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AXIAL NONLINEAR FIELD OF A VIBRATING CIRCULAR TRANSDUCER

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The on-axis cumulative growth of nonlinear effects resulting from a monochromatic excitation of a circular source mounted in an infinite rigid baffle is analyzed by perturbation analysis. The first order (linear) signal is the summation of two propagating planar waves: one emanating from the center of the source and the other originating from the source boundary. The mutual nonlinear interaction and propagation of these two waves are analyzed on the basis of the nonlinear wave equation governing the velocity potential. Nonuniform validity of the pressure expression is corrected by the method of renormalization and thereby obtaining uniformly accurate expression in the near as well as far fields. Asymptotic trends at long range are derived which resulted in a Fourier series description for the pressure signal. The results yield a computationally efficient model that can predict the spectral components as well as the temporal waveform. The predicted results are compared favorably with experimental observations over a wide range of variable parameters.

1. Introduction

The subject of this paper first came to our attention when considering the problem of distortion of two planar waves interacting at arbitrary angles [1] and the problems of nonlinear interaction and dispersion of higher order modes in waveguides [2–3]. We recognized that the linear signal on axis of symmetry of a harmonically vibrating circular plane transducer is a linear superposition of two simple planar waves of opposite amplitudes. One of these waves is propagating parallel to the face of the piston and its propagation distance is measured from the center of the piston along the axis of symmetry. The other wave propagates in a direction that makes a nonzero angle with the propagation direction of the first wave and its propagation distance is measured from the edge of the piston. The significant aspect of this exact simple representation for the linearized solution of the acoustic field on-axis provides a convenient frame work from which one may workout the second order potential.

The radiation problem of the harmonically, vibrating, plane circular transducer mounted in an infinite rigid baffle is one of several canonical problems in acoustics. Its linear solution is the most well known and the most fundamental one. The analytical efforts devoted to this problem is extensive. Although hundreds of paper are cited in literature, this problem still requires more work to obtain a simple and complete description of the acoustic field.

There are two basic formulations for the linear problem: the Rayleigh surface integral and the King integral [4–5]. The Rayleigh integral treats the signal as superposition of spherical wavelets which are generated by infinitesimal sources on the piston face. The King integral results from a Hankel (Fourier-Bessel) integral transformation transverse to the axis of symmetry. The acoustic medium in such formulation becomes a waveguide of infinite diameter. The Rayleigh surface integral is transformed into the SCHOCH line integral [6] by using observer related coordinates. Schoch solution is essentially a sum of the plane wave and diffraction integrals. Although these formulations ar straightforward, the basic difficulty with the linearized piston problem remains, i.e., while these exact integral forms are simple in appearance, they can not be written in other exact simpler forms which can be easily evaluated except by numerical integration.

The treatment of nonlinear effects which arise when the transducer is driven sinusoidally at a high amplitude has been analysed by several investigators. INGENITO and WILLIAMS [7] employed a perturbation series for the potential function in which the leading term was described by the Rayleigh integral. Their solution was not uniformly valid from the view point of the perturbation theory corresponding to a limitation to the field close to the transducer (Fresnel zone). In addition, it is only valid for situations where the axial wavelength is very small compared to the transducer radius ($ka > 100$). Aside from these restrictions, their formulation does not address higher harmonics and depletion of the fundamental. Consequently, it does not provide sufficient information to predict waveform.

GINSBERG [8–9] described the linearized signal by the King integral and used an asymptotic analysis to find the expression for the velocity potential. Only the cumulative part was retained in that analysis since the expression for the second order potential was quite intricate.

AANONSEN *et al.* [10] have used a finite difference method to calculate the harmonic contents of an axially symmetric acoustic beam by solving the parabolic wave equation in the frequency domain. The main limitions introduced by the parabolic approximation are the frequency should be high ($ka \gg 1$), the angle off-axis must be small and the distance from the source must not be small. BACON and BAKER [11] and BACON *et al.* [12] have compared the measured nearfield pressure with the numerical predictions of the parabolic approximation of the nonlinear wave equation. The numerical scheme is quite time consuming since the conditions required for step sizes and the number of retained harmonics to get a stable accurate solution are rather sever.

Recent work by TOO and GINSBERG [13] has modified the nonlinear progressive wave equation (NPE) and the associated computer code, which has been originally developed by MCDONALD and KUPERMAN [14], to describe the axisymmetric sound beams in the paraxial approximation. The basic assumption introduced in the derivation of this equation is that the particle velocity is in the direction in which the signal propagates. Therefore, like many previous models, NPE is inappropriate to the domain inside the Fresnel zone. Apart from this shortcoming, a suitable computer scheme is needed to initialize the window that is convected by NPE as the wave advances. The output results are dependent on the scheme used as well as on the choice of the boundaries for that window. The computational cost to implement NPE is deemed to be excessive.

Based on the quasi-linear approximations of the solution of Khokhlof-Zabolotskaya-Kuznetsov (KZK) equation for a Gaussian source, COULOUVART [15] has derived a uniform expression of the nonlinear effects in the sound beam by renormalizing the retarded time. The KZK equation is a modified Burgers-type equation which often referred as the paraxial parabolic equation. As alluded previously, several approximations must be made to derive the parabolic equation. Accordingly, the KZK equation is only suitable in the vicinity of the axis of the sound beam. Comparison between the experimental measurements on a circular transducer generating short pulses in water and the numerical solution of the KZK equation has been carried out by BAKER and HUMPHERY [16]. They used the computer code developed previously by AANONSEN *et al.* [10].

In this paper a perturbation analysis has been adopted to describe the distortion of the sound beams on axis of symmetry of a circular, plane, piston mounted in an infinite rigid baffle and driven sinusoidally at a high amplitude. The analytical model presented is derived from the prescribed boundary conditions on the source and the baffle, and from the nonlinear wave equation governing the velocity potential. The analysis consistenly accounts for the nonlinearity and diffraction. Dissipative effect has been discarded in the present analysis. The perturbation method of renormalization is invoked to eliminate secular terms from the pressure expression. The solution obtained is valid for the near, as well as the farfield, provided that the location is closer to the source than the shock formation distance.

Asymptotic trends, when the field point is distant compared to the radius of the transducer, are derived. The computational algorithm is simple and effecient. The predicted results are in good agreement with experimental works of several investigators over a wide range of variable parameters.

