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MODEL OF AN ACOUSTIC SOURCE WITH DISCONTINUOUS OPTIMAL ELEMENTS

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A new multi-elements model is considered. This model is composed of a sequence of discontinuous elements. They are constructed basing on the zeros of the Tchebycheff's polynomial. In this case, the discontinuous elements, and consequently the discontinuous model, are obtained. Such a model is particularly useful for modelling a source with corners and arbitrary boundary conditions.

It has been proved that the new model is of better quality than other ones applied in the BEM up to now. To confirm this conclusion, the error of the new model and their acoustic fields have been compared with those of different other models.

In order to clearly demonstrate the advantages of the new model, a plane and fully axisymmetric source has been taken into account, however the idea of the model with discontinuous elements can be applied to more practical problems.

Notations

x_i	nodes, to model $M_{\mathcal{W};n+1}$; $i = 0, 1, ..., n$,
$\mathcal{W}_q(x)$	polynomial of degree q,
$M_{\mathcal{W};n+1}$	one-element model of degree q with $n+1$ nodes; $q = n$,
μ_j	break points, j-subinterval (j-element) $\in [\mu_{j-1}, \mu_j)$, to $M_{\mathcal{P};n_j,n_{ij}+1}$; $j = 1, 2, ...n_j$,
ν_i	nodes separately numbered on each element, to model $M_{\mathcal{P};n_j,n_{ij}+1}$; $i = 1, 2, ...n_{ij}$,
$\mathcal{P}_q(x)$	piecewise polynomial of degree q,
$M_{\mathcal{P};n_j,n_{ij}+1}$	n_j-elements model of degree q with $n_{ij}+1$ nodes on each element; $q = \max\limits_j n_{ij}$,
$f(x), \widetilde{f}(x)$	given any function, interpolating function,
k	wave number,
$J_0(\ldots)$	Bessel function of zero order,
$G(r)$	infinite space solution of the Helmholtz equation for the point source [16] p. 641,
$f_{i..n}$	n-th divided difference of the function $f(x)$ at the nodes x_i, [3], [8] p. 193,
$P_i(x)$	finite product, [3], [8] p. 193,
x_b	radius of the membrane,
$[a, b] \equiv [0, x_P]$	from mathematical point of view.

1. Introduction

The first step of BEM is the discretization of the source boundary into elements [14] (boundary ≡ geometry and acoustic variables defined on the geometry). Next, applying the interpolation theory, the model of each element source is built. From the mathematical point of view, the model of the source constitutes the piecewise polynomials [11, 12]. Hereafter, for simplicity: model of the element $\equiv$ element, model of the source $\equiv$ model.

Because the elements may be connected in an arbitrary manner, the model, in general, is a discontinuous function at singular points, i.e. at the points of discontinuity of the physical variables and/or those of discontinuity of the geometry (corners). However, the BEM requires, at these points, the existence of derivatives of the boundary [7, 17]. Under these circumstances the derivatives may not be determined and they require special attention.

Two main techniques to circumvent to the singular points are proposed:

• by duplicating the singular point with a small gap between [15, 17]. However, the problem arises how large the gap should be to ensure a good quality of the model,

• by using discontinuous (nonconforming) elements in which the nodes are shifted inside the elements; in order words, the extreme nodes are not placed at the borders of element [7, 9, 15, 17]; the problem arises how large the displacement of the node from the corner should be. In [15] it was proposed to take a distance of 1/4 and 1/6 of the element length for linear and quadratic elements, respectively, but this was not theoretically justified.

The aim of this work is to construct a model with optimal elements denoted by $M_{\mathcal{P};O-N}$, i.e. optimal in the distribution of the nodes on each element. This was achieved by applying the zeros of the Tchebycheff's polynomial as the nodes; an idea of such elements may be found in [3, 4, 5]. The Tchebycheff's zeros are exactly distributed. Such a model consists of a set of discontinuous elements. The new model turned out to be better quality than other known models. To confirm this conclusion, the following quantities have been calculated: the error of the models, their directivity functions and the acoustic pressure near the boundary.

Two comparative models were taken into account: a one-element model of higher degree and a multi-elements continuous model with evenly spaced nodes on each element.

For simplicity of the calculations, a plane axisymmetric membrane has been chosen as the source.

2. Interpolation theory

Let $f(x)$ be any given function. It is required to construct an interpolating function, $\widetilde{f}(x)$, which satisfies any (here Lagrange's) interpolation condition, i.e. $f(\text{nodes}) = \widetilde{f}(\text{nodes})$.

In this paper two forms of $\widetilde{f}(x)$ are presented:

• a polynomial form; $\widetilde{f}(x) \equiv \mathcal{W}(x)$,

• a piecewise polynomial form; $\widetilde{f}(x) \equiv \mathcal{P}(x)$.

The $\mathcal{W}(x)$ and $\mathcal{P}(x)$ are two standard models for the $f(x)$ function.

2.1. Polynomial interpolation

Let Δ be any partition of the interval $[a, b]$, such that

$$\Delta: \quad a = x_0 < x_1 < \cdots < x_{i-1} < x_i \cdots x_n = b \tag{2.1}$$

and the set of nodal values is $\{f(x_i)\}_0^n \equiv \{f_i\}_0^n$.

