THE REAL ACOUSTIC POWER OF A PLANAR ANNULAR MEMBRANE RADIATION FOR AXIALLY-SYMMETRIC FREE VIBRATIONS

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The real part of the acoustic power radiated by a planar annular membrane is considered for axially-symmetric free vibrations. The membrane is located in a planar, rigid baffle and radiate acoustic wave into a lossless and homogeneous fluid medium. Sinusoidal in time processes are examined. The real power is obtained as elementary form for high-frequency radiated waves.

Notations

c propagation velocity of an acoustic wave in a fluid medium of density ϱ_0,

$J_m(x)$ Bessel function of the m-th order,

k $= r_2/r_1$,

k_n $= \omega_n\sqrt{\sigma/T}$,

k_0 $= 2\pi/\lambda$,

N acoustic power radiated by the membrane (3.1),

N' normalised power radiated by the membrane,

$N_m(x)$ Neumann function of the m-th order,

p acoustic pressure,

r radial variable,

r_1, r_2 radii of the annular membrane,

S $= \pi(r_2^2 - r_1^2)$,

T stretching force of the membrane, referred to unit length,

t time,

v normal component of vibration velocity of points of the membrane surface,

v_n vibration velocity of points on the surface of the membrane for mode $(0, n)$,

W characteristic function of planar annular source for $(0, n)$ vibration mode $(6')$,

x_n n-th root of the equation (2.2),

α_n $= J_0(x_n)/J_0(kx_n)$,

β $= k_0 r_1$,

δ_n $= x_n/\beta$,

η transverse displacement of points of the membrane surface,

λ wavelength in a fluid medium,

ϱ_0 density of a fluid medium,

σ surface density of the membrane,

ω_n angular frequency of free vibrations, corresponding to mode $(0, n)$.

1. Introduction

Planar vibrating sources are important for problems of generation and propagation of acoustic waves in a fluid medium. Most of research concentrates on analyse of rectangular and axially-symmetric sources. Extensive universal research of axially-symmetric sources are realised on acoustic wave radiation by: vibrating circular pistons (e.g. PRITCHARD [5], PORTER [4]), planar angular pistons (e.g. THOMPSON [8], MERRIWEATHER [3]) and membranes and circular plates (e.g. LEVINE and LEPPINGTON [2], RDZANEK [6] and [7]). Those papers, concerning membranes and circular plates, includes problems: the energetic aspect of radiating sources, acoustic interactions of particular elements of the source surface, constituent elements of sources array, vibration form influence on the resultant field radiated by vibrating array.

Up to now there were no elementary equations of acoustic power radiated by the planar annular vibrating membrane.

The classical mathematical method was used and the equation of the form of the Bouwkamp's integral [8] for the real part of acoustic power radiated by a planar annular membrane in case of axially-symmetric free vibrations. The considered processes were varying sinusoidally with time. Use of LEVIN and LEPINGTON'S mathematical method [2] based on Cauchy's theorem of residua allowed the derivation of the equation of real part of acoustic power of elementary form in special case for high-frequency waves' radiation. Frequency characteristics of described acoustic power are also presented graphically.

2. The annular membrane's free vibrations

The membrane is tight on two circles with radii r_1 and r_2, and $r_1 < r_2$. We consider axially-symmetric free vibrations sinusoidal in time. The transverse deflection of points of the membrane surface $\eta(r,t) = \eta(r)\exp(i\omega t)$ with boundary conditions $\eta(r_2,t) = \eta(r_1,t) = 0$ is represented by the n-th radial form of free vibrations

$$\eta_n(r)/A_n = J_0\left(x_n\frac{r}{r_1}\right) - \frac{J_0(x_n)}{N_0(x_n)}N_0\left(x_n\frac{r}{r_1}\right), \tag{2.1}$$

where J_0, N_0 are cylindrical functions of null order correspondingly Bessel's and Neumann's. The value $x_n = k_n r_1$ is the n-th frequency equation's root

$$\frac{J_0(kx_n)}{J_0(x_n)} = \frac{N_0(kx_n)}{N_0(x_n)}, \tag{2.2}$$

where $k = r_2/r_1 \geq 1$ and $k_n = \omega_n\sqrt{\sigma/T}$, ω_n is n-th free frequency, σ is surface density of the membrane, T is the stretching force of the membrane. The Table 1 includes some values of the frequency equation's roots (2.2) (compare [1]).