2. Formulation

Consider a circular plane transducer source of radius a lies in the plane $z = 0$ and centered at $x = y = 0$. The rest of the source plane is a rigid baffle. The transducer is driven continuously at a monochromatic angular frequency ω and radiates a sound beam symmetric about the z-axis into a dissipationless fluid half-space $z > 0$. Denote the nondimensional time variable as t. The corresponding dimensional position coordinates $(x/k,\, z/k)$ and the dimensional time is t/ω, where $k = \omega/c_0$ is the wavenumber of a nominal planar wave. c_0 is the small signal speed of sound in the linear theory. The dimensionless velocity potential ϕ is related to the particle velocity components such that $v_z = c_0(\partial\phi/\partial z)$, $v_x = c_0(\partial\phi/\partial x)$. The continuity of the particle velocity at the interface must be imposed at the displaced location of the transducer in the direction normal to the deformed surface. By making use of the Taylor series expansion

$$\left(\frac{\partial\phi}{\partial z}\bigg|_{z=0+w} = \frac{\partial\phi}{\partial z}\bigg|_{z=0} + \frac{\partial^2\phi}{\partial z^2}\bigg|_{z=0} w + \dots\right)$$

one transfers the boundary condition to a stationary boundary at $z = 0$. Therefore, for axisymmetric constant amplitude displacement of the transducer (uniform velocity distribution), the boundary condition can be written as

$$\frac{\partial \phi}{\partial z} = \frac{\partial w}{\partial t} - w\frac{\partial^2 \phi}{\partial z^2} + O(w^3) \quad \text{at} \quad z = 0, \tag{2.1}$$

where the dimensionless displacement w is given by

$$w = -\frac{\varepsilon}{2}e^{it} + \text{c.c.} \tag{2.2}$$

For weakly nonlinear waves, the acoustic Mach number ε is a finite parameter with $|\varepsilon| \ll 1$. In general, c.c will denote the complex conjugate of the preceding term. The nonlinear wave equation governing ϕ is [17]

$$\nabla^2\phi - \frac{\partial^2 \phi}{\partial t^2} = 2(\beta_0 - 1)\frac{\partial \phi}{\partial t}\nabla^2\phi + \frac{\partial}{\partial t}(\nabla\phi\cdot\nabla\phi) + O(\phi^3), \tag{2.3}$$

where β_0 is the coefficient of nonlinearity. For ideal gas $\beta_0 = (\gamma + 1)/2$, where γ is the ratio of specific heats. The acoustic pressure is related to the velocity potential by the Bernoulli equation which can be written in a binomial expansion as follows

$$\frac{p}{\varrho_0 c_0^2} = -\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi - \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2\right] + O(\phi^3). \tag{2.4}$$

In addition to Eq. (2.1), the other boundary condition on ϕ is that the signal should appear to be coming from the source, not travelling towards it. The other requirement imposed on ϕ is that the physical state variables, such as the acoustic pressure or particle velocity, derived from it should be bounded for large z.

In accord with standard procedures, one expands ϕ in a straightforward perturbation series. A slight modification of such an expansion leads to a sequence of equations that more prominantly displays the role of β_0 in the formation of nonlinear distortion. Specifically, one lets

$$\phi = \varepsilon\phi_1 + \varepsilon^2\left[\frac{1}{2}\frac{\partial}{\partial t}(\phi_1^2) + \phi_2\right] + O(\varepsilon^3). \tag{2.5}$$

The equations governing ϕ_1 and ϕ_2 are found by collecting like powers of ε in Eqs. (2.1), (2.2) and (2.3). The first order equations are

$$\begin{aligned} \nabla^2\phi_1 - \frac{\partial^2 \phi_1}{\partial t^2} &= 0, \\ \left.\frac{\partial \phi_1}{\partial z}\right|_{z=0} &= \frac{1}{2i}e^{it} + \text{c.c.} \end{aligned} \tag{2.6}$$

The resulting second order equations are

$$\begin{aligned} \nabla^2\phi_2 - \frac{\partial^2 \phi_2}{\partial t^2} &= \beta_0\frac{\partial}{\partial t}\left(\frac{\partial \phi_1}{\partial t}\right)^2, \\ \left.\frac{\partial \phi_2}{\partial z}\right|_{z=0} &= -\left.\left[\frac{1}{2}\frac{\partial^2}{\partial z\partial t}(\phi_1^2) + \frac{1}{\varepsilon}w\frac{\partial^2 \phi_1}{\partial z^2}\right]\right|_{z=0}. \end{aligned} \tag{2.7}$$

It is a straightforward matter to solve the linearized Eqs. (2.6) by the Rayleigh integral or King integral [5] with the following simple results for the on-axis signal

$$\phi_1 = -\frac{1}{2}\left[e^{i(t-z_1)} - e^{i(t-z)}\right] + \text{c.c}, \tag{2.8}$$

where

$$z_1 = \left(z^2 + k^2a^2\right)^{1/2}. \tag{2.8$'$}$$

Equations (2.8), (2.8′) represent the exact first-order solution as a linear superposition of two planar waves of opposite amplitudes. The time delay of the first wave corresponds to the propagation time from the edge of the projector to the spatial point, while the time delay of the second wave corresponds to the propagation time from the center of the projector to the spatial point. This simple representation of the linarized solution on-axis is of crucial importance because from this solution one may work out the second-order potential.

3. Evaluation of the second order potential

The first step in deriving ϕ_2 is to use Eq. (2.8) to form the inhomogeneous terms in Eq. $(2.7)_1$. This is easily performed by considering the first wave that emanates from the edge of the transducer equivalent to a planar wave emanating from its center (similar to the second wave) with propagation direction making an angle $\theta = \tan^{-1}(ka/z)$ with the z-axis (see Fig. 1). Accordingly, the observation point is considered as $\mathbf{x} = (ka, z)$.

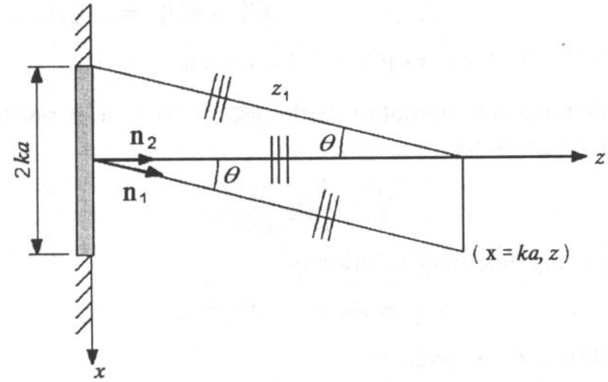

Fig. 1. Geometry of the superposition of the center wave and the edge wave.