One desires to find a q-degree polynomial $\mathcal{W}_q(x)$ which satisfies the condition:

$$f(x_i) = \mathcal{W}_q(x_i), \qquad q = n. \tag{2.2}$$

There are several ways to represent an interpolating polynomial. It seems that the Newton form is the most efficient one [1] p. 3,

$$\mathcal{W}_q(x) = \sum_{i=0}^{n} f_{0..i} P_i(x). \tag{2.3}$$

The interpolating polynomial $\mathcal{W}_q(x)$, among the nodes x_i, is not identical with the function $f(x)$. Therefore, one defines the error of interpolation as follows:

$$E_{\mathcal{W};n+1}(x) = f(x) - \mathcal{W}_q(x). \tag{2.4}$$

This error could be exactly expressed by one of the three formulas, [11] p. 118, but the most known one is

$$E_{\mathcal{W};n+1}(x) = \frac{f^{(n+1)}(x_c)}{(n+1)!} P_{n+1}(x), \qquad x_c \in \operatorname{int}(x, x_0, x_1, \ldots, x_n). \tag{2.5}$$

The value $E_{\mathcal{W};n+1}(x)$ cannot be calculated because $f^{(n+1)}(x_c)$ is unknown. Therefore two estimations of $E_{\mathcal{W};n+1}(x)$ are used. The first one is the estimation at the point x and the second one the estimation over the interval $[a, b]$. They are given respectively by

$$\|E_{\mathcal{W};n+1}(x)\|_{\infty,f} \le \frac{\mathfrak{M}_{f,n+1}}{(n+1)!} |P_{n+1}(x)|, \tag{2.6}$$

where

$$\mathfrak{M}_{f,n+1} = \|f^{(n+1)}(x)\|_\infty = \sup_{x\in[a,b]} |f^{(n+1)}(x)|, \tag{2.7}$$

and

$$\mathfrak{E}_{\mathcal{W};n+1} \le \frac{\mathfrak{M}_{f,n+1}}{(n+1)!} \mathfrak{M}_{P,n+1}\,, \tag{2.8}$$

where

$$\mathfrak{M}_{P,n+1} = \|P_{n+1}(x)\|_\infty = \sup_{x\in[a,b]} |P_{n+1}(x)|. \tag{2.9}$$

If the interpolation points are equi-spaced, $\mathfrak{M}_{P,n+1}$ can be computed by the closed formula, [10] p. 63:

$$\mathfrak{M}_{P,n+1} = \left(\frac{b-a}{n}\right)^{n+1} (n+1)!\,. \tag{2.10}$$

2.2. Piecewise polynomial interpolation

At first sight it would appear that by increasing the number of nodes, and hence the degree of the polynomial, better interpolations to the function $f(x)$ should be successively obtained. However, in practice, between the nodes at the ends of the interval $[a,b]$ the interpolating polynomial of higher degree may oscillate quite violently (see Runge's problem, [1] p. 22, [11] p. 96). Thus it may reflect not truly the behaviour of the function $f(x)$.

An alternative approach is to use piecewise polynomial interpolation. Instead of looking for a higher degree polynomial over all the interval $[a,b]$, a polynomial composed of a sequence of low degree polynomials is constructed that it is valid only locally.

Let Δ_μ be any partition of the interval $[a,b]$, i.e.,

$$\Delta_\mu : \quad a = \mu_0 < \mu_1 < \cdots < \mu_{j-1} < \mu_j < \cdots < \mu_{n_j} = b. \tag{2.11}$$

Furthermore, let Δ_ν be an arbitrary partition of the j-subinterval, $x \in [\mu_{j-1}, \mu_j)$, Fig. 1,

$$\Delta_\nu : \quad \mu_{j-1} \le \nu_0 < \nu_1 < \cdots < \nu_{i-1} < \nu_i < \cdots < \nu_{n_{ij}} \le \mu_j\,, \tag{2.12}$$

and the set of nodal values $\{f(\nu_i)\}_0^{n_{ij}} = \{f_i\}_0^{n_{ij}}$, $j = 1,2,...,n_j$, where n_{ij} may be different on each j-subinterval.

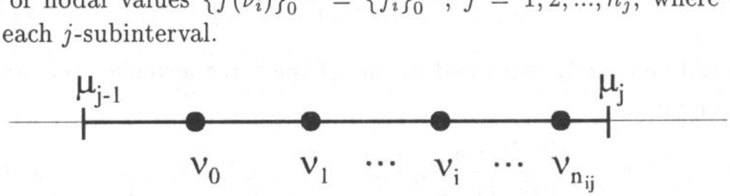

Fig. 1. General distribution of the nodes on the j-subinterval.

A q_j-degree polynomials $\mathcal{P}_{q_j}(x)$ and a q-degree piecewise polynomial $\mathcal{P}_q(x)$ are defined, on each j-subinterval and on the $[a,b]$ interval, respectively,

$$\mathcal{P}_{q_j}(x) \;\equiv\; \mathcal{W}_{q_j}(x), \qquad x \in [\mu_{j-1}, \mu_j), \quad q_j = n_{ij}\,, \tag{2.13}$$

$$\mathcal{P}_q(x) \;=\; \mathcal{P}_{q_j}(x), \qquad j = 1,2,...,n_j\,, \quad q = \max_j q_j\,, \tag{2.14}$$

The polynomial $\mathcal{P}_{q_j}(x)$ fulfils the interpolation condition

$$\widetilde{f}_{j,n_{ij}}(\nu_i) \equiv f(\nu_i) \equiv \mathcal{P}_{q_j}(\nu_i), \qquad \nu_i \in [\mu_{j-1}, \mu_j), \tag{2.15}$$

or in the $[a,b]$ interval

$$\widetilde{f}_{n_j,n_{ij}}(\nu_i) \equiv f(\nu_i) \equiv \mathcal{P}_q(\nu_i), \qquad \nu_i \in [a,b]. \tag{2.16}$$

The error of piecewise polynomial interpolation ought to be expressed similarly as pointed out above for polynomial interpolation. In this case, the error at every point of the j-subinterval and of the interval $[a,b]$ can be written respectively as follows

$$E_{\mathcal{P};j,n_{ij}+1}(x) \;=\; f(x) - \mathcal{P}_{q_j}(x), \qquad x \in [\mu_{j-1}, \mu_j), \tag{2.17}$$

$$E_{\mathcal{P};n_j,n_{ij}+1}(x) \;=\; f(x) - \mathcal{P}_q(x), \qquad x \in [a,b]. \tag{2.18}$$

