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HARMONIC COMPONENTS OF FINITE AMPLITUDE SOUND WAVES REFLECTED AT A SURFACE

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The current analysis addresses the reflection of finite amplitude sound waves at a stationary surface. The analysis develops a two-orders perturbation solution for the nonlinear wave equation governing the velocity potential. The expression for the acoustic pressure derived from the potential lacked uniform validity. Hence it was corrected by employing coordinate straining transformations and thereby a uniformly accurate expression was obtained. Then, the strained coordinate transformations are eliminated by a Fourier analysis and a series representation solution is derived.

1. Introduction

It has been known for some time that finite amplitude acoustic waves will distort as it propagates [1–2]. This means that many harmonics are progressively generated even if the initial wave is purely monochromatic. The analysis of nonlinear propagation and distortion of finite amplitude acoustic waves has been given much attention in the literature. This is because the role of nonlinearities in generation and propagation of acoustical waves has many practical applications in addition to theoretical interest [3].

The present paper aims at studying the reflection of finite amplitude plane sound waves at a stationary plane target for which the nonlinearities of the medium are sufficient to alter the linear propagation of the waves. These alterations may play an important role in the processes employing acoustic devices using purposely intense sound waves such as devices used in ultrasonic measurements and in medical diagnostics.

If a finite amplitude wave travelling in a homogeneous medium reaches an interface where the properties of the medium change abruptly, various frequency components generated before incidence on the interface might be in part reflected and transmitted. The finite amplitude sound beam generated by a piston source and reflected at a pressure-release surface was investigated experimentally by Mellen and Browning [4]. A pulse technique was used by Van Buren and Breazeale [5–6] to measure the propagation and the phase shift reflection. Experimental evidence of the nonlinear effects in the reflection of parametric radiation from a finite planar targets was reported by MUIR *et al.* [7] and by KARABUTOVA *et al.* [8]. GARRETT *et al.* [9] studied theoretically and experimentally the difference frequency wave that generated by the interaction of two finite amplitude primary waves and reflected from finite size planar targets at normal incidence. The propagation of a thin finite amplitude acoustic beam generated by an oscillating piston and its reflection from a plane pressure-release surface was considered by HAMILTON *et al.* [10]. Their analysis used The Kuzentosv' equation in sound pressure, which is the parabolic approximation of the wave equation. The reflected waveform was shown to be different from that of the incident signal.

In this paper we investigate the behaviour of finite amplitude plane sound waves reflected at a stationary surface. This problem has previously been treated by QUAIN [11]. The method applied by him was straight perturbation expansion of Heap's wave equation and, as a consequence, his solution contained secular terms. Apart from this shortcoming, the solution fails to satisfy the boundary condition at a pressure-release surface.

The current investigation treats the reflection of finite amplitude waves at a stationary surface. A regular perturbation expansion is employed to obtain the velocity potential as a solution of the nonlinear wave equation that satisfies the boundary conditions at rigid or pressure-release surface. Only the first two orders of the velocity potential are derived. The first order terms correspond to the linearized field. The terms that represent the cumulative distortional effects of nonlinearities and represent the significant part of the potential are included in the second order terms. The expression for the acoustic pressure that derived from the potential contains secular terms. The method of renormalization is invoked to correct the nonuniform validity by introducing two independent near identity coordinate straining transformations. Further, these transformations are eliminated with the aid of Fourier analysis. Thus a simple expression for the acoustic pressure in terms of the original coordinates is obtained.

2. Basic equations

Consider a monochromatic finite amplitude acoustic plane wave impinges on a stationary surface with an impedance possibly depending on the angle of incidence θ $(0 \leqslant \theta < \pi/2)$. The amplitude of the wave is characterized by a small parameter ε where $|\varepsilon| \ll 1$ in most practical situations. The subsequent analysis is confined for simplicity to two dimensional waves. The x-axis coincides with the surface that is located a t $y=0$-plane. The unit normal $\mathbf{n}_s$, which is pointing outward the surface, is in the direction $\mathbf{e}_y$.