Therefore Eq. (2.8) can be expressed in Cartesian coordinates. For example, let $\mathbf{n}_1 \cdot \mathbf{x} = x\sin\theta + z\cos\theta$ and $\mathbf{n}_2 \cdot \mathbf{x} = z$, where $\mathbf{n}_i$ is the unit vector in the direction of the propagation of wave i. $\mathbf{n}_1 \cdot \mathbf{x} \equiv z_1$ when $x = ka$. The resulting equation governing the second order potential is given by

$$\nabla^2\phi_2 - \frac{\partial^2\phi_2}{\partial t^2} = -\frac{i\beta_0}{2}\left[e^{2i(t-z_1)} + e^{2i(t-z)} - 2e^{i(2t-z-z_1)}\right] + \text{c.c.} \tag{3.1}$$

The first two exponentials in Eq. (3.1) excite second harmonics. Such signals propagate parallel to the two waves forming ϕ_1 which are the homogeneous solutions of the linearized wave equation. The last inhomogeneous term is due to the nonlinear interaction of the two waves forming ϕ_1. It excites a second harmonic whose propagation direction makes an angle $\theta/2$ with the z-axis.

The solution of Eq. (3.1) consists of the complementary solution and the particular solution. The form of the right-hand side of this equation suggests that the latter solution is the superposition of the solutions associated with each of these inhomogeneous terms. These solutions may be obtained by the method of variation of parameters, in which the amplitudes of the homogeneous solution is considered to be unknown functions. Thus let

$$\begin{aligned} \phi_2 &= u(z)e^{2it} + \text{c.c}, \\ u &= C_1(z)e^{-2iz_1} + C_2(z)e^{-2iz} + C_3(z)e^{-i(z_1+z)}. \end{aligned} \tag{3.2}$$

It should be noted that the unknown functions C_j depend on the axial distance only. The harmonic nature of the excitation eliminates the dependence of these functions on t. Similarly, the second order potential should depend on x through the phase variable z_1 since one only seeks the on-axis expression. This restriction could not be satisfied if C_j were functions of x.

The result of requiring that Eqs. (3.2) satisfy Eq. (3.1) is a set of uncoupled differential equations for the amplitude functions. These equations are found to be

$$\begin{aligned} C_1'' - 4i\cos\theta\, C_i' &= -\frac{i}{2}\beta_0\,, \\ C_2'' - 4C_2' &= -\frac{i}{2}\beta_0\,, \\ C_3'' - 2i(1+\cos\theta)C_3' + 2(1-\cos\theta)C_3 &= i\beta_0, \end{aligned} \tag{3.3}$$

where the prime denotes differentiation with respect to z. The particular solution of Eq. $(3.3)_1$ is readily found to be

$$C_1^P = \frac{1}{8}\beta_0\frac{z}{\cos\theta}\,. \tag{3.4$_1$}$$

The corresponding complementary solution is

$$C_1^h = A_1 + A_2e^{4iz\cos\theta}. \tag{3.4$_2$}$$

Therefore the amplitude C_1 is given by

$$C_1 = \frac{1}{8}\beta_0\frac{z}{\cos\theta} + A_1 + A_2e^{4iz\cos\theta}. \tag{3.5}$$

Similarly

$$C_2 = \frac{1}{8}\beta_0 z + A_3 + A_4e^{4iz}. \tag{3.6}$$

The particular solution of Eq. $(3.3)_3$ is given by

$$C_3^P = \frac{i\beta_0}{2(1-\cos\theta)}\,. \tag{3.7$_1$}$$

It is convenient to let C_3^P appear explicitly in the complementary solution which is therefore written as

$$C_3^h = \frac{i\beta_0}{2(1-\cos\theta)}\left[A_5 e^{\lambda_1 z} + A_6 e^{\lambda_2 z}\right], \tag{3.7$_2$}$$

where λ_1 and λ_2 are the roots of the characteristic equation

$$\lambda^2 - 2i(1+\cos\theta)\lambda + 2(1-\cos\theta) = 0. \tag{3.8$_1$}$$

These roots are found to be

$$\lambda_{1,2} = i\left[(1+\cos\theta) \mp (3+\cos^2\theta)^{1/2}\right]. \tag{3.8$_2$}$$

Combining Eqs. (3.7)$_1$ and (3.7)$_2$ yields

$$C_3 = \frac{i\beta_0}{2(1-\cos\theta)}\left[1 + A_5 e^{\lambda_1 z} + A_6 e^{\lambda_2 z}\right]. \tag{3.9}$$

The expresion for ϕ_2, obtained by substituting Eqs. (3.5), (3.6) and (3.9) into Eqs. (3.2) must satisfy the condition that ϕ_2 represents an outgoing wave in the z-direction. Consequently, ϕ_2 must only contain negative imaginary exponentials in the z-variable. Satisfaction of this condition requires that $A_2 = A_4 = A_6 = 0$. The remaining terms yield

$$u = \left(\frac{i\beta_0 z}{8\cos\theta} + A_1\right)e^{-2iz_1} + \left(\frac{i\beta_0 z}{8} + A_3\right)e^{-2iz} + \frac{1\beta_0}{2(1-\cos\theta)}\left[1 + A_5 e^{i\lambda z}\right]e^{-i(z_1+z)}, \tag{3.10}$$

where λ is redefined as the modulus of λ_1 ($\lambda = \lambda_1/i$).

Letting $\theta \to 0$ ($ka/z \to 0$) in Eq. (3.10) results in a singularity in the coefficient of the last term. This is similar to the behaviour obtained in the course of investigating the near resonant solution of the one-degree of freedom harmonic oscillator as discussed by GINSBERG [18]. When the forcing frequency approaches the natural frequency for this system, the amplitude of the particular solution increases as does the portion of the homogeneous solution that cancels the initial value of the particular solution. The combination of these two solutions is a temporal beating response that rises from zero at the initial time. As the difference between the forcing frequency and the natural frequency decreases further, the period of each beat increases, until ultimately, when the two frequencies are equal, only rising portion survives. The corresponding resonant response is a harmonic whose amplitude grows linearly with time.

