigg[\overline{\phi}_+(\nu,0^+)+\overline{\phi}_-(\nu,0^+)e^{-i\nu l}-\overline{\phi}_0(\nu,0)\\ -\frac{i\alpha}{2\sqrt{2\pi}}\left\{\frac{1}{\nu+\mu}-\frac{e^{-i(\nu-\mu)l}}{\nu-\mu}\right\}\Bigg].\end{aligned}\tag{3.24}$$

We note that

$$N(\nu)\approx 1+O(\beta),\qquad ik\beta/N(\nu)\approx O(\beta),$$

and assert that $(k\beta/\gamma)$ is very small provided that $|\nu/k|$ is not too near 1. This can be justified under small absorbing parameters β and low frequency of the acoustic wave. Thus using this Eqs. (3.16), (3.23) and (3.24) gives

$$\begin{aligned}N_\pm(\nu) &\approx 1\mp\frac{\nu\beta}{\pi\gamma},\\ A_1(\nu) &= -A_2(\nu)=\frac{1}{i\gamma}\left(\overline{\phi}'_+(\nu,0)+\overline{\phi}'_-(\nu,0)e^{-i\nu l}-\overline{\phi}'_0(\nu,0)\right).\end{aligned}\tag{3.25}$$

Note that in writing Eqs. (3.25), we have retained the terms of order $O(\beta/\gamma)$ and neglected the terms of $O(k\beta/\gamma)$.

Substitution of Eqs. (3.3) and (3.18) in Eq. $(3.25)_2$ yields

$$\begin{aligned}A_1\nu = -A_2(\nu) = \frac{kb\sin\vartheta_0}{\sqrt{2\pi}i\gamma(\nu-k\cos\vartheta_0)}\left\{\frac{S_+(\nu)}{S_+(k\cos\vartheta_0)}-\frac{S_+(-\nu)e^{-i(\nu-k\cos\vartheta_0)l}}{S_+(-k\cos\vartheta_0)}\right\}\\ -\frac{kb\sin\vartheta_0}{\sqrt{2\pi}i\gamma}\Big\{S_+(\nu)T(\nu)C_1-S_+(\nu)R_1(\nu)e^{ikl\cos\vartheta_0}\\ +S_+(-\nu)R_2(-\nu)e^{-i\nu l}+C_2T(-\nu)S_+(-\nu)e^{-i\nu l}\Big\}\\ +\frac{\alpha}{2\sqrt{2\pi}i\gamma}\Bigg\{k\beta\left[\frac{1}{\nu+\mu}+\frac{e^{-i\nu l}}{\mu-\nu}\right]-iS_+(\mu)\left[\frac{S_+(\nu)}{(\nu+\mu)}+\frac{e^{-i\nu l}S_+(-\nu)}{\mu-\nu}\right]\\ +\frac{C_3}{(k+\mu)}\left[T(\nu)S_+(\nu)+e^{-i\nu l}T(-\nu)S_+(-\nu)\right]\Bigg\}.\end{aligned}\tag{3.26}$$

Now putting the values of $A_1(\nu)$ in Eq. (3.6) and taking inverse Fourier transform the field $\phi(x,y)$ can be written as

$$\phi(x,y) = \phi^{\text{sep}}(x,y) + \phi^{\text{int}}(x,y), \tag{3.27}$$

where

$$\begin{aligned}
\phi^{\text{sep}}(x,y) = {} & \frac{kb\sin\vartheta_0}{2\pi}\int_{-\infty}^{\infty}\frac{S_+(\nu)e^{i\gamma y - i\nu x}}{i\gamma(\nu - k\cos\vartheta_0)S_+(k\cos\vartheta_0)}d\nu \\
& - \frac{kb\sin\vartheta_0}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i(\nu-k\cos\vartheta_0)l}S_+(-\nu)e^{i\gamma y-i\nu x}}{i\gamma(\nu-k\cos\vartheta_0)S_+(-k\cos\vartheta_0)}d\nu \\
& + \frac{\alpha}{4\pi}\left[k\beta\int_{-\infty}^{\infty}\frac{1}{i\gamma}\left\{\frac{1}{\nu+\mu}+\frac{e^{-i\nu l}}{\mu-\nu}\right\} - iS_+(\mu)\int_{-\infty}^{\infty}\frac{1}{i\gamma}\left\{\frac{S_+(\nu)}{\nu+\mu}\right.\right. \\
& \left.\left. + \frac{e^{-i\nu l}S_+(-\nu)}{\mu-\nu}\right\}\right]e^{i\gamma y - i\nu x}d\nu,
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
\phi^{\text{int}}(x,y) = {} & \frac{kb\sin\vartheta_0}{2\pi}\int_{-\infty}^{\infty}\frac{1}{i\gamma}\Big[S_+(\nu)R_1(\nu)e^{ikl\cos\vartheta_0} \\
& - S_+(-\nu)R_2(-\nu)e^{-i\nu l} - S_+(\nu)T(\nu)C_1 - T(-\nu)S_+(-\nu)e^{-i\nu l}C_2\Big]e^{i\gamma y - i\nu x}d\nu \\
& + \frac{\alpha C_3}{4\pi(k+\mu)}\int_{-\infty}^{\infty}\frac{1}{i\gamma}\left[T(\nu)S_+(\nu) + e^{-i\nu l}T(-\nu)S_+(-\nu)\right]e^{i\gamma y - i\nu x}d\nu.
\end{aligned} \tag{3.29}$$