Because of the reasons mentioned above, only the estimation of $E_{\mathcal{P};n_j,n_{ij}+1}(x)$ at the point x may be calculated, cf. Eq. (2.6),

$$\|E_{\mathcal{P};j,n_{ij}+1}(x)\|_{\infty,f} \le \frac{\mathfrak{M}_{f,n_{ij}+1}}{(n_{ij}+1)!}|P_{n_{ij}+1}(x)|, \tag{2.19}$$

where

$$\mathfrak{M}_{f,n_{ij}+1} = \|f^{(n_{ij}+1)}(x)\|_{\infty} = \sup_{x\in[\mu_{j-1},\mu_j]} |f^{(n_{ij}+1)}(x)|, \tag{2.20}$$

$\|\cdots\|_\infty$ – norm of $C[\mu_{j-1},\mu_j]$ space (Tchebycheff norm).

In practice, however, it plays a minor part. Two estimations are more important: the estimation over the j-subinterval and that over interval the $[a,b]$,

$$\begin{aligned} \|E_{\mathcal{P};j,n_{ij}+1}\|_\infty &= \frac{\mathfrak{M}_{f,n_{ij}+1}}{(n_{ij}+1)!}\mathfrak{M}_{P,n_{ij}+1}, \qquad x\in[\mu_{j-1},\mu_j), & (2.21)\\ \mathfrak{E}_{\mathcal{P};n_j,n_{ij}+1} &= \max_j \|E_{\mathcal{P};j,n_{ij}+1}\|_\infty, \qquad x\in[a,b], & (2.22) \end{aligned}$$

where

$$\mathfrak{M}_{P,n_{ij}+1} = \|P_{n_{ij}+1}(x)\|_\infty = \sup_{x\in[\mu_{j-1},\mu_j]} |P_{n_{ij}+1}(x)|. \tag{2.23}$$

For equi-spaced interpolation points on j-subintervals we have,

$$\mathfrak{M}_{P;n_{ij}+1} = \left(\frac{\mu_j-\mu_{j-1}}{n_{ij}}\right)(n_{ij}+1)!\,. \tag{2.24}$$

With regard to the distribution of the nodes on each j-subinterval, two cases should be considered: the first one, when two extreme nodes are placed at the ends of the j-subintervals; $\nu_0=\mu_{j-1}$, $\nu_{n_{ij}}=\mu_j$, and the second one, when these nodes are shifted inside; $\nu_0>\mu_{j-1}$, $\nu_{n_{ij}}<\mu_j$.

3. Models of the source

3.1. Acoustic source

For simplicity, let us consider the fully axisymmetric source, i.e. both the geometry and acoustic variables are independent of the angle of revolution. Here, the membrane placed in an infinite baffle is chosen as the source. In this case, the function $f(x)$ may be interpreted as a cross-section of the source, hence $a=0$, $b=x_b$.

An acoustic field of the source (exact acoustic field) has been described extensively in Refs. [16] p. 594, [18] p. 187 or [2, 6]; the final expressions for the directivity function and acoustic pressure near the source are given by,

$$Q(k,\gamma) = \int_0^{x_b} f(x)J_0(kx\sin\gamma)x\,dx, \tag{3.1}$$

$$p(k,H,x_P) = \int_0^{x_b} f(x)\left[\int_0^{2\pi} G(r)\,d\varphi\right]x\,dx, \tag{3.2}$$

respectively, where the geometric symbols are depicted in Fig. 2.

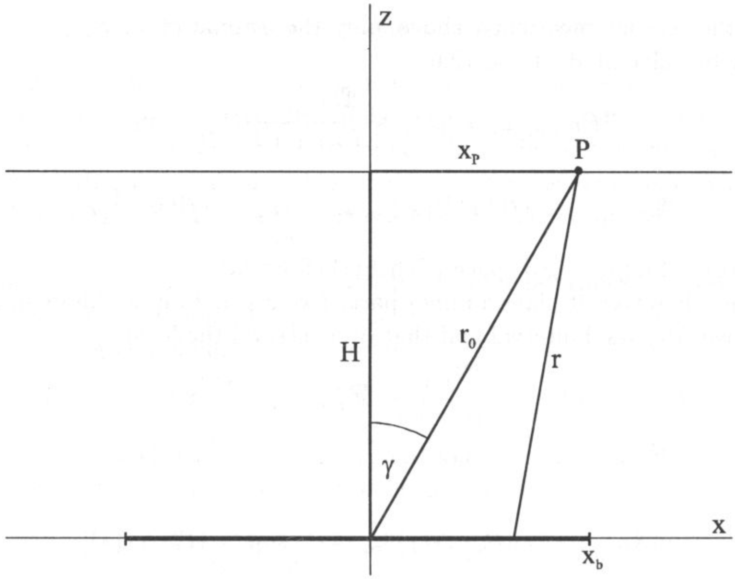

Fig. 2. Geometry of the problem.

3.2. Models of the source

Hereafter, the function $\widetilde{f}(x)$ ought to be interpreted as a cross-section of the model. The error of the model constitutes the error of interpolation, i.e. Eq. (2.4) or Eq. (2.18). Estimation of the model error on the interval $[a, b]$, Eq. (2.8) or Eq. (2.22), is assumed to be a direct measure of the quality of the model (see Sec. 5.1 below). However, in acoustics such a measure plays a minor part. It seems that the difference of the acoustic field may be more useful (indirect measure of the quality, see Sec. 5.2 below).

