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THE MUTUAL IMPEDANCE OF TWO CIRCULAR PLATES FOR HIGH FREQUENCY WAVE RADIATION

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In this paper the mutual impedance of two thin circular plates with non-axisymmetric, time-harmonic free vibrations is analyzed. It is assumed that plates clamped at the circumference are placed in a rigid, planar baffle and radiate into a lossless and homogeneous fluid medium. Damping in plates is ignored.

Using the Cauchy theorem on residues and asymptotic formulae for the Bessel functions, an approximate expression is derived for a normalized mutual resistance and reactance for high frequencies.

1. Introduction

The practical application of a system of two plates as a sound transmitter or receiver of acoustic waves requires the knowledge of frequency characteristics of its acoustic parameters. One of them is the mutual impedance describing the influence of the plates vibrations on each other. In general case the vibrations of plates are non-axisymmetric so the mutual impedance concerns non-axisymmetric modes.

Hitherto the problem of interactions of non-axisymmetric modes was considered only for one plate [4, 8].

The problem of the mutual impedance of two elastic circular pistons was investigated in 1964 by PORTER [5] and CHAN [1] in 1967. They expressed an axially symmetric distribution of velocity in terms of the radial variable by a power series. In the paper [6], the mutual impedance of two circular co-planar sources with nonuniform velocity distributions: gaussian, parabolic and bessel has been considered. In paper [2], expressions were presented for acoustic power of two sources with parabolic velocity distribution for high frequencies.

The present paper deals with the mutual impedance of circular plates supporting non-axisymmetric free vibrations. By using the LEVINE and LEPPINGTON'S method [3], which is based on the Cauchy theorem, an elementary formula is derived for a normalized mutual impedance.

2. Mutual impedance of two circular plates

An acoustic radiator vibrating in an elastic medium encounters a counteraction of the medium. The measure of the source loading is the acoustic impedance defined as follows:

$$Z_a = \frac{1}{\sigma\langle v^2\rangle} \int_\sigma p(r,\varphi)\, v^*(r,\varphi)\, d\sigma\,, \tag{2.1}$$

where

$$\langle v^2\rangle = \frac{1}{\sigma} \int_\sigma v(r,\varphi)\, v^*(r,\varphi)\, d\sigma \tag{2.2}$$

is the mean value of the second power of the normal velocity of points on the source surface σ. Values $p(r,\varphi)$, $v(r,\varphi)$ stand for the surface distribution of pressure and normal velocity, respectively, r, φ denote the radial and angular coordinates of a point of the source with respect to the polar reference system. Now we find the analytical form of the definition (2.1) for two plates on which the distribution of velocity is defined as a superposition of free vibrations.

Let us consider two thin plates of the radii a_1, a_2, fixed on the rim in a rigid and flat acoustic baffle. The plates radiate into lossless and homogeneous fluid medium. The distance between the centres of the plates is denoted by l (Fig. 1).

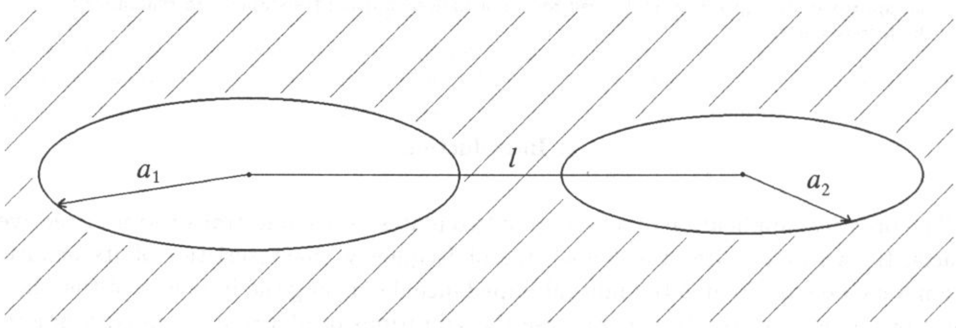

Fig. 1. The geometry of plates.