We calculate the constant A_n from the normalisation condition

$$\int_{r_1}^{r_2} \eta_n^2(r)r\,dr = \frac{1}{2}(r_2^2 - r_1^2). \tag{2.3}$$

Table 1. Roots x_n of equation $J_0(x_n)N_0(kx_n) - J_0(kx_n)N_0(x_n) = 0$.

n	k					
	1.1	1.2	1.5	2	3	5
1	31.412314	15.701344	6.270239	3.123029	1.548459	0.7631913
2	62.830045	31.412615	12.559773	6.273438	3.129084	1.5571072
3	94.246574	47.121681	18.845157	9.418203	4.703797	2.3464207
4	125.662802	62.830196	25.129431	12.561424	6.276664	3.1340324
5	157.078909	78.538490	31.413277	15.703999	7.848734	3.9208424
6	188.494956	94.246675	37.696903	18.846253	9.420391	4.7072157

We get than

$$A_n = \frac{\pi}{2} x_n (k^2 - 1)^{1/2} \left\{ \frac{1}{N_0^2(kx_n)} - \frac{1}{N_0^2(x_n)} \right\}^{-1/2}. \tag{2.3'}$$

3. An integral form of the acoustic power

Let the source of surface S of the normal component of the vibration velocity v radiate acoustic pressure p. Than $N = \frac{1}{2} \int_S pv^* \, dS$ is the acoustic power radiated by the source. v^* is here a value conjugate with a complex value v.

We calculate the acoustic power of the source of the axial symmetry on the basis of the integral equation (compare the Bouwkap's integral [8] and the paper [6])

$$N = \pi \varrho_0 c k_0^2 \int_0^{\pi/2 - i\infty} W^2(\vartheta) \sin \vartheta \, d\vartheta, \tag{3.1}$$

where c is the velocity of propagation of the wave in a fluid medium of density in rest stage ϱ_0, $k_0 = 2\pi/\lambda$ is a wave number, λ is wavelength and

$$W(\vartheta) = \int_{r_1}^{r_2} v(r) J_0(k_0 r \sin \vartheta) r \, dr \tag{3.2}$$

is the characteristic function of planar annular sound source, constraints radii of which are r_1 and r_2, $v_n(r) = -i\omega_n \eta_n(r)$ is the normal component of vibration velocity in the case of $(0, n)$ vibration mode. The integral (3.1) is calculated in the plane of the complex variable $\vartheta = \vartheta' + i\vartheta''$.

If $k_0 \to \infty$ then $p(r) = \varrho_0 c v(r)$, $N^{(\infty)} = \frac{1}{2} \varrho_0 c \int_S v^2 \, dS$ and than it is comfortable to use for calculations the normalised radiation power $N/N^{(\infty)}$.

Inserting the characteristic function

$$\frac{W_n(\vartheta)}{-i\omega_n A_n} = \frac{2}{\pi} \frac{r_1^2}{x_n^2 - \beta^2 \sin^2 \vartheta} \frac{1}{N_0(x_n)} \{\alpha_n J_0(k\beta \sin \vartheta) - J_0(\beta \sin \vartheta)\}, \tag{3.3}$$

calculated on the basis of equations (3.2) and (2.1), into the equation (3.1), we get

$$\frac{N_n}{N_n^{(\infty)}} = \frac{2\delta_n^2}{\alpha_n^2 - 1} \int\limits_0^{\pi/2 - i\infty} \left\{ \frac{\alpha_n J_0(k\beta \sin\vartheta) - J_0(\beta \sin\vartheta)}{\sin^2\vartheta - \delta_n^2} \right\}^2 \sin\vartheta \, d\vartheta, \tag{3.4}$$

where $\delta_n = x_n/\beta$, $\beta = k_0 r_1$, $\alpha_n = J_0(x_n)/J_0(kx_n)$.

If we confine integration in equation (3.4) to real values $0 \leq \mathrm{Re}(\vartheta) \leq \pi/2$ and substitute $\sin\vartheta' = x$, then we get integral equation

$$N_n' = \frac{2\delta_n^2}{\alpha_n^2 - 1} \int\limits_0^1 \left\{ \frac{\alpha_n J_0(k\beta x) - J_0(\beta x)}{x^2 - \delta_n^2} \right\}^2 \frac{x \, dx}{\sqrt{1 - x^2}}, \tag{3.4'}$$

where $N_n' = \mathrm{Re}\{N_n/N_n^{(\infty)}\}$ is the real component of the normalised acoustic power radiated by the n-th axially-symmetric mode of the planar annular membrane.