3.2.1. One-element model $M_{\mathcal{W}}$. Under the circumstances given above, the model $M_{\mathcal{W}}$ is given by Eq. (2.3): $M_{\mathcal{W}} \equiv \mathcal{W}_q(x) \equiv \widetilde{f}_n(x)$. If Eq. (2.3) is substituted into Eqs. (3.1) and (3.2), the acoustic field of $M_{\mathcal{W}}$ is obtained; it is denoted by $\widetilde{Q}_{\mathcal{W}}(k, \gamma)$, $\widetilde{p}_{\mathcal{W}}(k, H, x_P)$. The error of the model $M_{\mathcal{W}}$ is given by Eq. (2.4) and the direct measure of the quality by Eq. (2.8). In numerical calculations, the one-element model $M_{\mathcal{W}}$ with equi-spaced nodes $x_{e,i}$ is used as a comparative model. It is marked by $M_{\mathcal{W};R}$ and $x_{e,i}$ – ■ in Fig. 3. All the symbols ought to be completed by index $n+1$; e.g. $M_{\mathcal{W};R;n+1}$.

3.2.2. Multi-elements model $M_{\mathcal{P}}$. The model $M_{\mathcal{P}}$ is given by Eq. (2.14): $M_{\mathcal{P}} \equiv \mathcal{P}_q(x) \equiv \widetilde{f}_{n_j n_{ij}}(x)$. If Eq. (2.14) is substituted into Eqs. (3.1) and (3.2), the acoustic field of $M_{\mathcal{P}}$ is obtained; it is marked by $\widetilde{Q}_{\mathcal{P}}(k, \gamma)$, $\widetilde{p}_{\mathcal{P}}(k, H, x_P)$. The error of the model $M_{\mathcal{P}}$ is given by Eq. (2.18) and the direct measure of the quality by Eq. (2.22).

The multi-elements model $M_{\mathcal{P}}$ with equi-spaced break points $\mu_{e,j}$ (all elements have the same length) and equi-spaced nodes $\nu_{e,i}$ is called multi-elements regular model $M_{\mathcal{P};R}$ (in Fig. 3 $\mu_{e,j}$ – ∇ and $\nu_{e,i}$ – ■). It is a comparative model in numerical calculations.

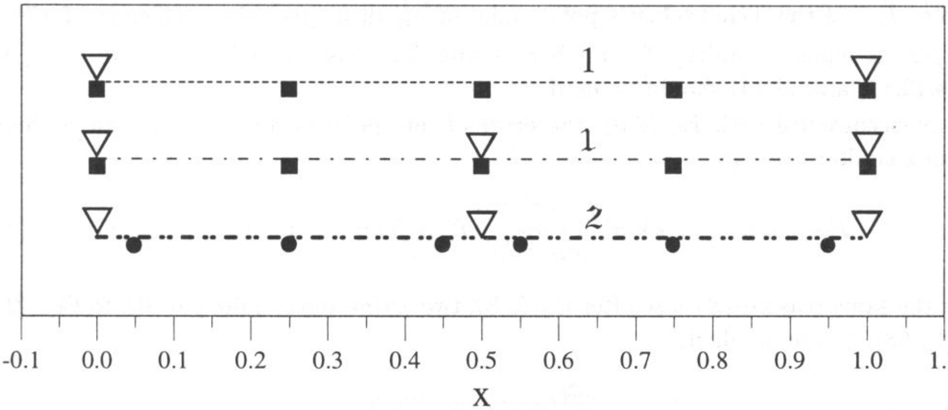

Fig. 3. Distribution of the nodes and break points: 1 – $M_{\mathcal{W};R;5}$, I – $M_{\mathcal{P};R;2,3}$, 2 – $M_{\mathcal{P};O-N;2,3}$.

Under conditions given above, instead of Eq. (2.3), Eqs. (2.13) and (2.14), $M_{\mathcal{P};R}$ may be expressed by a simpler formula,

$$\mathcal{P}_{q_j}(x) = \sum_{i=0}^{n_{ij}} f_{e,i}\Re_i(x), \qquad x \in [\mu_{j-1}, \mu_j), \tag{3.3}$$

where $f_{e,i} = f(\nu_{e,i})$ and $\Re_i(x)$ are the shape functions; they can be easily found elsewhere, [7] p. 71. Up to now, this is a fundamental equation applied in modelling the source in BEM. In this section the additional index n_j, $n_{ij}+1$ has been dropped to simplify the notation; e.g. the full symbol is $M_{\mathcal{P};R;n_j,n_{ij}+1}$.

3.2.3. Multi-elements model with optimal elements $M_{\mathcal{P};O-N}$. The feature of $M_{\mathcal{P};O-N}$ are the equi-spaced break points $\mu_{e,j}$, but the nodes are the Tchebycheff's zeros.

Analysing formula (2.8), one can see that the error is a product of two factors. One of them, $\mathfrak{M}_{f,n+1}$, depends on the properties of the function $f(x)$ and is not amenable to regulation, while the other one $\mathfrak{M}_{P,n+1}$, is determined by the choice of the nodes x_i. Thus the question of an optimal choice of x_i arises so that $\mathfrak{M}_{P,n+1}$ deviates less than any other polynomial on interval $[a,b]$. This problem was solved by Tchebycheff, [3], [8] p. 540, who pointed out optimal nodes, here marked as $x_{T,i}$. Detailed discussions of the mathematical aspect of the nodes $x_{T,i}$ can be found in [3, 4, 5, 6]. In these papers, $x_{T,i}$ were applied to construct the one-element optimal model.

Hereafter the idea of $x_{T,i}$ is utilized to construct a multi-elements optimal model $M_{\mathcal{P};O-N}$ and $x_{T,i}$ ought to be replaced by $\nu_{T,i}$. Note, that the $\nu_{T,i}$ are non-uniformly spaced. Furthermore, the external $\nu_{T,i}$ are shifted from the borders of the j-subinterval and these displacements are mathematically proved.