In a similar manner, the singularity of Eq. (3.10) at $\theta \to 0$ may be removed by appropriate selection of the coefficient of the homogeneous solution A_5. First, λ is expanded in a Taylor series for small $(1-\cos\theta)$

$$\begin{aligned} \lambda &= (1+\cos\theta) - (1+\cos\theta)\left[1 + \frac{2(1-\cos\theta}{(1+\cos\theta)^2}\right]^{1/2} = -\frac{1-\cos\theta}{1+\cos\theta} + \dots, \\ \exp(i\lambda z) &= \exp\left[-i\frac{1-\cos\theta}{1+\cos\theta}z\right] = 1 - \frac{i(1-\cos\theta)z}{1+\cos\theta} + \dots. \end{aligned} \tag{3.11}$$

Threfore the corresponding asymptotic form for the last term in Eq. (3.10) is

$$u_3 = \frac{i\beta_0}{2(1-\cos\theta}\left\{1 + A_5\left[1 - \frac{i(1-\cos\theta)z}{1+\cos\theta}\right]\right\} e^{-i(z_1+z)}. \tag{3.12}$$

The singularity for $\theta \to 0$ is cancelled if one chooses the leading term in $A_5 = -1$. Thus let $A_5 = -1 + A_5^*$, where the coefficient A_5^* may depend on $(1 - \cos\theta)$ in any manner such that it is not singular as $\theta \to 0$. The second order potential is now found from Eqs. (3.2) and (3.10) to be

$$\phi_2 = \left(\frac{i\beta_0 z}{8\cos\theta} + A_1\right) e^{2i(t-z_1)} + \left(\frac{i\beta_0 z}{8} + A_3\right) e^{2i(t-z)} + \frac{i\beta_0}{2(1-\cos\theta)}\left[1 - (1 - A_5^*)e^{i\lambda z}\right] e^{i(2t-z_1-z)} + \text{c.c.} \tag{3.13}$$

The foregoing expression for ϕ_2 should satisfy the boundary condition given by Eq. $(2.7)_2$. This could be achieved by the appropriate selection of the coefficients A_1, A_3 and A_5^*. Each of them describes a homogeneous solution of the wave equation associated with the second order potential. Thus, they represent effects that $O(\varepsilon^2)$ at all locations. In contrast, observable distortion phenomena are associated with the second order terms that grow with increasing distance. However, the bounded $O(\varepsilon^2)$ effects might be significant near the transducer. Therefore, satisfying Eq. $(2.7)_2$ and combining the resulting expression for ϕ_2 with the linearized solution given by Eq. (2.8), according of Eq. (2.5), one arrives at the following expression for the potential

$$\begin{aligned}\phi = &-\frac{\varepsilon}{2}\left[e^{i(t-z_1)} - e^{i(t-z)}\right] + \frac{\varepsilon^2}{8}\left\{\left(\beta_0\frac{z}{\cos\theta} + 2i\right) e^{2i(t-z_1)}\right. \\ &+ \left(\beta_0 z - \frac{i\beta_0}{2} - i\right) e^{2i(t-z)} + 4i\left[\frac{\beta_0}{(1-\cos\theta)}\left(1 - e^{i\lambda z}\right)\right. \\ &+ \left.\left.\left(1 - \frac{1}{\sqrt{3}}\right)(\beta_0 - 1)\right] e^{i(2t-z_1-z)} - 2z\right\} + \text{c.c} + O(\varepsilon^3).\end{aligned} \tag{3.14}$$

4. Evaluation of the pressure

The pressure is related to the potential function by Eq. (2.4). The quadratic products in that relation represent effects that are uniformly $O(\varepsilon^2)$ at all locations. These bounded effects might be significant near the projector. Thus differentiating Eq. (3.14) according to Eq. (2.4) yields the following expression for the pressure

$$\begin{aligned}\frac{p}{\varrho_0 c_0^2} = &\,\varepsilon\frac{i}{2}\left[e^{i(t-z_1)} - e^{i(t-z)}\right] - \varepsilon^2\frac{i\beta_0}{4}\left[\frac{z}{\cos\theta}e^{(2i(t-z_1)} + ze^{2i(t-z)}\right. \\ &+ \left.\frac{4}{1-\cos\theta}\left(1 - e^{i\lambda z}\right) e^{i(2t-z_1-z)}\right] + p_{NS} + \text{c.c} + O(\varepsilon^3),\end{aligned} \tag{4.1}$$

where

$$p_{NS} = \varepsilon^2 \left[\frac{1}{2} e^{2i(t-z_1)} - \frac{1}{8}(\beta_0 + 2) e^{2i(t-z)} + B e^{i(2t-z_1-z)} - \frac{1}{4}(1 - \cos\theta) e^{i(z_1 - z)} \right], \quad (4.2)$$

and

$$B = \frac{1}{\sqrt{3}} \left[\left(\sqrt{3} - 1\right) \beta_0 + 1 \right] e^{i\lambda z} - \frac{1}{4}(3 + \cos\theta). \quad (4.3)$$

Equation (4.1) reveals that the cumulative growth of the $O(\varepsilon^2)$ signal originates from the first two terms of $O(\varepsilon^2)$. This is manifested by the increase in the magnitudes of the second harmonics with increasing z. In contrast, the amplitude of the third term of $O(\varepsilon^2)$ oscillates in the z-direction with period $2\pi/\lambda$. However, this amplitude grows with increasing z when θ is very small ($ka \ll z$). The nonsecular terms p_{NS} given by Eq. (4.2) do not grow with increasing z and therefore are bounded at all locations. In general they become unimportant at large z. However, their contribution in the nearfield should be taken ino consideration.

The basic concern when growth is encountered in a regular perturbation series, such as Eq. (4.1), is that the second order term might exceed the estimate of its magnitude. Such behaviour is known as nonuniform validity. In this section we will derive an expression for the pressure that behaves properly at all locations.

In order to render the pressure expression uniformly valid, the renormalization version of the method of strained coordinates [19] will be employed. Therefore, one seeks a coordinate transformation whose form is

$$\begin{aligned} z_1 &= \alpha_1 + \varepsilon \left[S_1 e^{i(t-\alpha_1)} + S_2 e^{i(t-\alpha_2)} + \text{c.c} \right], \\ z &= \alpha_2 + \varepsilon \left[S_3 e^{i(t-\alpha_2)} + S_4 e^{i(t-\alpha_1)} + \text{c.c} \right]. \end{aligned} \quad (4.4)$$

In accord with standard procedures, the above coordinate transformations are substituted into Eq. (4.1), and a Taylor series in ascending powers of ε is employed. The undetermined functions S_j, $j = 1, 4$, are then selected on the basis of removing the nonuniformly accurate terms. This procedure yields the following expression for the pressure

$$\frac{p}{\varrho_0 c_0^2} = \varepsilon \frac{i}{2} \left[e^{i(t-\alpha_1)} - e^{i(t-\alpha_2)} \right] + p_{NS} + O(\varepsilon^3). \quad (4.5)$$