The nonlinear wave equation governing the velocity potential under isentropic conditions in invicid and irrotational fluid motion is [12]

$$c_0^2\nabla^2\phi-\frac{\partial^2\phi}{\partial t^2}=\frac{\partial}{\partial t}\left[\frac{1}{c_0^2}(\beta_0-1)\left(\frac{\partial\phi}{\partial t}\right)^2+\nabla\phi\cdot\nabla\phi\right]+O(\varepsilon^3), \tag{2.1}$$

where β_0 is the coefficient of nonlinearity and c_0 is the small signal speed of sound in linear theory. The fluid velocity is defined such that $\mathbf{v}=\nabla\phi$. Whilst the acoustic pressure is obtained from the Bernoulli equation using the binomial expansion and the fact that ϕ is $O(\varepsilon)$. It can be written, valid to the second order, as

$$p=-\rho_0\left[\frac{\partial\phi}{\partial t}+\frac{1}{2}\nabla\phi\cdot\nabla\phi-\frac{1}{2c_0^2}\left(\frac{\partial\phi}{\partial t}\right)^2\right]+O(\varepsilon^3). \tag{2.2}$$

The response must satisfy the appropriate boundary conditions, which depend on the reflectivity of the surface. For a rigid surface, the magnitude of its specific acoustic impedance $|z|\to\infty$ which corresponds to the reflection coefficient $R=1$. This requires that the component of the fluid velocity normal to the surface vanishes ($\mathbf{v}\cdot\mathbf{n}_s=0$ at $y=0$). For the ideal pressure-release reflector $|z|\to 0$, corresponding to $R=-1$. This condition is satisfied when the amplitude of the acoustic pressure vanishes at the surface regardless of the value of the fluid velocity [13].

The velocity potential is expanded in a perturbation series

$$\phi=\varepsilon\phi_1+\varepsilon^2\phi_2+\dots \tag{2.3}$$

The equation governing ϕ_1 and ϕ_2 are obtained by collecting like powers of ε in Eq. (2.1)

$$c_0^2\nabla^2\phi_1-\frac{\partial^2\phi_1}{\partial t^2}=0, \tag{2.4}$$

$$c_0^2\nabla^2\phi_2-\frac{\partial^2\phi_2}{\partial t^2}=\frac{\partial}{\partial t}\left[\frac{1}{c_0^2}(\beta_0-1)\left(\frac{\partial\phi_1}{\partial t}\right)^2+\nabla\phi_1\cdot\nabla\phi_1\right]. \tag{2.5}$$

The solution of the first order equation, Eq. (2.4), that satisfies the boundary conditions on either solid or pressure-release surfaces can be written as

$$\phi_1=\frac{c_0^2}{2\omega}\left[e^{i(\omega t-k_x x+k_y y)}+Re^{i(\omega t-k_x x-k_y y)}\right]+\text{c.c}, \tag{2.6}$$

where R is the pressure amplitude reflection coefficient in linear theory. It can depend on the incident angle θ as well as the angular frequency ω. Without loss of generality, R is considered to be real. Here c.c denotes the complex conjugate of all preceding terms. The wavenumber components k_x and k_y are found from satisfying the linear wave equation to be

$$k=\frac{\omega}{c_0},\qquad k_x=(k^2-k_y^2)^{1/2}\equiv k\sin\theta,\qquad k_y=k\cos\theta. \tag{2.7}$$

3. Description of the second order potential

The substitution of the first order solution ϕ_1 given by Eq. (2.6) into the right hand side of Eq. (2.5) yields the inhomogeneous equation for ϕ_2

$$\nabla^2\phi_2-\frac{1}{c_0^2}\frac{\partial^2\phi_2}{\partial t^2}=-\frac{i\omega}{2}\left[\beta_0 e^{2i(\omega t-\psi_1)}+\beta_0 R^2 e^{2i(\omega t-\psi_2)}+\right.$$
$$\left.+\frac{4k_y^2 R}{k^2}\left(\frac{\beta_0 k^2}{2k_y^2}-1\right)e^{i(2\omega t-\psi_1-\psi_2)}\right]+\text{c.c,} \tag{3.1}$$

where

$$\psi_1=k_x x-k_y y,$$
$$\psi_2=k_x x+k_y y. \tag{3.2}$$

The first two inhomogeneous terms on the right hand side of Eq. (3.1) are the result of self-action of the incident and of the reflected waves respectively. They excite the second harmonics of the corresponding waves. Such signals propagate parallel to the corresponding waves forming ϕ_1. The third inhomogeneous term is due to the nonlinear interaction of the incident and the reflected waves. It is independent of y since $\psi_1+\psi_2=2k_x x$. Therefore it excites a second harmonic wave whose propagation direction is parallel to the surface.