In the case under consideration the interpolating polynomial can be expressed by Eq. (2.3). At such a particular distribution of the $\nu_{T,i}$, this formula takes a particular form of

$$\mathcal{P}_{q_j}(x) = \sum_{i=0}^{n_{ij}} f_{0..i}\widehat{T}_i(x), \qquad x \in [\mu_{j-1}, \mu_j), \tag{3.4}$$

where $\widehat{T}_{n_{ij}}$ is the Tchebycheff's polynomial of n_{ij}-th degree (the coefficient of $x^{n_{ij}}$ in $\widehat{T}_{n_{ij}}(x)$ is equal to unity); for further details, see Refs. [3] and [8] p. 540. Next, the Eqs. (2.13) and (2.14) should be used.

In accordance with Eq. (2.5), the error of interpolation by Eq. (3.4) can be found from a similar formula

$$E_{\mathcal{P};O-N;j,n_{ij}+1}(x) = \frac{f^{(n_{ij}+1)}(x)}{(n_{ij}+1)!}\widehat{T}_{n_{ij}+1}(x), \qquad x \in [\mu_{e,j-1}, \mu_{e,j}). \tag{3.5}$$

For the same reasons as those after Eq. (2.5), two estimations, quite parallel to Eqs. (2.6) and (2.8), can be applied,

$$\|E_{\mathcal{P};O-N;j,n_{ij}+1}(x)\|_{\infty,f} \le \frac{\mathfrak{M}_{f,n_{ij}+1}}{(n_{ij}+1)!}\left|\widehat{T}_{n_{ij}}(x)\right\|, \qquad x \in [\mu_{e,j-1}, \mu_{e,j}). \tag{3.6}$$

where $\mathfrak{M}_{f,n_{ij}+1}$ is expressed by formula (2.6) and

$$\|E_{\mathcal{P};O-N;j,n_{ij}+1}\|_{\infty} \le \frac{\mathfrak{M}_{f,n_{ij}+1}}{(n_{ij}+1)!}\mathfrak{M}_{T,n_{ij}+1}, \qquad x \in [\mu_{e,j-1}, \mu_{e,j}). \tag{3.7}$$

The $\mathfrak{M}_{T,n_{ij}+1}$ can be calculated analytically [13], p. 94,

$$\mathfrak{M}_{T,n_{ij}+1} = (\mu_{e,j} - \mu_{e,j-1})^{n_{ij}+1}\, 2^{-(2n_{ij}+1)}. \tag{3.8}$$

The estimation of the error (3.5) over the $[a,b]$ interval can be written as

$$\mathfrak{E}_{\mathcal{P};O-N;n_j,n_{ij}+1} = \max_j \|E_{\mathcal{P};O-N;j,n_{ij}+1}\|_{\infty}, \qquad x \in [a,b]. \tag{3.9}$$

The $M_{\mathcal{P};O-N}$ model is given by Eq. (2.14) via (3.4): $M_{\mathcal{P};O-N} \equiv \mathcal{P}_q(x) \equiv \widetilde{f}_{n_j,n_{ij}}$. Note that this is a discontinuous model. If Eq. (3.4) is substituted into Eqs. (3.1), (3.2), an acoustic field of $M_{\mathcal{P};O-N}$ is obtained that is denoted by $\widetilde{Q}_{\mathcal{P};O-N}(k,\gamma)$, $\widetilde{p}_{\mathcal{P};O-N}(k,H,x_P)$. The error of the $M_{\mathcal{P};O-N}$ model is given by Eq. (3.5) and a direct measure of the quality by Eq. (3.9).

The multi-elements model $M_{\mathcal{P}}$ with equi-spaced break points $\mu_{e,j}$ and nodes $\nu_{T,i}$ is called the multi-elements model with discontinuous optimal elements $M_{\mathcal{P};O-N}$ (in Fig. 3 $\mu_{e,j} - \nabla$ and $\nu_{T,i} - \bullet$).

In this section all the symbols should be completed by an additional index $n_j, n_{ij}+1$; e.g. $M_{\mathcal{P};O-N;n_j,n_{ij}+1}$.

4. Numerical implementation

The aim of numerical calculations is to compare the quality of the new model $M_{\mathcal{P};O-N}$ with those of other models, i.e. with those of $M_{\mathcal{W};R}$ and $M_{\mathcal{P};R}$. To do this the same total number of nodes and/or of the same degrees are assumed, see Fig. 3. To realise the task, one takes into account:

$M_{\mathcal{W};R;5}$ $\rightarrow$ 1-element regular model of degree 4 with 5 nodes; the graphs concerning this model are denoted by short dashed lines marked by 1,

$M_{\mathcal{P};R;2,3}$ $\rightarrow$ 2-element regular model of degree 2 with 3 nodes on each element; lines 1: short + long,

$M_{\mathcal{P};O-N;2,3}$ $\rightarrow$ 2-element model degree 2 with 3 nodes on each optimal element; lines 2: short + short + long (bolted).

In all the figures the same kind of lines relates to the same model.