The coordinate transformations are given by

$$\begin{aligned} z_1 &= \alpha_1 + \varepsilon \frac{i\beta_0}{2} \left[z_1 e^{i(t-\alpha_1)} - \frac{2}{1 - \cos\theta} \left(1 - e^{i\lambda\alpha_2}\right) e^{i(t-\alpha_2)} + \text{c.c} \right], \\ z &= \alpha_2 - \varepsilon \frac{i\beta_0}{2} \left[z e^{i(t-\alpha_2)} - \frac{2}{1 - \cos\theta} \left(1 - e^{i\lambda\alpha_2}\right) e^{i(t-\alpha_1)} + \text{c.c} \right]. \end{aligned} \quad (4.6)$$

Evaluation of the pressure at a selected location z and time t requires simultaneous solution for the transcendental equations for the coordinate straining transformations, Eqs. (4.6). This can be accomplished by using a numerical procedure such as the Newton-Raphson's method. The frequency content of the pressure waveform may be evaluated from the Fourier analysis. It is to be remarked that the terms in Eqs. (4.6) that couple the strained coordinates α_1 and α_2 do not show explicit growth with increasing z. However as $\theta \to 0$, the magnitudes of these terms increase and in the limit they have explicit dependence on z as can be shown in the next section.

5. Asymptotic trends

Equations (4.5) and (4.6) are generally valid. Examination of the behaviour at the limiting value of $\theta \to 0$ $(ka/z \to 0)$ provides important insights when the field point is distant compared to the radius of the transducer. For small ka/z, Eq. $(2.8)_2$ can be expanded in a Taylor series

$$z_1 \cong z + \frac{1}{2}\frac{k^2a^2}{z} + \ldots . \tag{5.1}$$

Substitution of Eq. $(3.11)_1$ in the argument of the exponential functions ($e^{i\lambda z}$) in Eqs. (4.6) followed by expansion in a Taylor series in $(1 - \cos\theta)$, making us of Eq. (5.1), simplifying by deleting higher order terms and then converting the results to real forms by accounting for the complex conjugate of each term, yields the following common forms for the coordinate transformations

$$\begin{aligned} z_1 &\sim \alpha_1 - \varepsilon\beta_0 z\,[\sin(t-\alpha_1) - \sin(t-\alpha_2)] \\ &\equiv \alpha_1 + 2\varepsilon\beta_0 z\cos\left(t - \frac{\alpha_1+\alpha_2}{2}\right)\sin\left(\frac{\alpha_1-\alpha_2}{2}\right), \\ z &\sim \alpha_2 + 2\varepsilon\beta_0 z\cos\left(t - \frac{\alpha_1+\alpha_2}{2}\right)\sin\left(\frac{\alpha_1-\alpha_2}{2}\right). \end{aligned} \tag{5.2}$$

From which it follows

$$\begin{aligned} z_1 - z &\sim \alpha_1 - \alpha_2 , \\ z_1 + z &\sim (\alpha_1+\alpha_2) + 4\varepsilon\beta_0 z\cos\left(t - \frac{\alpha_1+\alpha_2}{2}\right)\sin\left(\frac{\alpha_1-\alpha_2}{2}\right). \end{aligned} \tag{5.3}$$

As alluded previously, P_{NS} can be neglected at large distances and the pressure expression Eq. (4.5) is written in a real form as

$$\begin{aligned} \frac{p}{\varrho_0 c_0^2} &\sim \varepsilon\,[\sin(t-\alpha_1) - \sin(t-\alpha_2)] + O(\varepsilon^2) \\ &= 2\varepsilon\cos\left(t - \frac{\alpha_1+\alpha_2}{2}\right)\sin\left(\frac{\alpha_1-\alpha_2}{2}\right). \end{aligned} \tag{5.4}$$

The next step is to replace $z_1 - z$ by $\frac{1}{2}\frac{k^2a^2}{z}$ by making use of Eq. (5.1) and then substitute the first of Eqs. (5.3) in Eq. (5.4), and use the resulting expression for p to eliminate $\alpha_1 + \alpha_2$ between the second of Eqs. (5.3) and (5.4). The pressure expression that is derived in this manner is

$$\frac{p}{\varrho_0 c_0^2} \sim 2\varepsilon \sin\left(\frac{1}{4}\frac{k^2a^2}{z}\right)\cos\left(t - z + \beta_0 z \frac{p}{\varrho_0 c_0^2}\right)$$

$$\equiv \varepsilon \frac{z_0}{z} D \cos\left(t - z + \beta_0 z \frac{p}{\varrho_0 c_0^2}\right). \tag{5.5}$$

where

$$D = \sin M/M,$$

and

$$M = \frac{z_0}{2z}, \tag{5.6}$$

in which z_0 is the Rayleigh distance nondimensionalized by the scale factor k.

Except for the fact that z and t are nondimensional here and the amplitude shows spherical spreading, Eq. (5.5) is identical to Earnshaw's implicit closed form solution for the finite amplitude planar wave [20] in the case of harmonic excitation at a boundary.

In order to obtain the spectral analysis of the pressure signal, one could implement procedures that are similar to that used in [21–22] and will not be repeated here. Specifically, the spectral representation for the pressure is

$$p = \frac{2}{\beta_0 z}\sum_{m=1}^{\infty}\frac{1}{m}J_m(mc)\sin[m(t - z + \pi/2)], \tag*{(5.7)$_1$}$$

where

$$c = 2\varepsilon\beta_0 z \sin\left(\frac{k^2a^2}{4z}\right), \tag*{(5.7)$_2$}$$

and J_m are the Bessel functions of the first kind of order m.

The description given by Eq. (5.7) is valid if no shock form. That is up to the place where discontinuity of the pressure wave profile occurs. The smallest value of z at which multivaluedness of the waveform occurs is obtained when $|c| = 1$. That is

$$z = \frac{1}{2\varepsilon\beta_0}. \tag{5.8}$$

This result is the same as that for the one dimensional nonplanar wave except that β_0 is replaced by $2\beta_0$. In otherwords, the shock formation distance for the piston problem is half that of the planar finite amplitude wave.

Expanding the sine function in Eq. (5.5) in a Taylor series expansion for a small argument and deleting higher order terms yields

$$\frac{p}{\varrho_0 c_0^2} \sim \varepsilon \frac{z_0}{z}\cos\left(t - z + \beta_0 z \frac{p}{\varrho_0 c_0^2}\right). \tag{5.9}$$

Like in the linear theory, Eq. (5.9) shows that the pressure signal appears as though it was coming from a spherical sound source of radius z_0.