The normal velocity of the first plate is given in the form of a double, infinite sum:

$$v^{(1)}(r,\varphi) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_{mn}^{(1)}\, v_{mn}^{(1)}(r,\varphi), \tag{2.3}$$

where $c_{mn}^{(1)}$ is an expansion coefficient of velocity in a series of eigenfunctions for the first plate and $v_{mn}^{(1)}(r,\varphi)$ are the normal velocity mode functions

$$v_{mn}^{(1)}(r,\varphi) = \mathcal{V}_{mn}^{(1)} \left\{ \begin{array}{c} \sin(m\varphi) \\ \cos(m\varphi) \end{array} \right\} \left[J_m\left(\gamma_{mn}\frac{r}{a_1}\right) - \frac{J_m(\gamma_{mn})}{I_m(\gamma_{mn})} I_m\left(\gamma_{mn}\frac{r}{a_1}\right)\right], \tag{2.4}$$

where $J_m(\,\cdot\,)$ denotes the Bessel function of the m-th order, $I_m(\,\cdot\,)$ is the modified Bessel function of the m-th order, γ_{mn} stands for the roots of the characteristic equation $J_m(\gamma_{mn})I_m'(\gamma_{mn}) - I_m(\gamma_{mn})J_m'(\gamma_{mn}) = 0$.

In general case, the expansion coefficients of velocity in a series of eigenfunctions are complex, e.g. when we take into account the losses into material of the plates or the influence of radiated wave on the vibrations of the plates. The normalization factors

$$\mathcal{V}_{mn}^{(1)} = \frac{\sqrt{\varepsilon_m}}{\sqrt{2\pi} a_1 J_m(\gamma_{mn})}, \qquad \varepsilon_m = \begin{cases} 1, & m = 0, \\ 2, & m \geq 1, \end{cases} \tag{2.5}$$

are chosen such that the eigenfunctions are orthonormal.

Each of the normal velocity modes $v_{kl}^{(2)}$ of the second plate gives rise to an extra acoustic pressure on the surface of the first plate, p_{kl}^{21}. The total such a pressure is equal to an infinite sum of particular pressures p_{kl}^{21}

$$p^{21}(r, \varphi) = \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} c_{kl}^{(2)} \, p_{kl}^{21}(r, \varphi), \tag{2.6}$$

where $c_{kl}^{(2)}$ is an expansion coefficient of the velocity on the second plate. Substituting the velocity (2.3) and the pressure (2.6) into the definition (2.1), we get:

$$Z_a^{21} = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} c_{mn}^{(1)*} \, c_{kl}^{(2)} \sqrt{\frac{\langle v_{kl}^{(2)2} \rangle}{\langle v_{mn}^{(1)2} \rangle}} \, Z^{21}_{\substack{kl \\ mn}}, \tag{2.7}$$

where

$$Z^{21}_{\substack{kl \\ mn}} = \frac{1}{\sigma \sqrt{\langle v_{mn}^{(1)2} \rangle \langle v_{kl}^{(2)2} \rangle}} \int_{\sigma_1} p_{kl}^{21}(r, \varphi) \, v_{mn}^{(1)*}(r, \varphi) \, d\sigma, \tag{2.8}$$

$\sigma = \pi a_1 a_2$, $\sigma_1 = \pi a_1^2$.

The quantity $Z^{21}_{\substack{kl \\ mn}}$ is the mutual impedance of two circular plates excited with radiating non-axisymmetric modes.