4. The membrane's radiation for the high frequency range

If $\delta_n < 1$ ($\delta_n^2 \ll 1$) then analysing equation (3.4') is much more easy. We use the mathematical method of LEVIN and LEPPINGTON [2] and we introduce a function of a complex variable

$$F(z) = \left\{ \alpha_n^2 J_0(k\beta z) - 2\alpha_n J_0(\beta z) \right\} H_0^{(1)}(k\beta z) + J_0(\beta z) H_0^{(1)}(\beta z) \tag{4.1}$$

such that

$$\mathrm{Re}\, F(z) = \left\{ \alpha_n J_0(k\beta x) - J_0(\beta x) \right\}^2, \tag{4.1'}$$

where x is a real variable, $H_0^{(1)}$ is the Hankel's function of 1-st kind and null order.

The base of analysis is the equation which left side is the contour integral

$$\int\limits_C \frac{z F(z) \, dz}{\sqrt{1 - z^2}(z^2 - \delta_n^2)^2} = 0 \tag{4.2}$$

calculated for contour C (Fig. 1) inside which the integrand is single-valued and regular (comp. [9]). There are no contributions during integration both over a big circle when its radius increases infinitely and over arcs of small circles around the points of branching ($z = 0$, $z = 1$), when their radii decreases tending to null. At the point $z = \delta_n$ the integrand (4.2) has a pole of 2-nd order.

On applying the Cauchy's theorem concerning residua, we get the integral (4.2) of the form

$$\mathcal{P} \int\limits_0^1 \frac{x F(x) \, dx}{\sqrt{1 - x^2}(x^2 - \delta_n^2)^2} - \pi i \, \mathcal{F}'(\delta_n) + \int\limits_1^\infty \frac{x F(x) \, dx}{-i\sqrt{x^2 - 1}(x^2 - \delta_n^2)^2}$$

$$+ \int\limits_\infty^0 \frac{-y F(iy) \, dy}{\sqrt{y^2 + 1}(y^2 + \delta_n^2)^2} = 0, \tag{4.2'}$$
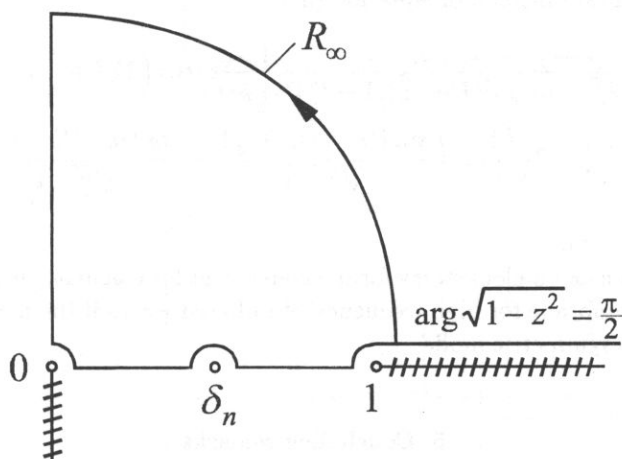
Fig. 1. The integration contour C for pattern (4.2) (comp. the paper [2]).

where the auxiliary function is introduced

$$\mathcal{F}(z) = \frac{z F(z)}{\sqrt{1-z^2}(z+\delta_n)^2} \tag{4.3}$$

and the integral $\mathcal{P} \int_0^1$ is interpreted as a principal value. We take into account that $\operatorname{Re} F(iy) = 0$, then

$$\operatorname{Re}\left\{\mathcal{P} \int_0^1 \frac{x F(x)\,dx}{\sqrt{1-x^2}(x^2-\delta_n^2)^2}\right\} = \int_0^1 \left\{\frac{\alpha_n J_0(k\beta x) - J_0(\beta x)}{x^2-\delta_n^2}\right\}^2 \frac{x\,dx}{\sqrt{1-x^2}}$$

$$= \int_0^1 \frac{\alpha_n N_0(k\beta x)\,[\alpha_n J_0(k\beta x) - 2 J_0(\beta x)] + J_0(\beta x) N_0(\beta x)}{(x^2-\delta_n^2)^2} \frac{x\,dx}{\sqrt{x^2-1}}$$

$$+ \operatorname{Re}\{\pi i\,\mathcal{F}'(\delta_n)\}. \tag{4.4}$$

We also take into account that $F(\delta_n) = 0$, $\operatorname{Re} F'(\delta_n) = 0$, $\operatorname{Im} F'(\delta_n) = \dfrac{2}{\pi \delta_n}(1-\alpha_n^2)$ and finally

$$\operatorname{Re}\{\pi i\,\mathcal{F}'(\delta_n)\} = \frac{\alpha_n^2 - 1}{2\delta_n^2 \sqrt{1-\delta_n^2}}. \tag{4.5}$$