6. Results and discussion

Well documented exprimental data describing nonlinear effects in the nearfield is quite sparse. GOULD *et al.* [23] measured the field generated by a transducer vibrating at 2.58 MHz when $c_0 = 1475$ m/s which corresponds to $k = 10990\,\mathrm{m}^{-1}$. The geometrical radius was 0.0101 m, but subsequent analysis of the linearized field caused INGENITO and WILLIAMS [7] to suggest that $a = 0.01042$ m is more appropriate. The results were presented in Gould's paper as selected traces of the amplitudes of the fundamental and second harmonic either along or transverse to the axis of the beam. Such traces were obtained by photographing an oscilloscope screen. So they are difficult to read accurately. However travelling microscope readings of the axial distribution of the second harmonic were reported by INGENITO and WILLIAMS [7]. Figure 2 compares the measured axial distribution of the second harmonic with the predicted results. The transducer was driven at source pressure level of 5 atmosphers (506.6 KPa). The nondimensionalized Rayleigh distance is 6657 which corresponds to 0.5966 m, whereas $ka = 114.52$. The overall agreement between theory and experiment is good. It is to be noted that the prediction for the farthest dip, near the nondimensional $z = 1300$, is somewhat less deep than that was predicted by INGENITO and WILIAMS (Fig. 2 in Ref. [7]), while the dip near $z = 600$ is deeper than their prediction and the one near $z = 800$ is substantially deeper.

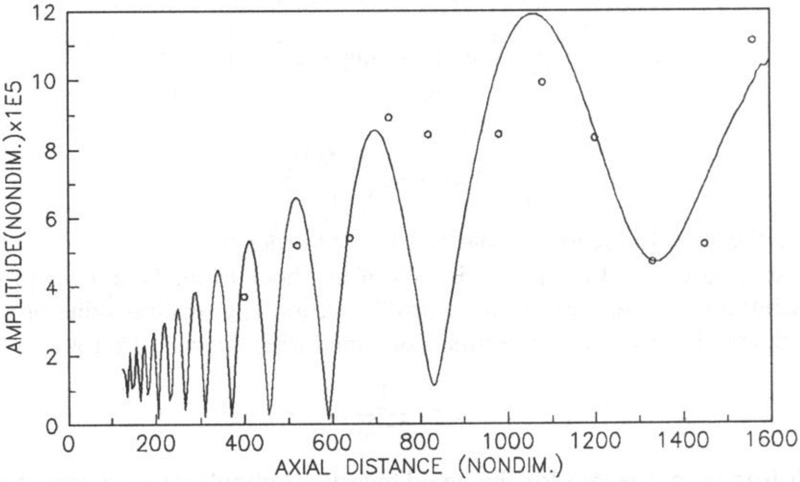

Fig. 2. Axial variation of the amplitude of the second harmonic in the Fresnel region. $f = 2.58$ MHz, $k = 10990\,\mathrm{m}^{-1}$, $a = 0.01042$ m, source pressure = 5 atm (506.6 KPa) —— : predicted, o: measured values (Ref. [23]).

The experiments recently reported by BAKER *et al.* [12] for propagation in a water tank provide useful data for validating the analysis in the Fresnel region. The average pressure across the transducer face was 100 KPa, the transducer radius was $a = 0.019$ m and the frequency was 2.25 MHz. This corresponds to $ka = 180.7$ when $c_0 = 1486$ m/s. The Rayleigh distance is 1.717 m whereas the last axial pressure maximum ocurred at 0.5462 m. Comparing the exprimental and computed results will, therefore, indicate how well the nearfield propagation properties are predicted. Figures 3–5 show the variations of harmonic amplitudes obtained by analyzing the waveform at numerous axial locations. In Figs. 4 and 5, the experimental data have been smoothed slightly near the transducer due to the difficulty in following small-scale fluctuations when published curves were digitized in order to be presented here.

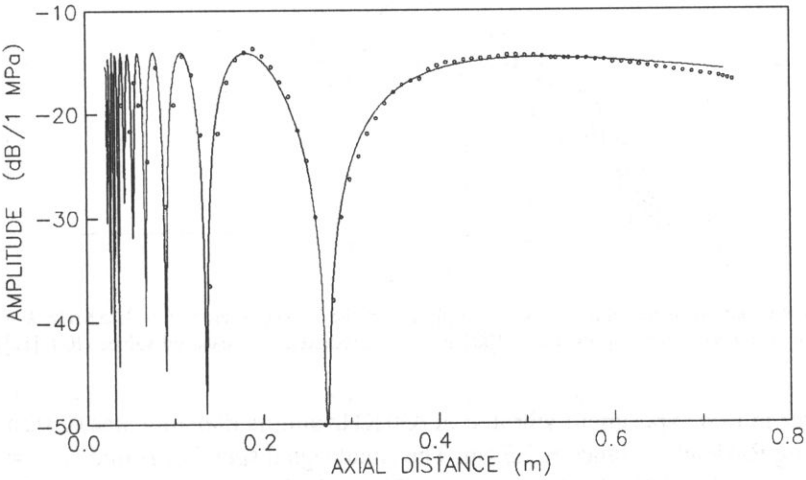

Fig. 3. Axial variation of the first harmonic amplitude in the Fresnel region. $f = 2.25\,\text{MHz}$, $k = 9514\,\text{m}^{-1}$, $a = 0.019\,\text{m}$, source pressure = 100 KPa —— : predicted, o: measured values (Ref. [12]).

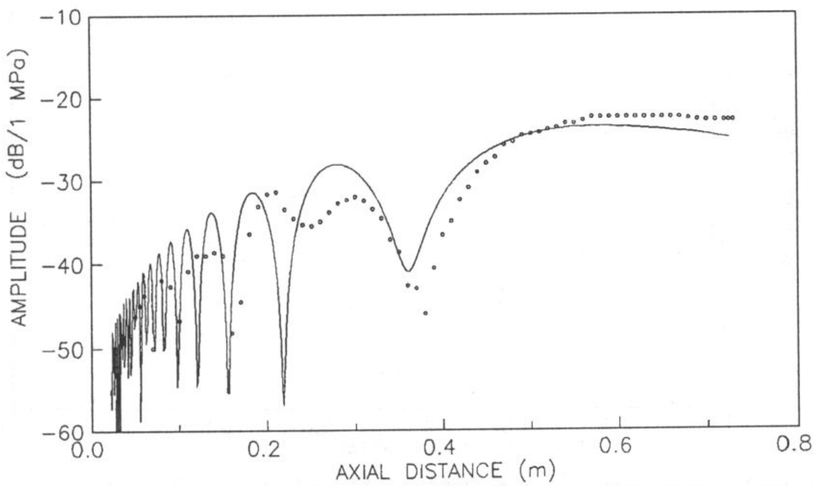

Fig. 4. Axial variation of the second harmonic amplitude in the Fresnel region. $f = 2.25\,\text{MHz}$, $k = 9514\,\text{m}^{-1}$, $a = 0.019\,\text{m}$, source pressure = 100 KPa —— : predicted, o: measured values (Ref. [12]).