The pressure p_{kl}^{21} is calculated with using the Huygens-Rayleigh formula [6]. It has the following integral form:

$$p_{kl}^{21}(r, \varphi) = \frac{k_0 \varrho_0 \omega (i)^k}{2\pi} \int_0^{\frac{\pi}{2} - i\infty} \int_0^{2\pi} W_{kl}^{(2)}(\vartheta) e^{i k_0 r \sin\vartheta \cos(\varphi - \gamma)} \times \begin{Bmatrix} \sin(k\gamma) \\ \cos(k\gamma) \end{Bmatrix} e^{-i k_0 l \sin\vartheta \cos\gamma} \sin\vartheta \, d\vartheta \, d\gamma, \tag{2.9}$$

where

$$W_{kl}^{(2)}(\vartheta) = \mathcal{V}_{kl}^{(2)} \int_0^{a_2} J_k(k_0 r_2 \sin\vartheta) \left[J_k\left(\gamma_{kl} \frac{r_2}{a_2}\right) - \frac{J_k(\gamma_{kl})}{I_k(\gamma_{kl})} I_k\left(\gamma_{kl} \frac{r_2}{a_2}\right) \right] r_2 \, dr_2, \tag{2.10}$$

k_0 denotes the wave number in the gaseous medium, ϱ_0 is the equilibrium density of the gaseous medium, ω stands for the angular frequency of the vibrations.

Upon performing the integration in (2.8) with p_{kl}^{21} and $v_{mn}^{(1)}$ replaced by (2.9) and (2.4), respectively, and referring the impedance $Z^{21}_{\substack{kl \\ mn}}$ to the specific resistance of a fluid medium $\varrho_0 c_0$, the normalized mutual impedance between (k,l) and (m,n) modes is obtained as follows

$$\zeta^{21}_{\substack{kl\\mn}} = \frac{\sqrt{\varepsilon_k\varepsilon_m}\,2k_0^2a_1a_2}{\gamma_{mn}\gamma_{kl}}(-1)^k \int\limits_0^{\frac{\pi}{2}-i\infty} \frac{\alpha_{mn}J_m(k_0a_1\sin\vartheta) - \dfrac{k_0a_1\sin\vartheta}{\gamma_{mn}}J_{m+1}(k_0a_1\sin\vartheta)}{1-\left(\dfrac{k_0a_1\sin\vartheta}{\gamma_{mn}}\right)^4}$$

$$\times \frac{\alpha_{kl}J_k(k_0a_2\sin\vartheta) - \dfrac{k_0a_2\sin\vartheta}{\gamma_{kl}}J_{k+1}(k_0a_2\sin\vartheta)}{1-\left(\dfrac{k_0a_2\sin\vartheta}{\gamma_{mn}}\right)^4}$$

$$\times\left[(-1)^m J_{k-m}(k_0l\sin\vartheta) \pm J_{k+m}(k_0l\sin\vartheta)\right]\sin\vartheta\,d\vartheta, \tag{2.11}$$

where $\alpha_{mn} = J_{m+1}(\gamma_{mn})/J_m(\gamma_{mn})$. The signs plus and minus in the last term correspond to the choice of the cosine and sine functions, respectively, in the normal velocity distribution function (2.4).

This solution (2.11) is a generalization of the pattern obtained in [7] where $k = m = 0$, $a_1 = a_2$.

3. Acoustic resistance for high frequencies

The normalized mutual impedance (2.11) has no exact analytical solution. But there is a possibility to calculate its value using an approximate method.

As shown below, one can obtain an approximate representation for the mutual resistance by replacing the Bessel functions with their asymptotic expansions and then making use of the method of stationary phase.

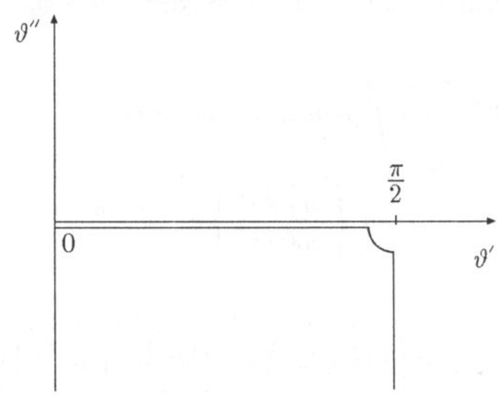

Fig. 2. The integration contour in the complex plate $\vartheta = \vartheta' + i\vartheta''$.