The solution of Eq. (3.1) consists of the complementary solution and the particular solution. The form of the right hand side of Eq. (3.1) suggests that the later solution is the superposition of the solutions associated with each of these inhomogeneous terms. These solutions may be found by the aid of the variation of parameters method. To this end one lets

$$\phi_2^p=C_1(y)e^{2i(\omega t-\psi_1)}+C_2(y)\,e^{2i(\omega t-\psi_2)}+C_3(x)\,e^{i(2\omega t-\psi_1-\psi_2)}+\text{c.c.} \tag{3.3}$$

The result of requiring that Eq. (3.3) satisfies Eq. (3.1) is a set of uncoupled differential equations for the unknown amplitude functions C_j. Making use of Eq. (3.2) these equations are found to be

$$C_1''+4ik_y C_1'=-\frac{i}{2}\omega\beta_0,$$
$$C_2''-4ik_y C_2'=-\frac{i}{2}\omega\beta_0 R^2, \tag{3.4}$$
$$C_3''-4k_y^2 C_3=-2i\omega R\frac{k_y^2}{k^2}\left(\frac{k^2\beta_0}{2k_y^2}-1\right),$$

The prime identifies differentiation with respect to the argument.

Without much analysis, one may solve Eqs. (3.4) and form the particular solution according to Eq. (3.3). Then, of course, one adds the complementary solution and requires that the total solution must satisfy the boundary conditions for a solid surface as well as for a pressure-release surface, and thereby chooses the arbitrary constants contained in the complementary solution. After performing these steps, one thus combines the resulting expression for ϕ_2 with the linearized solution given by Eq. (2.6) to arrive at the following expression for the potential

$$\phi=\varepsilon\frac{c_0^2}{2\omega}\left[e^{i(\omega t-\psi_1)}+Re^{i(\omega t-\psi_2)}\right]-$$

$$-\varepsilon^2\frac{c_0^2}{8\omega}\left\{\frac{k^2\beta_0 y}{k_y}\left[e^{2i(\omega t-\psi_1)}-R^2e^{2i(\omega t-\psi_2)}\right]+\right.$$

$$\left.+\frac{iRk_y^2}{k^2}\left[2(\mu-1)\,e^{i(2\omega t-\psi_1-\psi_2)}-\mu e^{2i(\omega t-\psi_1)}-\mu e^{2i(\omega t-\psi_2)}+2i\omega t\right]\right\}+\text{c.c}+O(\varepsilon^3), \tag{3.5}$$

where

$$\mu=\frac{2k^2}{k_y^2}\left(\frac{\beta_0 k^2}{2k_y^2}-1\right)+1. \tag{3.6}$$

The potential functions given by Eq. (3.5) contain secular terms and, consequently, the resulting expressions for the acoustic pressure and the velocity components will not be uniform. Nayfeh and Kluwick [14] and Ginsberg [15] have proved that the secular terms should be removed from the physical response variables such as acoustic pressure and velocity components, because the potential does indeed contain growing terms and an analysis that removes such terms is therefore conceptually in error. Accordingly, the method of renormalization will be applied to the acoustic pressure.

4. Application of the renormalization method

The acoustic pressure is linked to the potential by Eq. (2.2). The quadratic products in that relation will introduce non-secular terms. Thus differentiating Eq. (3.5) yields

$$\frac{p}{\rho_0 c_0^2}=\frac{\varepsilon}{2i}\left[e^{i(\omega t-\psi_1)}+Re^{i(\omega t-\psi_2)}\right]-$$

$$-\frac{\varepsilon^2}{4}\left\{\frac{\beta_0 k^2 y}{ik_y}\left[e^{2i(\omega t-\psi_1)}-R^2e^{2i(\omega t-\psi_2)}\right]-\right. \tag{4.1}$$

$$-\frac{k_y^2 R\mu}{k^2}\Big[e^{2i(\omega t-\psi_1)}+e^{2i(\omega t-\psi_2)}-2e^{i(2\omega t-\psi_1-\psi_2)}+ \qquad (4.1)\ \text{[cont.]}$$

$$+\frac{2}{\mu}(e^{i(\psi_1-\psi_2)}-1]\Big\}+\text{c.c}+O(\varepsilon^3)\,.$$

Inspection of Eq. (4.1) shows that at $O(\varepsilon^3)$ the first set of terms represents the sound in the second harmonics of the incident and of the reflected signals, respectively. The amplitude of each signal grows linearly with increasing y (secular behaviour). In contrast, the amplitude of the last set of terms remains bounded at all locations and these terms represent local effects. This means that the cumulative distortion only originates from the self-action of the incident and of the reflected waves.