5. Calculations, results, conclusions

As a preliminary check of the quality of the models, the cross-sections of the source and of the models are shown in Fig. 4; $x \in [0, x_b/4]$, where $x_b = 1$ is assumed. Inspection of the figure reveals two conclusions:

- *The model $M_{\mathcal{P};O-N}$ consists of discontinuous elements.*

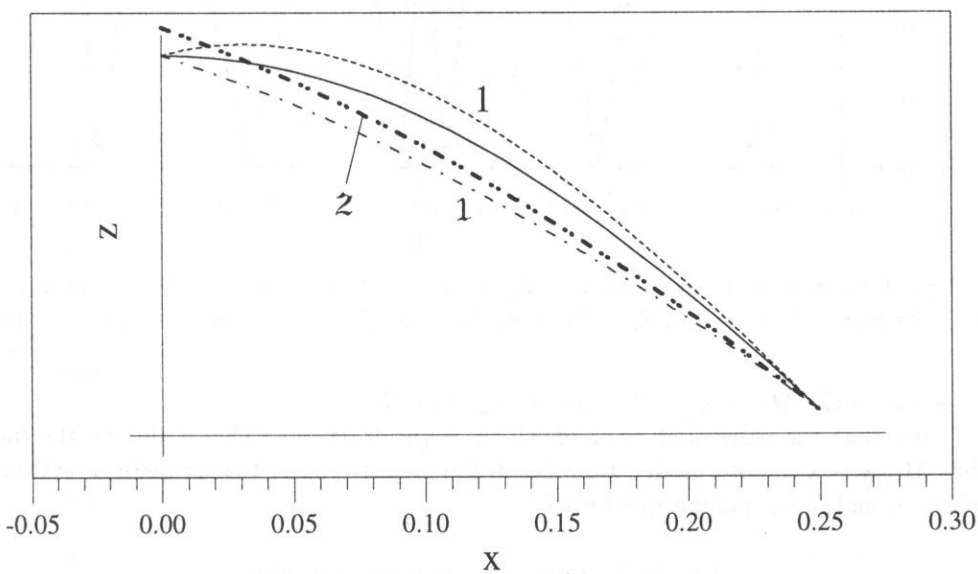

Fig. 4. Cross-sections of the source and the models: solid line – membrane, 1 – $M_{\mathcal{W};R;5}$, 1 – $M_{\mathcal{P};R;2,3}$, 2 – $M_{\mathcal{P};O-N;2,3}$.

Hereby it is quite suitable to modelling the boundary with singular points, see Ref. [7] p. 87, p. 237. Furthermore, the displacements of the nodes inside the element from singular points are exactly determined.

- *The model $M_{\mathcal{P};O-N}$ is more convergent to the source than the others are.*

In order to confirm this conclusion, two measures of the model quality are examined.

5.1. *Direct measure of the model quality*

As mentioned in Sec. 3.2, the estimation of the model error on the interval $[a, b]$ makes up a measure of the model quality, Eqs. (2.8) and (2.22). Here they are plotted in Fig. 5. As expected, the estimated error of $M_{\mathcal{P};O-N}$ is less than that of $M_{\mathcal{P};R}$. Because the error estimation on the $[a, b]$ interval is a direct measure of the model quality.

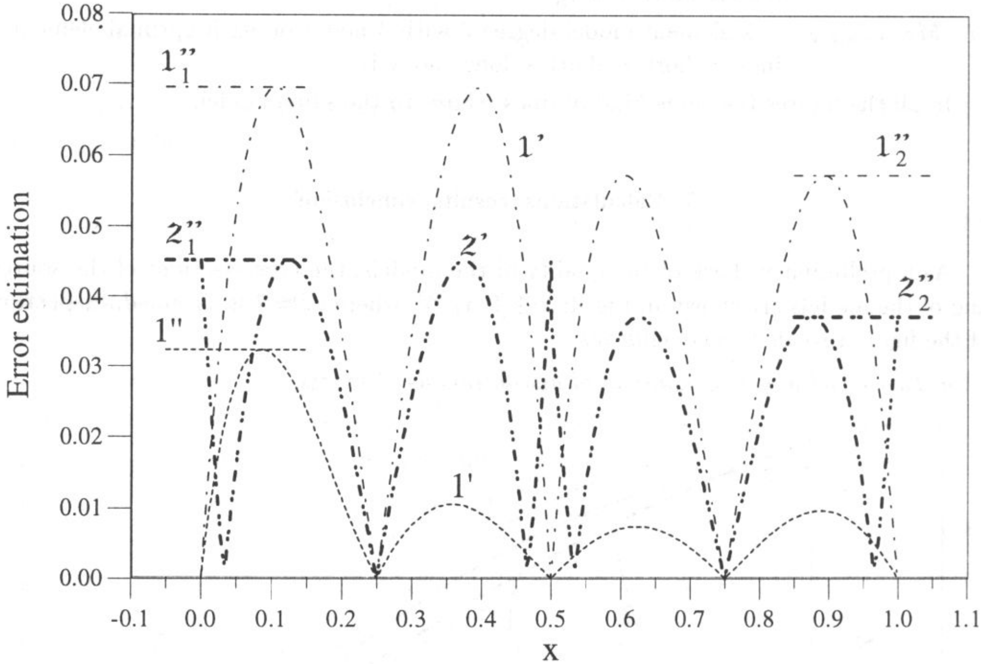

Fig. 5. Estimations of the errors: line $1'$ – $\|E_{\mathcal{W};R;5}(x)\|_{\infty,f}$, $1'' \equiv \mathfrak{E}_{\mathcal{W};R;5}$, $1'_j$ – $\|E_{\mathcal{P};R;j,3}(x)\|_{\infty,f}$, $1''_j$ – $\|E_{\mathcal{P};R;j,3}\|_{\infty}$, $1''_1 \equiv \mathfrak{E}_{\mathcal{P};R;2,3}$, $2'_j$ – $\|E_{\mathcal{P};O-N;j,3}(x)\|_{\infty,f}$, $2''_j$ – $\|E_{\mathcal{P};O-N;j,3}\|_{\infty}$, $2''_1 \equiv \mathfrak{E}_{\mathcal{P};O-N;2,3}$.