In order to separate the mutual resistance from impedance (2.11) let us substitute $\vartheta = \vartheta' + i\vartheta''$ and consider the integral along the segment $(0, \pi/2)$ of the real axis (Fig. 2). Let us substitute $x = \sin\vartheta'$ and use the abbreviations: $s = a_1/a_2$, $p = l/a_1$, $\beta = k_0a_1$,

$\delta_{mn} = \gamma_{mn}/k_0 a_1$, $\delta_{kl} = \gamma_{kl}/k_0 a_1$. Then we get the expression:

$$\Theta^{21}_{\substack{kl\\mn}} = 2\sqrt{\varepsilon_k \varepsilon_m}\, s^2 \delta_{kl}^2 \delta_{mn}^2 (-1)^k \int\limits_0^1 \frac{\alpha_{mn}\delta_{mn} J_m(\beta x) - x J_{m+1}(\beta x)}{x^4 - \delta_{mn}^4}$$

$$\times \frac{s\alpha_{kl}\delta_{kl} J_k\left(\dfrac{\beta}{s}x\right) - x J_{k+1}\left(\dfrac{\beta}{s}x\right)}{x^4 - (s\delta_{kl})^4} \left[(-1)^m J_{k-m}(\beta p x) \pm J_{k+m}(\beta p x)\right] \frac{x\, dx}{\sqrt{1-x^2}} . \tag{3.1}$$

Let us introduce the function of a complex variable z [3]

$$F(z) = \left[\alpha_{mn}\delta_{mn} J_m(\beta z) - z J_{m+1}(\beta z)\right] \left[s\alpha_{kl}\delta_{kl} J_k\left(\frac{\beta}{s}z\right) - z J_{k+1}\left(\frac{\beta}{s}z\right)\right]$$

$$\times \left[(-1)^m H^{(1)}_{k-m}(\beta p z) \pm H^{(1)}_{k+m}(\beta p z)\right] . \tag{3.2}$$

Now, let us consider the complex integral:

$$\int\limits_C \frac{F(z)\, z\, dz}{\sqrt{1-z^2}\,\left[z^4 - \delta_{mn}^4\right]\left[z^4 - (s\delta_{kl})^4\right]} . \tag{3.3}$$

The contour (Fig. 3) by-passes singular points of the integrand δ_{mn}, $s\delta_{kl}$, $i\delta_{mn}$, $is\delta_{kl}$ and branch point at $z = 0$ (of the Hankel function $H^{(1)}_{k\pm m}(\cdot) = J_{k\pm m}(\cdot) + iN_{k\pm m}(\cdot)$). Part of the contour follows the upper side of the branch cut between $z = 1$ and $z = \infty$ (of the function $\sqrt{1-z^2}$).

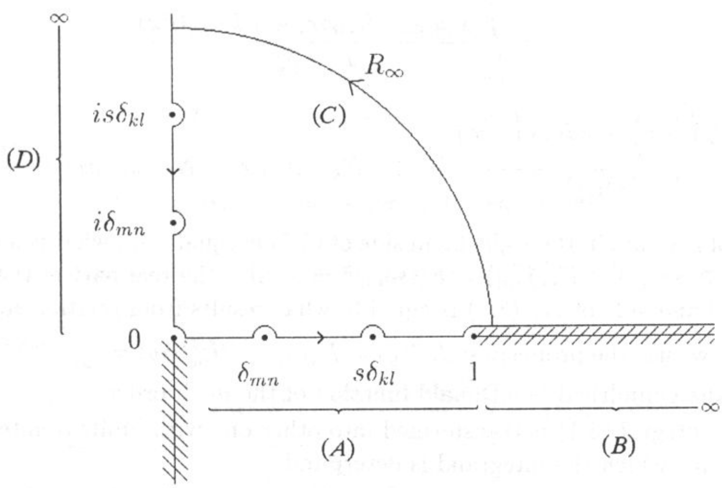

Fig. 3. The integration contour C.