In order to solve the integrals appearing in Eqs. (3.28) and (3.29), we put $x = r\cos\vartheta$, $y = r\sin\vartheta$ and deform the contour by the transformation $\nu = -k\cos(\vartheta + i\xi)$, $(0 < \vartheta < \pi$, $-\infty < \xi < \infty)$. Hence after using Eqs. (2.6) and (3.4), we have for large kr

$$\begin{aligned}
\phi^{\text{sep}}(x,y) = {} & \frac{ie^{ik(r+r_0)}}{4\pi k(\cos\vartheta + \cos\vartheta_0)(rr_0)^{1/2}}f_1(-k\cos\vartheta) \\
& + \frac{\alpha}{2(2\pi kr)^{1/2}}\left[\beta e^{i\pi/4}\left\{\frac{1}{(\cos\vartheta_1 - \cos\vartheta)} + \frac{e^{ikl\cos\vartheta}}{(\cos\vartheta_1 + \cos\vartheta)}\right\}\right. \\
& \left. + \frac{e^{-i\pi/4}S_+(k\cos\vartheta_1)}{k}\left\{\frac{S_+(-k\cos\vartheta)}{(\cos\vartheta_1 - \cos\vartheta)} + \frac{e^{ikl\cos\vartheta}S_+(k\cos\vartheta)}{(\cos\vartheta_1+\cos\vartheta)}\right\}\right]e^{ikr},
\end{aligned} \tag{3.30}$$

$$\phi^{\text{int}}(x,y) = \frac{ie^{ik(r+r_0)}}{4\pi(rr_0)^{1/2}}f_2(-k\cos\vartheta) + \frac{\alpha e^{i(kr+\pi/4)}}{2(2\pi kr)^{1/2}}f_3(-k\cos\vartheta). \tag{3.31}$$

In Eqs. (3.30) and (3.31)

$$f_1(-k\cos\vartheta) = -\sin\vartheta_0\left[\frac{S_+(-k\cos\vartheta)}{S_+(k\cos\vartheta_0)} - \frac{S_+(k\cos\vartheta)e^{ikl(\cos\vartheta+\cos\vartheta_0)}}{S_+(-k\cos\vartheta_0)}\right],$$

$$f_2(-k\cos\vartheta) = \sin\vartheta\Big[S_+(-k\cos\vartheta)R_1(-k\cos\vartheta)e^{ikl\cos\vartheta_0} - S_+(k\cos\vartheta)R_2(k\cos\vartheta)e^{ikl\cos\vartheta} - S_+(-k\cos\vartheta)T(-k\cos\vartheta)C_1 - S_+(k\cos\vartheta)T(k\cos\vartheta)C_2e^{ikl\cos\vartheta}\Big],$$

$$f_3(-k\cos\vartheta) = \frac{C_3}{(k+k\cos\vartheta_1)}\Big[T(-k\cos\vartheta)S_+(-k\cos\vartheta) + e^{ikl\cos\vartheta}T(k\cos\vartheta)S_+(k\cos\vartheta)\Big].$$

From Eqs. (3.27), (3.30) and (3.31), we obtain

$$\begin{aligned}\phi(x,y) &= \frac{ie^{ik(r+r_0)}}{4\pi(rr_0)^{1/2}k}\left[\frac{f_1(-k\cos\vartheta)}{(\cos\vartheta+\cos\vartheta_0)} + kf_2(-k\cos\vartheta)\right]\\ &+ \frac{\alpha e^{i(kr+\pi/4)}}{2(2\pi kr)^{1/2}}\left[\beta\left\{\frac{1}{(\cos\vartheta_1-\cos\vartheta)} + \frac{e^{ikl\cos\vartheta}}{(\cos\vartheta_1+\cos\vartheta)}\right\} + f_3(-k\cos\vartheta)\right.\\ &\left. - \frac{iS_+(k\cos\vartheta_1)}{k}\left\{\frac{S_+(-k\cos\vartheta)}{(\cos\vartheta_1-\cos\vartheta)} + \frac{e^{ikl\cos\vartheta}S_+(k\cos\vartheta)}{(\cos\vartheta_1+\cos\vartheta)}\right\}\right]. \end{aligned} \tag{3.32}$$