- *The model $M_{\mathcal{P};O-N}$ is of better quality than $M_{\mathcal{P};R}$.*

Note that the estimated error of $M_{\mathcal{W};R}$ is the least one. This is due to the fact that $M_{\mathcal{W};R}$ is a smooth model. However in the present paper, the comparison between $M_{\mathcal{P};O-N}$ and $M_{\mathcal{P};R}$ plays a greater role.

5.2. *Indirect measure of the model quality*

To confirm the last conclusion, the models are compared in a different manner. For this purpose one defines:

- $\Delta Q(k,\gamma) = Q(k,\gamma) - \widetilde{Q}(k,\gamma)$,
- $\Delta p(k, H, x_P) = p(k, H, x_P) - \widetilde{p}(k, H, x_P)$.

These differences may be interpreted as indirect measures of the model quality.

The differences in the directivity functions are presented in Fig. 6. The results clearly confirm the last conclusion. For a comprehensive study, Figs. 7 and 8 show the differences

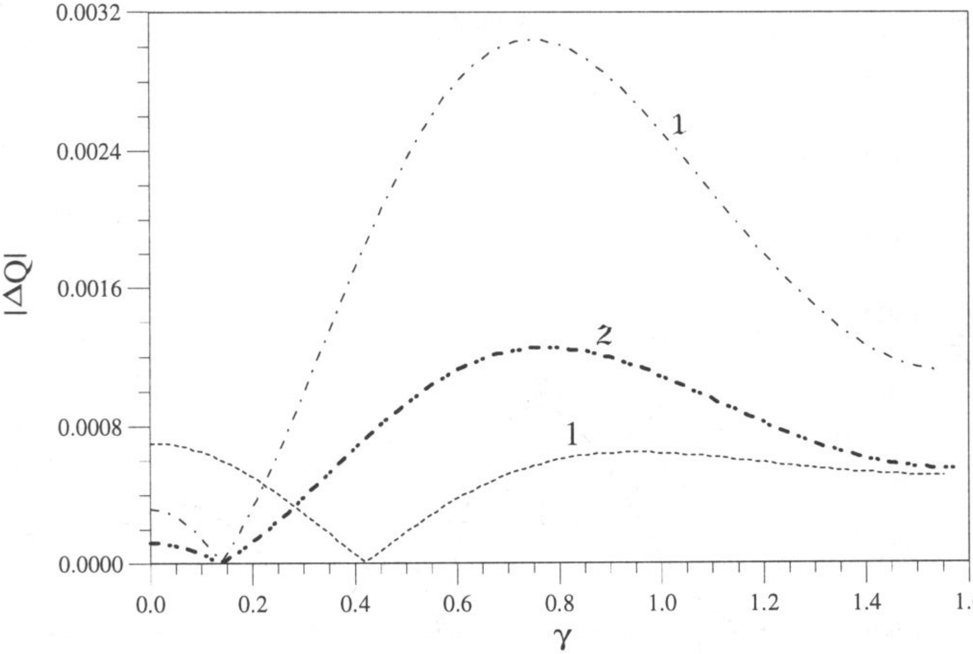

Fig. 6. Difference of the directivity functions $\Delta Q(k,\gamma)$, $k=5$; line 1 – $|\Delta Q_{\mathcal{W};R;5}|$, 1 – $|\Delta Q_{\mathcal{P};R;2,3}|$, 2 – $|\Delta Q_{\mathcal{P};O-N;2,3}|$.

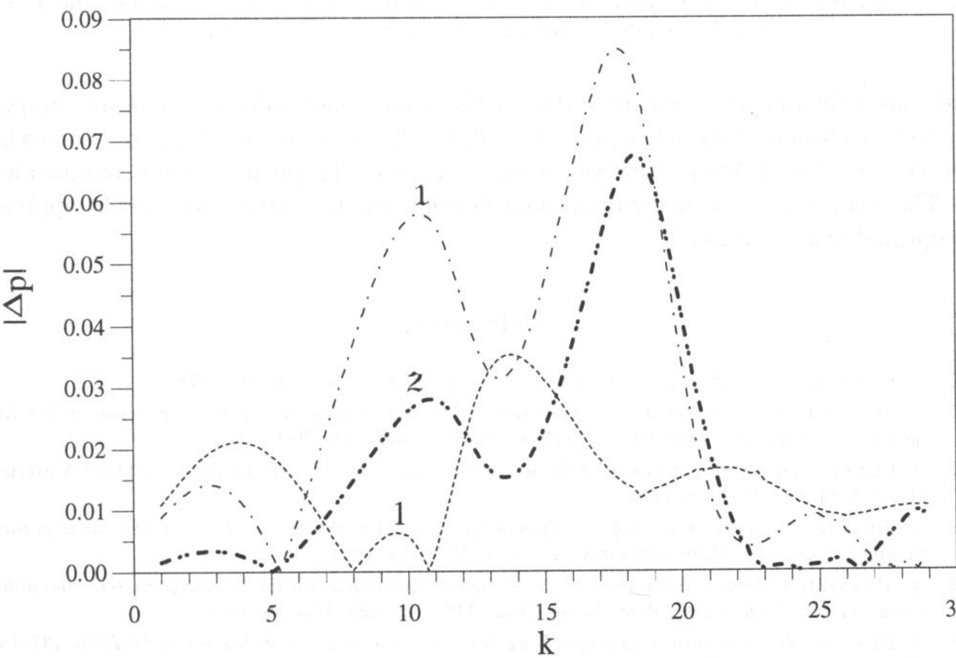

Fig. 7. Difference in the acoustic pressures $\Delta p(k,H,x_P)$ at the fixed point of the axis, $H=0.1x_b$, $k=5$; line 1 – $|\Delta p_{\mathcal{W};R;5}|$, 1 – $|\Delta p_{\mathcal{P};R;2,3}|$, 2 – $|\Delta p_{\mathcal{P};O-N;2,3}|$.