In order to eliminate the secular terms which produce the nonuniformity in the expansion given by Eq. (4.1) the renormalization version of the method of strained coordinates is employed [16]. Different transformation is introduced for each of the wave variable ψ_j, $j=1$, 2. Further examination of Eq. (4.1) suggests that the trial transformations are

$$\psi_j=\alpha_j+\varepsilon[F_j e^{i(\omega_j t-\alpha_j)}+\text{c.c}], \quad j=1,\,2, \tag{4.2}$$

where the complex conjugate term is introduced to ensure that each transformation is real.

The aforementioned transformations are substituted into Eq. (4.1) and the result is expanded in a Taylor series in ascending powers of ε. Then the functions F_j are selected on the basis of removing the second order secular terms. This procedure yields the following expression for the acoustic pressure in real functional forms after accounting for the complex conjugate of each term

$$\frac{p}{\rho_0 c_0^2}=\varepsilon[\sin(\omega t-\alpha_1)+R\sin(\omega t-\alpha_2)]+p_{Ns}+O(\varepsilon^3)\,, \tag{4.3}$$

where

$$p_{Ns}=-\varepsilon^2\frac{k_y^2 R}{k^2}\{\mu[1-\cos(2k_y y)]\cos(2\omega t-2k_x x)+1\}\,. \tag{4.4}$$

The coordinate transformation are given by

$$\begin{aligned}\psi_1&=\alpha_1-\varepsilon c\sin(\omega t-\alpha_1)\,,\\ \psi_2&=\alpha_2+\varepsilon R\,c\sin(\omega t-\alpha_2)\,,\end{aligned} \tag{4.5}$$

where

$$c=\frac{\beta_0 k^2 y}{k_y}\,.$$

Equations (4.3) and (4.5) reveal that the response consists of two non-interactive waves, each is reminiscent of that for a planar wave with an important exception. The linear effect is measured by the difference between the nonlinear and linear spatial phases $\alpha_j - \psi_j$. In an isolated planar wave, this difference is proportional to the propagation distance which would be $(k_x x \pm k_y y)/k$ for the oblique wave. Instead, the distance parameter for each wave in Eq. (4.3) is ky/k_y. Therefore, it is to be concluded that although Eq. (4.3) specify a superposition of two waves, the response of one affects the other by altering the spatial dependence for the difference $\alpha_j - \psi_j$.

To this end, calculating the acoustic pressure at set of values (x, y, t) requires solution of each of the transcendental equations for the coordinate straining transformations, given by Eqs. (4.5). This can be accomplished by using a numerical procedure such as the Newton-Raphson's method. The frequency content of the temporal pressure waveform may be evaluated from its spectral analysis. However, with the aid of Fourier analysis, the acoustic pressure can be expressed in terms of the physical coordinates to avoide the solution of the transcendental equations for the coordinate transformations. The procedures are similar to that used in [17–18] and will not be repeated here. Specifically, the series representations for the acoustic pressure is

$$\frac{p}{\rho_0 c_0^2} = \frac{2}{c} \sum_{m=1}^{\infty} \left\{ \frac{(-1)^{m-1}}{m} J_m(\varepsilon mc) \sin[m(\omega t - \psi_1)] + \frac{1}{m} J_m(\varepsilon Rmc) \sin[m(\omega t - \psi_2)] \right\} + p_{Ns} + O(\varepsilon^3), \tag{4.6}$$

where J_m are the Bessel functions of first kind of order m.

The Fourier series solution given by Eq. (4.6) is analogous to the FUBINI-GHIRON [19] representation of a planar wave. It is valid if no shock forms. This occurs at distances less than the first location where multivaluedness of the waveform occurs. From Eq. (4.6) it is seen that each of the individual Fourier harmonics will undergo the same phase shift. This was noted experimentally by previous investigators [4–6].

5. Conclusion

The reflection of finite amplitude planar waves at a rigid or a pressure-release surface is investigated. The reflection takes place as though there were no coupling among harmonics. The nonlinear interaction between the incident and the reflected waves results in a wave with propagation direction that is parallel to the plane surface no matter what the incident angle is. It represents a local effect. In other words, it has no contribution to the distortion process. As a corollary, the cumulative distortion originates only from the self-action of the incident and of the reflected waves.

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