The Cauchy theorem implies that the following is true for the integrals:

$$\wp \underset{(A)}{\int\limits_0^1} + \underset{(B)}{\int\limits_1^\infty} + \underset{(C)}{\int\limits_{R_\infty}} + \wp \underset{(D)}{\int\limits_\infty^0} = \pi i \sum_{j=1}^{4} \operatorname{res} f(z_j), \tag{3.4}$$

where $\wp\int$ denotes the principal value of an integral, $f(z)$ is equal to the integrand in (3.3), $z_j = \delta_{mn}$, $i\delta_{mn}$, $s\delta_{kl}$, $is\delta_{kl}$ are the first order poles.

The integral along a large circle vanish with increasing its radius. Also, integrals along small circles around the points $z = 0$ and $z = 1$ vanish when their radii tend to zero.

Then it remains:

$$\wp\int\limits_0^1 \frac{F(x)x\,dx}{\sqrt{1-x^2}\,[x^4-\delta_{mn}^4]\,[x^4-(s\delta_{kl})^4]} + \int\limits_1^\infty \frac{F(x)x\,dx}{-i\sqrt{x^2-1}\,[x^4-\delta_{mn}^4]\,[x^4-(s\delta_{kl})^4]}$$
$$+\wp\int\limits_\infty^0 \frac{F(iy)iy\,d(iy)}{\sqrt{1+y^2}\,[y^4-\delta_{mn}^4]\,[y^4-(s\delta_{kl})^4]} = \pi i\sum_{j=1}^4 \operatorname{res} f(z_j). \tag{3.5}$$

Taking the real part of the left-hand side of (3.5), we arrive at the integral (3.1)

$$\int\limits_0^1 \frac{\alpha_{mn}\delta_{mn}J_m(\beta x) - xJ_{m+1}(\beta x)}{x^4-\delta_{mn}^4}$$
$$\times \frac{s\alpha_{kl}\delta_{kl}J_k\left(\dfrac{\beta}{s}x\right) - xJ_{k+1}\left(\dfrac{\beta}{s}x\right)}{x^4-(s\delta_{kl})^4}\left[(-1)^m J_{k-m}(\beta p x) \pm J_{k+m}(\beta p x)\right]\frac{x\,dx}{\sqrt{1-x^2}}$$
$$= \int\limits_1^\infty \frac{\alpha_{mn}\delta_{mn}J_m(\beta x) - xJ_{m+1}(\beta x)}{x^4-\delta_{mn}^4}$$
$$\times \frac{s\alpha_{kl}\delta_{kl}J_k\left(\dfrac{\beta}{s}x\right) - xJ_{k+1}\left(\dfrac{\beta}{s}x\right)}{x^4-(s\delta_{kl})^4}\left[(-1)^m N_{k-m}(\beta p x) \pm N_{k+m}(\beta p x)\right]\frac{x\,dx}{\sqrt{x^2-1}}. \tag{3.6}$$

The sum of residues in the right-hand side of (3.5) is equal zero, what is a consequence of $F(\delta_{mn}) = F(s\delta_{mn}) = F(i\delta_{mn}) = F(is\delta_{mn}) = 0$. Also the real part of the third integral in the left-hand side of Eq. (3.5) is equal 0, what results from relation $\Re e F(iy) = 0$. To prove this, we use the properties: $J_m(iy) = I_m(y)i^m$, $H_m^{(1)}(iy) = \dfrac{2}{\pi}i^{-(m+1)}K_m(y)$, where $K_m(y)$ is the cylindrical MacDonald function of the m-th order.

So, the integral (3.1) is transformed into other one with limits of integration from 1 to infinity, for which the integrand is determind.