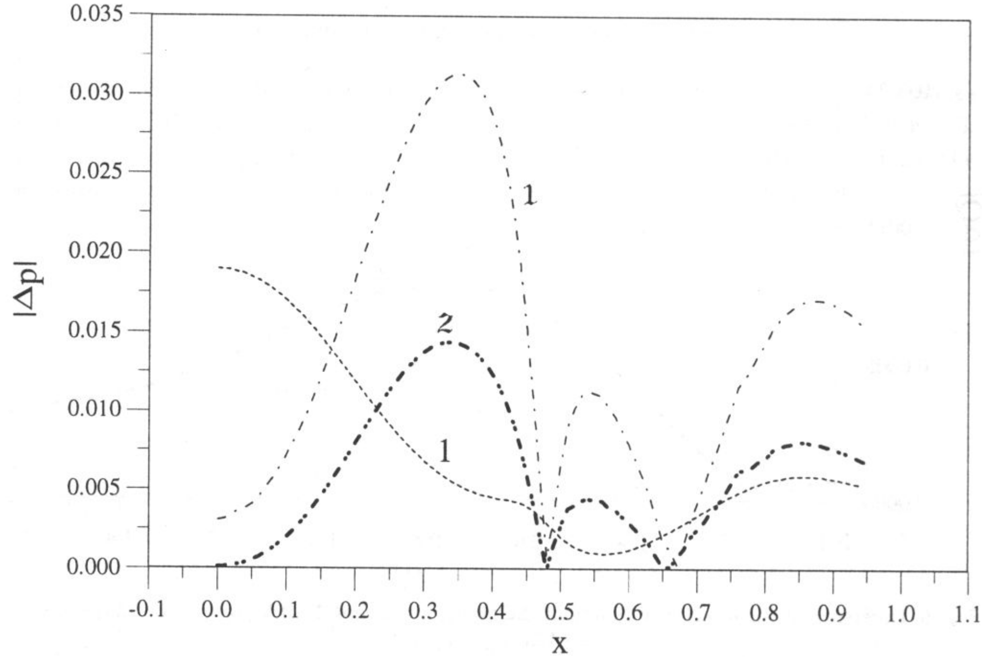

Fig. 8. Difference in the acoustic pressures $\Delta p(k, H, x_p)$ on the line parallel to the radius of the source, $H = 0.1x_b$, $k = 5$; line 1 – $|\Delta p_{\mathcal{W};R;5}|$, 1 – $|\Delta p_{\mathcal{P};R;2,3}|$, 2 – $|\Delta p_{\mathcal{P};O-N;2,3}|$.

of the acoustic pressures near the surface of the models. Generally, all the results support the last conclusion. Only in the range $k > 20$, Fig. 7, the quality of $M_{\mathcal{P};O-N}$ is somewhat poorer than that of $M_{\mathcal{P};R}$. However an explanation of this phenomenon is impossible.

The quality of the model with optimal elements can be further improved by applying an optimal discretization

References

[1] C. de Boor, *A practical guide to splines*, Springer-Verlag, N.Y., Berlin 1978.

[2] A. Brański, M. Ratuszniak, *Modelowanie płaskiego źródła drgającego z prędkością bez linii węzłowych*, Materiały OSA'93, 107–110, Rzeszów - Polańczyk 1993.

[3] A. Brański, *Optymalny, z punktu doboru węzłów, model akustyczny membrany kołowej*, Materiały OSA'94, 61–64, Wrocław 1994.

[4] A. Brański, A. Grzebyk, *Błędy optymalnego, z punktu doboru węzłów, modelu akustycznego membrany kołowej*, Materiały OSA'94, 65–68, Wrocław 1994.

[5] A. Brański, *Optimal, from the choice of nodes, acoustical model of axisymmetric membrane vibrating with high modes*, Proc. Noise Control'95, 123–126, Warsaw 1995.

[6] A. Brański, *Zastosowanie rozsuniętych węzłów Czebyszewa w BEM*, Materiały OSA'96, 137–142, Gliwice - Ustroń 1996.

[7] C.A. Brebbia, J. Dominguez, *Boundary elements. An introductory course*, Comp. Mech. Publ., McGraw-Hill Book Company, Southampton 1992.

[8] C.F. Gerald, P.O. Wheatley, *Applied numerical analysis*, Addison-Wesley Publ. Comp., USA 1984.

[9] L.J. Gray, E. Lutz, *On the treatment of corners in the boundary element method*, J. Comput. Appl. Math., **32**, 369–386 (1990).

[10] J.M. Jankowscy, *Przegląd metod i algorytmów numerycznych*, WNT, Warszawa 1981.

[11] J.R. Rice, *Numerical methods, software, and analysis*, IMSL Refrence Edition, Mc Graw-Hill Book Company, N.Y. 1983.

[12] L.L. Schumaker, *Splines functions: basic theory*, John Wiley & Sons, N.Y. 1981.

[13] B. Sendow, Stare i nowe w metodach numerycznych, PWN, 1976.

[14] A.F. Seybert, C.Y.R. Cheng, *Application of the boundary element method to acoustic cavity response and muffler analysis*, ASME Trans., 1988, private communication.

[15] J.J. do Rego Silva, *Acoustic and elastic wave scattering using boundary elements*, Computational Mechanics Publications, Southampton – Boston 1994.

[16] E. Skudrzyk, *The foundations of acoustics*, Springer-Verlag, Wien – New York 1971.

[17] W. Wang *et al.*, *A boundary integral approach for acoustic radiation of axisymmetric bodies with arbitrary boundary conditions valid for all wave number*, J.A.S.A., **101**, 3, 1468–1478 (1997).

[18] R. Wyrzykowski, *Liniowa teoria pola akustycznego ośrodków gazowych*, WSP, Rzeszów 1972.