Now we calculate the integral (4.4) inside of interval $[1, \infty)$, applying the asymptotic expressions

$$J_0(ax) N_0(bx) \sim \frac{1}{\pi x \sqrt{ab}}\left\{\sin(b-a)x - \cos(b+a)x\right\}, \tag{4.6}$$

$$\int_1^\infty \frac{e^{imx}\,dx}{\sqrt{x^2-1}(x^2-\delta_n^2)^2} = \sqrt{\frac{\pi}{2m}}\left\{(1-\delta_n^2)^{-2} + O\left(\frac{1}{m}\right)\right\} e^{i(m+\pi/4)}. \tag{4.7}$$

In this way, we obtain instead of equation $(3.4')$

$$
N'_n = \frac{1}{\sqrt{1-\delta_n^2}} + \frac{\delta_n^2}{\beta\sqrt{\pi\beta}(1-\alpha_n^2)(1-\delta_n^2)^2} \left\{ \frac{\alpha_n^2}{k\sqrt{k}} \cos\left(2k\beta + \frac{\pi}{4}\right) \right.
$$
$$
\left. + \cos\left(2\beta + \frac{\pi}{4}\right) + 2\sqrt{\frac{2}{k}}\alpha_n \left(\frac{\sin\left((k-1)\beta + \frac{\pi}{4}\right)}{\sqrt{k-1}} - \frac{\cos\left((k+1)\beta + \frac{\pi}{4}\right)}{\sqrt{k+1}} \right) \right\} \qquad (4.8)
$$

with error $o(\delta_n^2\beta^{-3/2})$.

This equation is of an elementary form – convenient for calculations of the real power of the annular membrane for high frequency of radiated waves if the membrane vibrates with n-th axially-symmetric mode.

5. Concluding remarks

As result of theoretical analysis of the problem of radiation of a planar annular membrane the elementary expression was derived for normalised real acoustic power of axially-symmetric modes of free vibrations. This expression can be used for digital calculations only if the condition $x_n < \beta = k_0 r_1$ is satisfied.

There were given proper components which have essentially an "oscillating" character of changes of the real component of power of annular membrane (Fig. 2 and Fig. 3).

In case when the condition $x_n < k_0 r_1$ is not satisfied or when we need high accuracy of results, one should perform numerical calculations based on the integral equation $(3.4')$.
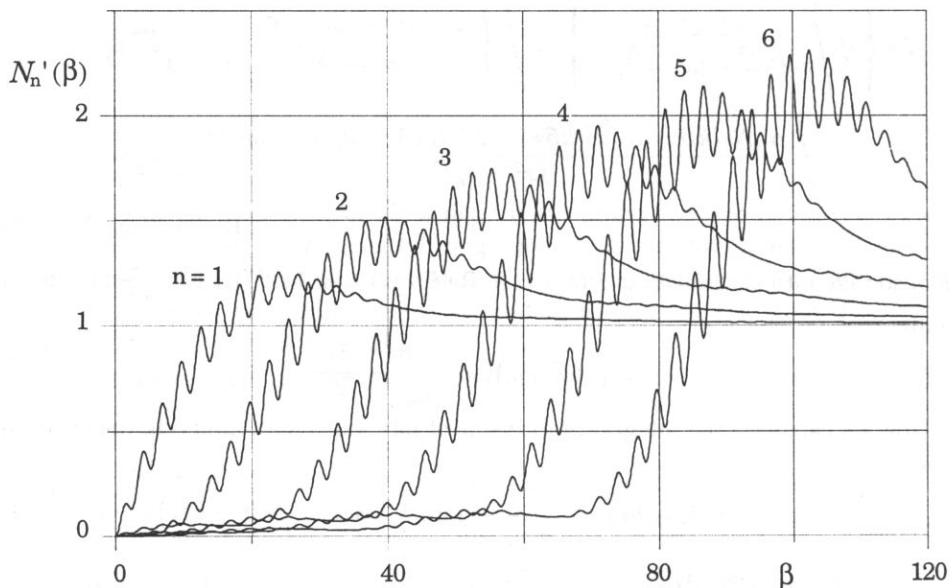