The theoretical prediction is compared to Moffett's farfield measurements (Fraunhofer region) [24] of the fundamental and second harmonic in a fresh water lake. The

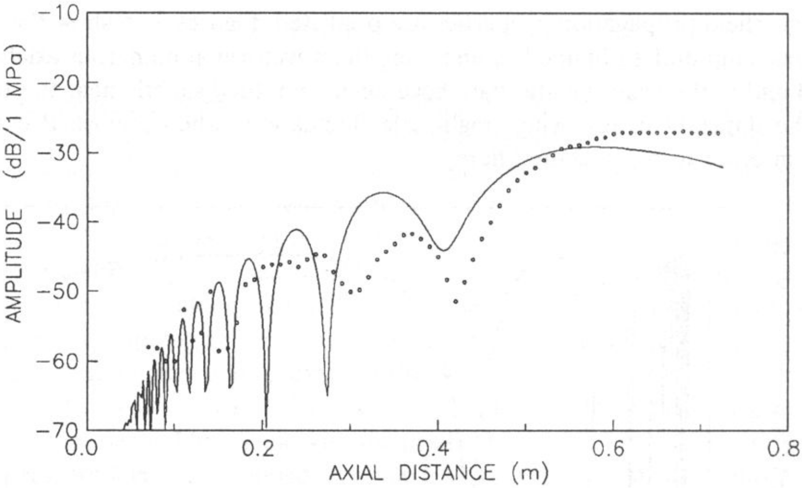

Fig. 5. Axial variation of the third harmonic amplitude in the Fresnel region. $f = 2.25$ MHz, $k = 9514\,\mathrm{m}^{-1}$, $a = 0.019$ m, source pressure = 100 KPa —— : predicted, o: measured values (Ref. [12]).

transducer in that experiment vibrated at 450 KHz and its diameter was 0.102 m; the corresponding Rayleigh distance is 2.59 m. The small signal speed of sound is $c_0 = 1418$ m/s. The nondimensional Rayleigh distance is 5171 which corresponds to 2.593 m whereas $ka = 101.7$. The source level SL_0 is 215 dB/1 μPa m. The theoretical predictions shown in Fig. 6 compare favorably with Moffett's measurements.

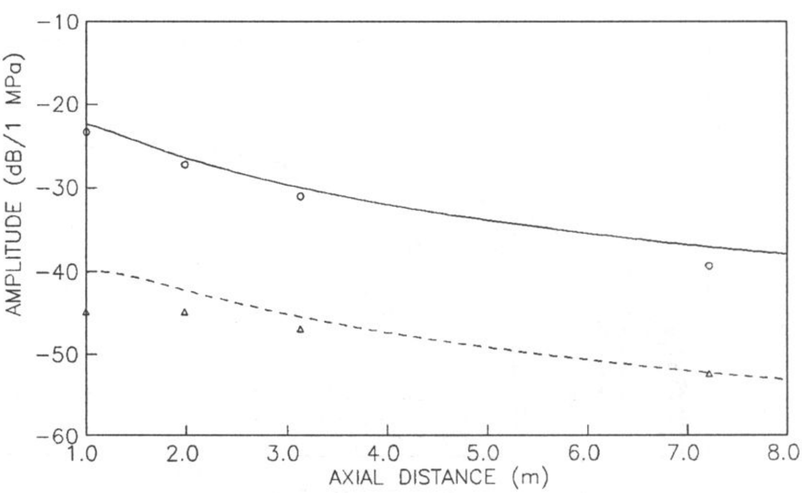

Fig. 6. Axial variations of the amplitude of the first and second harmonic in the Fraunhofer region. $f = 450$ KHz, $k = 1994\,\mathrm{m}^{-1}$, $a = 0.051$ m, source pressure = 0.447 atm (45.25 KPa). First harmonic; —— : predicted, o: measured; second harmonic; - - - - : predicted, △: measured (Ref. [12]).

At ranges $z = 0.4005$ m and 0.6007 m Figs. 7 and 8 exhibit the time waveforms for BAKER'S [12] data. In comparison to the linearized signal, the wave distorted with the

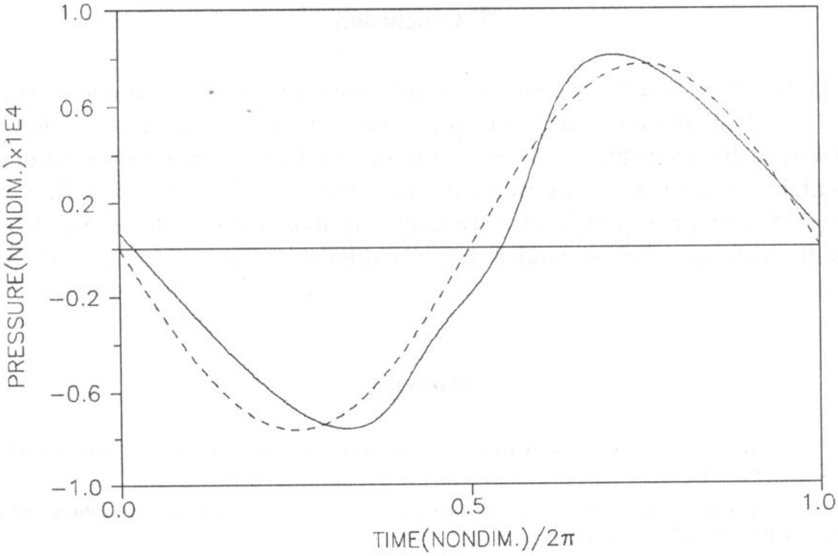

Fig. 7. Temporal waveform at z = 0.4005 m, f = 2.25 MHz, k = 9514 m^{-1}, a = 0.019 m, source pressure = 100 KPa —— : nonlinear signal, - - - - : linear signal (Ref. [12]).