All cylindrical functions in the integrand on the right-hand side of (3.6) we expand asymptotically [4] as x tends to infinity

$$\begin{aligned} J_m(x) &\simeq \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{2m+1}{4}\pi\right),\\ N_m(x) &\simeq \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{2m+1}{4}\pi\right). \end{aligned} \tag{3.7}$$

After computing the integral on the right-hand side of (3.6) by using the method of stationary phase, we get finally:

$$\begin{aligned}\Theta^{21}_{\substack{kl\\mn}} &= 2\sqrt{\varepsilon_k\varepsilon_m}\frac{s^3\delta_{kl}^2\delta_{mn}^2}{\pi\beta^2\sqrt{p}\,[1-\delta_{mn}^4]\,[1-(s\delta_{kl})^4]}\\
&\times\left[\mathcal{A}\left(\frac{\sin\frac{\beta}{s}(ps-s+1)}{\sqrt{ps-s+1}}+(-1)^{k+m}\frac{\sin\frac{\beta}{s}(ps+s-1)}{\sqrt{ps+s-1}}\right)\right.\\
&-\mathcal{B}\left(\frac{\cos\frac{\beta}{s}(ps-s+1)}{\sqrt{ps-s+1}}-(-1)^{k+m}\frac{\cos\frac{\beta}{s}(ps+s-1)}{\sqrt{ps+s-1}}\right)\\
&-\mathcal{C}\left((-1)^m\frac{\cos\frac{\beta}{s}(ps+s+1)}{\sqrt{ps+s+1}}-(-1)^k\frac{\cos\frac{\beta}{s}(ps-s-1)}{\sqrt{ps-s-1}}\right)\\
&\left.+\mathcal{D}\left((-1)^m\frac{\sin\frac{\beta}{s}(ps+s+1)}{\sqrt{ps+s+1}}+(-1)^k\frac{\sin\frac{\beta}{s}(ps-s-1)}{\sqrt{ps-s-1}}\right)\right],\end{aligned}\tag{3.8}$$

where

$$\begin{aligned}\mathcal{A} &= (s\alpha_{kl}\alpha_{mn}\delta_{kl}\delta_{mn}+1),\\
\mathcal{B} &= (s\alpha_{kl}\delta_{kl}-\alpha_{mn}\delta_{mn}),\\
\mathcal{C} &= (s\alpha_{kl}\alpha_{mn}\delta_{kl}\delta_{mn}-1),\\
\mathcal{D} &= (s\alpha_{kl}\delta_{kl}+\alpha_{mn}\delta_{mn}).\end{aligned}\tag{3.9}$$

4. Acoustic reactance for high frequencies

Acoustic reactance, which is the imaginary part of impedance (2.11), has the form:

$$\begin{aligned}\chi^{21}_{\substack{kl\\mn}} &= \frac{2\sqrt{\varepsilon_k\varepsilon_m}k_0^2a_1a_2}{\gamma_{mn}\gamma_{kl}}(-1)^k\\
&\times\int_0^\infty\frac{\alpha_{mn}J_m(k_0a_1\cosh\vartheta'')-\dfrac{k_0a_1\cosh\vartheta''}{\gamma_{mn}}J_{m+1}(k_0a_1\cosh\vartheta'')}{1-\left(\dfrac{k_0a_1\cosh\vartheta''}{\gamma_{mn}}\right)^4}\\
&\times\frac{\alpha_{kl}J_k(k_0a_2\cosh\vartheta'')-\dfrac{k_0a_2\cosh\vartheta''}{\gamma_{kl}}J_{k+1}(k_0a_2\cosh\vartheta'')}{1-\left(\dfrac{k_0a_2\cosh\vartheta''}{\gamma_{kl}}\right)^4}\\
&\times[(-1)^mJ_{k-m}(k_0l\cosh\vartheta'')\pm J_{k+m}(k_0l\cosh\vartheta'')]\cosh\vartheta''\,d\vartheta''.\end{aligned}\tag{4.1}$$