In the limit $r \to 0$, Eq. (3.32) shows that

$$\begin{aligned}\phi(x,y) \approx 2r^{1/2}\Bigg[&-\frac{e^{ikr_0}}{4\pi(r_0)^{1/2}}\left\{f_1(-k\cos\vartheta) + \frac{k}{2}f_2(-k\cos\vartheta)\right\}\\ &+ \frac{i\alpha e^{i\pi/4}k}{(2\pi k)^{1/2}}\Bigg\{\beta\left(1+e^{ikl\cos\vartheta}\right)\\ &- \frac{iS_+(k\cos\vartheta_1)}{k}\left(S_+(-k\cos\vartheta) + e^{ikl\cos\vartheta}S_+(k\cos\vartheta)\right) + \frac{f_3(-k\cos\vartheta)}{2}\Bigg\}\Bigg],\end{aligned}$$

where we have neglected the terms which are constant and $O(r)$. Therefore, the velocity will remain bounded at the edge if and only if the co-efficient of $r^{1/2}$ vanishes. Hence the Kutta–Joukowski condition requires that

$$\alpha = \frac{e^{ikr_0-3i\pi/4}}{(2\pi kr_0)^{1/2}}\mathcal{G}_1(-k\cos\vartheta), \tag{3.33}$$

where

$$\begin{aligned}\mathcal{G}_1(-k\cos\vartheta) = &\left\{f_1(-k\cos\vartheta) + \frac{k}{2}f_2(-k\cos\vartheta)\right\}\Bigg\{\beta\left(1+e^{ikl\cos\vartheta}\right)\\ &- \frac{iS_+(k\cos\vartheta_1)}{k}\left(S_+(-k\cos\vartheta) + e^{ikl\cos\vartheta}S_+(k\cos\vartheta)\right) + \frac{f_3(-k\cos\vartheta)}{2}\Bigg\}^{-1}.\end{aligned}$$

Using Eq. (3.33) in Eq. (3.32) the far field is given by

$$\phi = \phi_A + \phi_W, \tag{3.34}$$

where ϕ_A denotes that part of ϕ that arises when there is no wake and ϕ_W the part that arises when there is a wake. They are explicitly given by

$$\phi_A = \frac{ie^{ik(r+r_0)}}{4\pi(rr_0)^{1/2}k}\mathcal{G}_2(-k\cos\vartheta), \tag{3.35}$$

$$\phi_W = \frac{ie^{ik(r+r_0)}}{4\pi(rr_0)^{1/2}k}\mathcal{G}_3(-k\cos\vartheta). \tag{3.36}$$

In Eqs. (3.35) and (3.36)

$$\mathcal{G}_2(-k\cos\vartheta) = \left[\frac{f_1(-k\cos\vartheta)}{(\cos\vartheta+\cos\vartheta_0)} + kf_2(-k\cos\vartheta)\right],$$

$$\begin{aligned}\mathcal{G}_3(-k\cos\vartheta) = -\mathcal{G}_1(-k\cos\vartheta)\Bigg[f_3(-k\cos\vartheta) &+ \beta\left\{\frac{1}{(\cos\vartheta_1-\cos\vartheta)} + \frac{e^{ikl\cos\vartheta}}{(\cos\vartheta_1+\cos\vartheta)}\right\}\\ &- \frac{iS_+(k\cos\vartheta_1)}{k}\left\{\frac{S_+(-k\cos\vartheta)}{(\cos\vartheta_1-\cos\vartheta)} + \frac{e^{ikl\cos\vartheta}S_+(k\cos\vartheta)}{(\cos\vartheta_1+\cos\vartheta)}\right\}\Bigg].\end{aligned}$$

4. Conclusion

We have solved a new diffraction problem using a method invented by Jones. As far as we know, this is the first new problem to be solved by this method. We also note from Eqs. (3.30) and (3.31) that ϕ^{sep} consists of two parts each representing the diffracted field produced by the edges at $x = 0$ and $x = -l$ respectively, as though the other edges were absent while ϕ^{int} gives the interaction of one edge upon the other. Further, from Eq. (3.34), it is observed that the field caused by the Kutta–Joukowski condition will be substantially in excess of that in its absence when the source is near the edge. The results for no wake situation can be obtained by taking $\alpha = 0$. Finally, the results correspond to the rigid barrier if we put $\beta = 0$ in Eq. (3.34). Thus the consideration of absorbing strip with wake presents a more generalized model in the theory of diffraction.

References

[1] U.J. Kurze, J. Acoust. Soc. Amer., **55**, 504 (1974).

[2] A.D. Rawlins, Proc. Roy. Soc. London, **A346**, 469 (1975).

[3] J.E. Ffowcs–Williams and L.H. Hall, J. Fluid Mech., **40**, 657 (1970).

[4] J.B. Alblas, Appl. Sci. Res., **A6**, 237 (1957).

[5] D.S. Jones, J. Inst. Maths. Applic., **9**, 114 (1972).

[6] R. Balasubramanyam, IMA J. Appl. Maths., **33**, 1, 71 (1984).

[7] P.M. Morse and K.U. Ingard, *Encyclopedia of Phys. Acoustic I*, Springer Verlag, Berlin 1961.

[8] D.S. Jones, *The theory of electromagnetism*, Pergamon, Oxford 1964.

[9] B. Noble, *Methods based on the Wiener–Hopf technique*, Pergamon Press, London 1958.