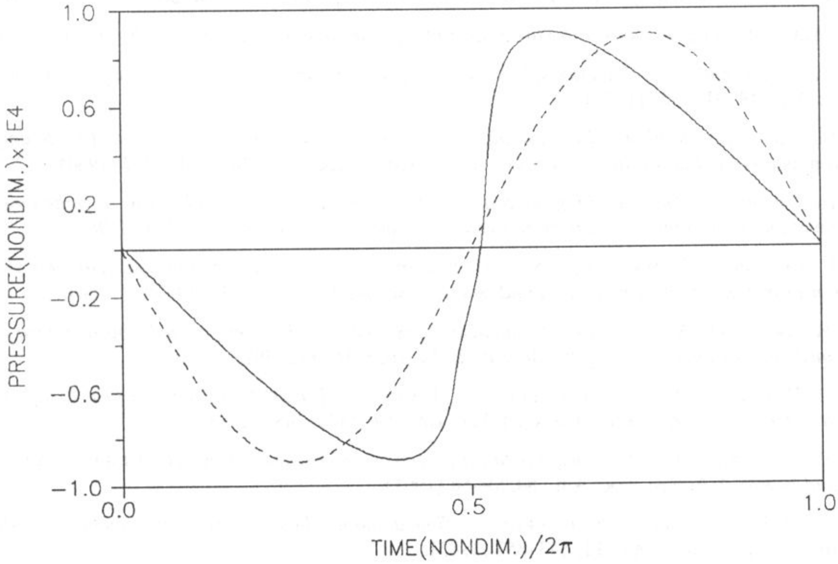

Fig. 8. Temporal waveform at z = 0.6007 m, f = 2.25 Mhz, k = 9514 m^{-1}, a = 0.019 m, source pressure = 100 KPa —— : nonlinear signal, - - - - : linear signal (Ref. [12]).

compressional phase being steeper than the rarefaction. The waveform has also developed a marked top-bottom asymmetry with a positive peak being higher and sharper than the negative one.

7. Conclusion

An analytical representation of the on-axis finite amplitude continuous wave signal radiated by a baffled transducer undergoing monochromatic excitation is derived. The face velocity at the transducer is restricted to be constant. A uniformly valid description that is suitable at any location up to shock formation distance is obtained. An asymptotic analysis yields a simple expression for the long rang approximaton that is easy to evaluate. The results form an efficient model that can predict the waveform and the harmonic contents.

References

[1] M. FODA, *Nonlinear propagation and distortion of two plane waves interacting at arbitrary angles*, Acustica, **82**, 213–219 (1996).

[2] M. FODA, *Distortion and dispersion of nonlinear waves in a rectangular duct due to a bifrequency excitation*, Acustica, **82**, 411–4222 (1996).

[3] M. FODA, *Analysis of nonlinear propagation and interactions of higher order modes in a circular waveguide*, ACTA Acustica (1996).

[4] M. GREENSPAN, *Piston radiator: some extensions of the theory*, J. Acoust. Am. Soc., **65**, 608–621 (1979).

[5] G.R. HARRIS, *Review of transient field theory for baffled piston*, J. Acoust. Soc. Am., **70**, 10–20 (1981).

[6] A. SCHOCH, *Considerations in regard to field of a piston diaphragm*, Akust. Z., **6**, 318–326 (1941).

[7] F. INGENITO and A.O. WILLIAMS, *Calculations of second-harmonic generation in a piston beam*, J. Acoust. Soc. Am., **49**, 319–328 (1971).

[8] J.H. GINSBERG, *Nonlinear King integral for arbitrary axisymmetric sound beams at finite amplitudes: I. Asympotic evaluation of the velocity potential*, J. Acoust. Soc. Am., **76**, 1201–1207 (1984).

[9] J.H. GINSBERG, *Nonlinear King integral for arbitrary axisymmetric sound beams at finite amplitudes: II. Derivation of uniformly accurate expressions*, J. Acoust. Soc. Am., **76**, 1208–1214 (1984).

[10] S.I. AANONSEN, T. BARKVE, J.N. TOJOTTA and S. TOJOTTA, *Distortion and harmonic generation in the nearfield of a finite amplitude sound beam*, J. Acoust. Soc. Am., **75**, 749–768 (1984).

[11] D.R. BACON and A.C. BARKER, *Comparison of two theoretical models for predicting nonlinear propagation in medical ultrasonic fields*, Phys. Med. Biol., **34**, 1633–1643 (1989).

[12] A.C. BAKER, K. ANASTASIADIS and V.F. HUMPHERY, *The nonlinear pressure field of a planar circular piston: theory and experiment*, J. Acoust. Soc. Am., **84**, 1483–1487 (1988).

[13] G.P. TOO and J.H. GINSBERG, *Nonlinear progressive wave equation model for transient and steady-state sound beams*, J. Acoust. Soc. Am., **84**, 59–68 (1991).

[14] B.E. MCDONALD and W.A. KUPERMAN, *Time domain solution of parabolic equation including nonlinearity*, Comp. Math. Appl., **11**, 843–851 (1985).

[15] F.Y. COULOUVART, *An analytical approximation of strong nonlinear effects in bounded sound beam*, J. Acoust. Soc. Am., **90**, 1592–1600 (1991).

[16] A.C BAKER and V.F. HUMPHERY, *Distorion and high-frequency generation due to nonlinear propagation of short ultrasonic pulses from a plane circular piston*, J. Acoust. Soc. Am., **92**, 1699–1705 (1992).

[17] S. GOLDSTEIN, *Lectures in fluid Mechanics*, Wiley-Interscience, New York 1960, Chap. 4.

[18] J.H. GINSBERG and H.C. MIAO, *Finite amplitude distortion and dispersion of a nonlinear mode in waveguide*, J. Acoust. Soc. Am., **80**, 911–920 (1986).

[19] A.H. NAYFEH, *Introduction to perturbation techniques*, Wiley-Interscience, New York 1981.

[20] A.D. PIRCE, *Acoustics*, McGraw-Hill, New York 1981, Chap. 5.

[21] M. FODA, *Analysis of nonlinear propagation of waves induced by a vibrating flat plane*, Acustica, **72**, 118–130 (1990).

[22] M. FODA, *Uniformly accurate expressions for sound waves induced by a vibrating planar boundary*, Acustica, **74**, 254–263 (1991).

[23] R.K. GOULD, C.W. SMITH, A.O. WILLIAMS, JR. and R.P. RYAN, *Measured structure of harmonics self-generated in an acoustic beam*, J. Acoust. Soc. Am., **40**, 421–427 (1966).

[24] M.B. MOFFETT, *Measurement of fundamental and second harmonic pressures in the field of a circular piston source*, J. Acoust. Soc. Am., **65**, 318–323 (1979).