By substituting $x = \cosh \vartheta''$ and using the same notations as for resistance, we get the expression:

$$\chi^{21}_{\substack{kl\\mn}} = 2\sqrt{\varepsilon_k \varepsilon_m}\, s^2 \delta^2_{kl} \delta^2_{mn} (-1)^k$$

$$\times \int_0^\infty \frac{\alpha_{mn}\delta_{mn} J_m(\beta x) - x J_{m+1}(\beta x)}{x^4 - \delta^4_{mn}} \, \frac{s\alpha_{kl}\delta_{kl} J_k\left(\dfrac{\beta}{s}x\right) - x J_{k+1}\left(\dfrac{\beta}{s}x\right)}{x^4 - (s\delta_{kl})^4}$$

$$\times \left[(-1)^m J_{k-m}(\beta p x) \pm J_{k+m}(\beta p x)\right] \frac{x\, dx}{\sqrt{1-x^2}} \,. \tag{4.2}$$

The calculation of this integral is much easier then that of the resistance because we do not have to change the limits of integrations. We can immediately change all cylindrical functions in (4.2) by inserting their asymptotic forms (3.7) and using the stationary phase method. In this way we arrive at the following equation:

$$\chi^{21}_{\substack{kl\\mn}} = 2\sqrt{\varepsilon_k \varepsilon_m} \frac{s^3 \delta^2_{kl} \delta^2_{mn}}{\pi \beta^2 \sqrt{p}\,[1 - \delta^4_{mn}]\,[1 - (s\delta_{kl})^4]}$$

$$\times \left[\mathcal{A} \left(\frac{\cos \dfrac{\beta}{s}(ps - s + 1)}{\sqrt{ps - s + 1}} + (-1)^{k+m} \frac{\cos \dfrac{\beta}{s}(ps + s - 1)}{\sqrt{ps + s - 1}} \right) \right.$$

$$+ \mathcal{B} \left(\frac{\sin \dfrac{\beta}{s}(ps - s + 1)}{\sqrt{ps - s + 1}} - (-1)^{k+m} \frac{\sin \dfrac{\beta}{s}(ps + s - 1)}{\sqrt{ps + s - 1}} \right)$$

$$+ \mathcal{C} \left((-1)^m \frac{\sin \dfrac{\beta}{s}(ps + s + 1)}{\sqrt{ps + s + 1}} - (-1)^k \frac{\sin \dfrac{\beta}{s}(ps - s - 1)}{\sqrt{ps - s - 1}} \right)$$

$$\left. + \mathcal{D} \left((-1)^m \frac{\cos \dfrac{\beta}{s}(ps + s + 1)}{\sqrt{ps + s + 1}} + (-1)^k \frac{\cos \dfrac{\beta}{s}(ps - s - 1)}{\sqrt{ps - s - 1}} \right) \right], \tag{4.3}$$

where

$$\begin{aligned} \mathcal{A} &= (s\alpha_{kl}\alpha_{mn}\delta_{kl}\delta_{mn} + 1), \\ \mathcal{B} &= (s\alpha_{kl}\delta_{kl} - \alpha_{mn}\delta_{mn}), \\ \mathcal{C} &= (s\alpha_{kl}\alpha_{mn}\delta_{kl}\delta_{mn} - 1), \\ \mathcal{D} &= (s\alpha_{kl}\delta_{kl} + \alpha_{mn}\delta_{mn}). \end{aligned} \tag{4.4}$$

5. Conclusions

The theoretical analysis makes it possible to obtain an integral formula for normalized mutual impedance with non-axisymmetric modes of free vibrations. It can be calculated for short acoustic waves with approximate methods. The obtained formulae for acoustic resistance (3.8) and reactance (4.3) are similar to each other in the form and have "oscillatory" character of variations.

The expression obtained for normalized impedance (2.11) can be used in the analysis of more complicated vibrations, e.g. with taking into account losses in the plate material [3].

References

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