Fig. 2. Normalised real component of acoustic power radiated by the planar annular membrane versus β for modes $(0, n)$ and $k = 1.2$.
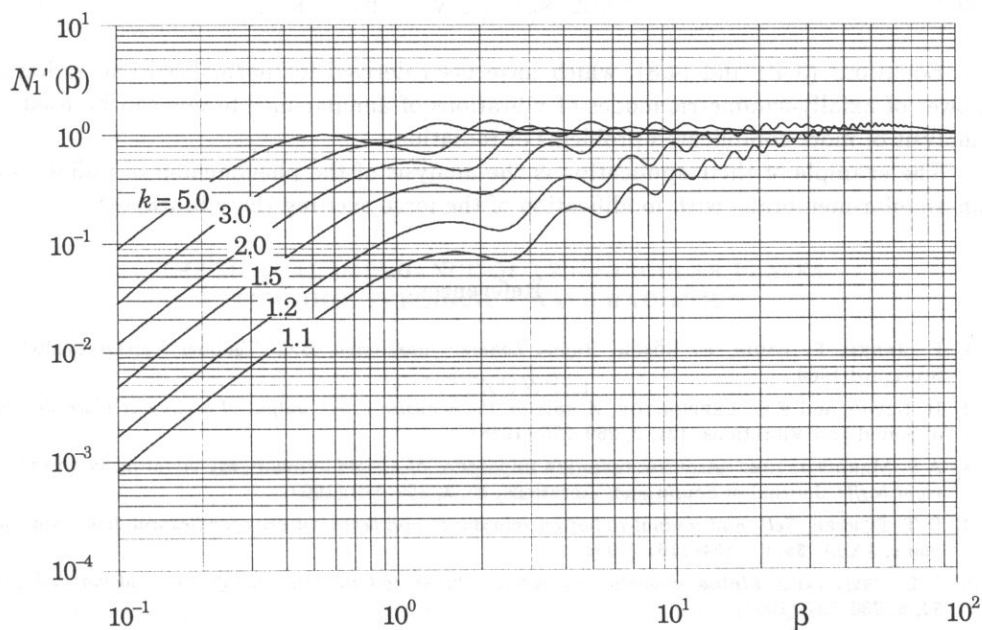
Fig. 3. Normalised real component of acoustic power radiated by the planar annular membrane versus β for different k and mode $(0,1)$.
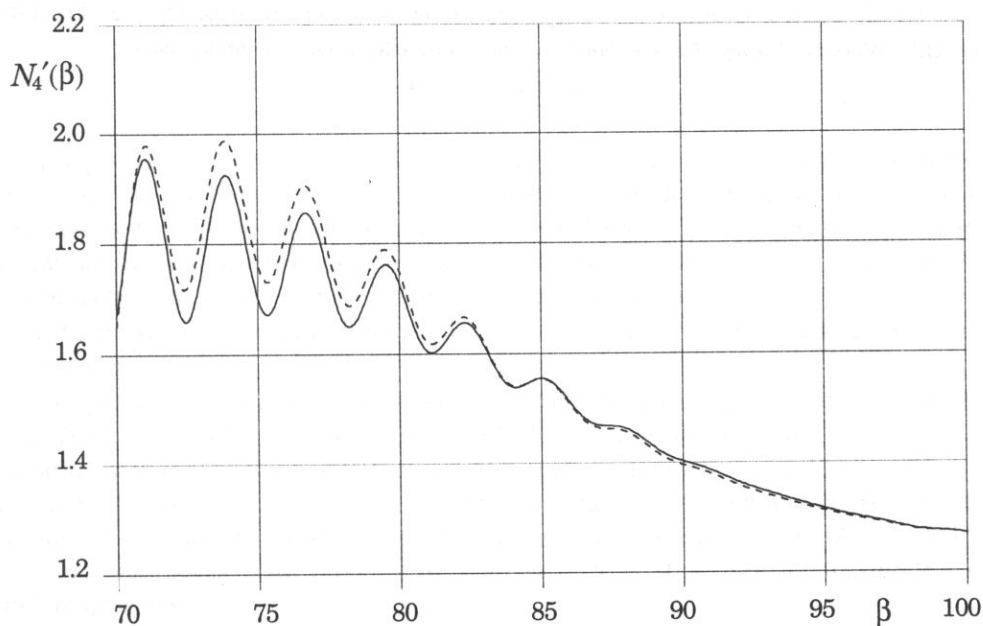

Fig. 4. Normalised real component of acoustic power radiated by the planar annular membrane versus β for the mode $(0,4)$ and $k = 1.2$ – obtained from the formula $(3.4')$ (the solid line). The curve obtained from the formula (4.8) is dashed.

Equations (3.4') and (4.8), which have been derived for normalised real radiation power of axially-symmetric modes of vibrations of annular membrane, can be used for analysis of more complicated problems of radiation.

The example of their application is the analysis of the phenomenon of radiation of an annular membrane with modification of the force exciting the vibrations.

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